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A CONVERSE OF SCHWARZ'S INEQUALITY

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Abstract. A converse of Schwarz's inequality was studied by Bellman [2]. He gave an explicit bound for a continuous case. We aim to give a bound of Schwarz's inequality for the discrete case. Our method is to use the ratio of the value of biharmonic Green's functions.

1. Introduction

A converse of Schwarz's inequality was studied by Bellman [2] and other types of results are by many authors, for example, [5] and [4]. Actually, Bellman showed the following.

Theorem 1.1 (Bellman)**.** *Let u and v be concave functions on the interval* $[0, 1]$ *with* $\int_0^1 u(x)^2\,dx = \int_0^1 v(x)^2\,dx = 1$ *and* $u(0) = u(1) = v(0) = v(1) = 0$*. Then* $\int_0^1 u(x)v(x)\,dx \geq 1/2$*.*

Our aim is to obtain a discrete version of the above. Namely, we study some properties of the discrete biharmonic Green's functions and give a bound for the inequality by an elementary calculus.

More precisely, let $\mathcal{N} = \{X, Y, K, r\}$ be a finite network which is connected and has no self-loops. Denote by X the set of nodes, by Y the set of arcs, by r the resistance, which is a strictly positive function on Y, and by $K(x, y)$ the node-arc incident matrix as in [3]; i.e., K is a function defined on $X \times Y$ such that $K(x, y) = -1$ if x is the initial node of y; $K(x, y) = 1$ if x is the terminal node of y; $K(x, y) = 0$ otherwise. Let $L(X)$ ($L(Y)$ resp.) be the set of all real-valued functions on X (Y resp.). As for the discrete potential theory, we refer to Anandam [1].

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Let $X_0 \subsetneq X$ and call a node in $X \setminus X_0$ a boundary node. For $u \in L(X)$, we define the Laplacian Δu by

$$\Delta u(x) = -\sum_{y \in Y} K(x,y) r(y)^{-1} \sum_{z \in X} K(z,y) u(z).$$

Denote by $\mathbf{U}$ the set of $u \in L(X)$ such that

$$\Delta u(x) \le 0 \text{ on } X_0, \quad u(x) = 0 \text{ on } X \setminus X_0, \quad \sum_{x \in X_0} u(x)^2 = 1.$$

Our problem is to find the minimum of $\sum_{x \in X_0} u(x)v(x)$ for $u, v \in \mathbf{U}$. This is a discrete type of the converse to Schwarz's inequality studied in [2].

The condition $\Delta u(x) \le 0$ on X_0 implies that u is superharmonic on X_0 in the discrete potential theory. Let $g_a(x)$ be the Green's function of $\mathcal{N}$ with pole at $a \in X_0$, i.e., $g_a(x) = 0$ on $X \setminus X_0$ and $\Delta g_a(x) = -\epsilon_a(x)$ on X_0, where $\epsilon_a(a) = 1$ and $\epsilon_a(x) = 0$ if $x \ne a$. It is easily seen that every $u \in \mathbf{U}$ is uniquely represented as Green's potential of $\mu = -\Delta u$:

$$u(x) = \sum_{z \in X_0} g_z(x) \mu(z);$$

see Lemma 2.5 for the proof. For $u, v \in \mathbf{U}$, let $\mu = -\Delta u$ and $\nu = -\Delta v$. We have

$$\begin{aligned} \sum_{x \in X} u(x)v(x) &= \sum_{x \in X} \sum_{a \in X_0} g_a(x)\mu(a) \sum_{b \in X_0} g_b(x)\nu(b) \\ &= \sum_{a \in X_0} \sum_{b \in X_0} \mu(a)\nu(b) \sum_{x \in X} g_a(x) g_b(x). \end{aligned}$$

Let us put

$$q_a(b) = \sum_{x \in X} g_a(x) g_b(x)$$

and call it the biharmonic Green's function of $\mathcal{N}$ with pole at a. Then we see that

$$\sum_{x \in X} u(x)v(x) = \sum_{a \in X} \sum_{b \in X} q_a(b)\mu(a)\nu(b) =: Q(\mu, \nu)$$

is the biharmonic Green's mutual energy of μ and ν.

Our problem is reduced to find the minimum of $Q(\mu, \nu)$ subject to $\mu, \nu \ge 0$ and $Q(\mu, \mu) = Q(\nu, \nu) = 1$. This setting is the same as in [2]. Following the method of Bellman, we arrive at the following:

Theorem 1.2. *For $u, v \in \mathbf{U}$,*

$$\sum_{x \in X} u(x)v(x) \ge \min \left\{ \frac{q_a(b)}{\sqrt{q_a(a)}\sqrt{q_b(b)}}; a, b \in X_0 \right\}.$$

Our aim is to estimate the value of the right-hand side. We show the above theorem in Section 2. We shall estimate the ratio $q_a(b)^2/q_a(a)q_b(b)$ in Sections 3 and 4 for the finite linear network. As an application of the results, we give in Section 5 an explicit bound of the converse of Schwartz's inequality for discrete concave sequences.

2. Discrete Bellman's Theorem

First, we remind some properties of harmonic functions and Green's functions. Let $\{X, Y, K, r\}$ be a finite network and let $X_0 \subsetneq X$. We say that $u \in L(X)$ is harmonic at $x \in X$ if $\Delta u(x) = 0$. The following result is well-known as the maximum principle.

Lemma 2.1. *If u is harmonic on X_0 and $u = 0$ on $X \setminus X_0$, then $u = 0$ on X.*

For $u, v \in L(X)$, we define the discrete derivative du, the Dirichlet mutual sum $D(u, v)$, and the Dirichlet sum $D(u)$ by

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y)u(x),$$

$$D(u, v) = \sum_{y \in Y} r(y)[du(y)][dv(y)],$$

$$D(u) = D(u, u) = \sum_{y \in Y} r(y)[du(y)]^2.$$

Since $\mathcal{N}$ is a finite network, we can easily see by the change of summation that

Lemma 2.2. *For $u, v \in L(X)$,*

$$D(u, v) = -\sum_{x \in X}[\Delta u(x)]v(x) = -\sum_{x \in X} u(x)[\Delta v(x)].$$

Let g_a be the Green's function with pole at $a \in X_0$; i.e.,

$$\Delta g_a = -\epsilon_a \text{ on } X_0 \qquad \text{and} \qquad g_a = 0 \text{ on } X \setminus X_0.$$

Lemma 2.3. *For each $a \in X_0$, there exists a unique Green's function. Moreover, $g_a(b) = g_b(a)$ for $a, b \in X_0$.*

Proof. It is easy to see the contraction property, which means $D(|u|) \leq D(u)$ and $D(\min(u, 1)) \leq D(u)$ for $u \in L(X)$. Let us put $L_0(X) = \{u \in L(X); u = 0 \text{ on } X \setminus X_0\}$. We know that $L_0(X)$ is a Hilbert space with respect to the inner product $D(\cdot, \cdot)$ and that if $u_n, u \in L_0(X)$ and $D(u_n - u) \to 0$ as $n \to \infty$, then $u_n(x) \to u(x)$ as $n \to \infty$ for every $x \in X_0$.

For $a \in X_0$, we consider the following extremum problem:

$$d(a, B) = \min\{D(u); u \in L_0(X), u(a) = 1\}.$$

We can easily see that the minimizing sequence $\{u_n\}$ is a Cauchy sequence. Thus there exists $u^* \in L_0(X)$ such that $D(u_n - u^*) \to 0$ as $n \to \infty$. By the standard variational technique, $D(u^*, f) = 0$ for evey $f \in L_0(X)$ with $f(a) = 0$. We can conclude that $d(a, B) = D(u^*)$, $\Delta u^*(x) = 0$ for $x \neq a$, and $\Delta u^*(a) = -D(u^*)$.

Let us put

$$g_a(x) = \frac{u^*}{D(u^*)}.$$

By the contraction property, $0 \leq u^* \leq 1$ on X. We have $0 \leq g_a(x) \leq g_a(a)$ on X and $\Delta g_a(x) = -\varepsilon_a(x)$ on X. Namely, g_a satisfies the properties of Green's function with pole at a.

Let g'_a be another Green's function of $\mathcal{N}$ with pole at a and let $f = g_a - g'_a$. Then f is harmonic on X_0 and $f = 0$ on $X \setminus X_0$. Lemma 2.1 shows $f = 0$ on X. This means $g_a(x) = g'_a(x)$ on X.

Now we shall prove that $g_a(b) = g_b(a)$ for $a, b \in X_0$. In the above discussion, we see that for any $v \in L_B(X)$

$$D(v, g_a) = -\sum_{x \in X} v(x)[-\Delta g_a(x)] = v(a).$$

Taking $v = g_b$, we have $D(g_b, g_a) = g_b(a)$. Since $D(g_b, g_a) = D(g_a, g_b)$, we conclude that $g_a(b) = g_b(a)$. □

Let

$$\mathbf{U} = \left\{ u \in L(X); \Delta u(x) \leq 0 \text{ on } X_0,\ u(x) = 0 \text{ on } X \setminus X_0,\ \sum_{x \in X_0} u(x)^2 = 1 \right\}.$$

Let

$$q_a(b) = \sum_{x \in X_0} g_a(x) g_b(x)$$

for $a, b \in X_0$. We introduce an auxiliary lemma.

Lemma 2.4 ([2])**.** *Let $\sum_{i,j=1}^N a_{ij} x_i x_j$ be a positive definite quadratic form and let $b_i \geq 0$. Suppose $\sum_{i,j=1}^N a_{ij} x_i x_j = 1$. Then*

$$\sum_{i=1}^N b_i x_i \geq \min \left\{ \frac{b_i}{\sqrt{a_{ii}}}; 1 \leq i \leq N \right\} \qquad \text{for } x_j \geq 0;$$

the minimum is attained at the point of the form $(x_1, \dots, x_i, \dots, x_N) = (0, \dots, 1/\sqrt{a_{ii}}, \dots, 0)$ for some i.

Lemma 2.5. *Let $u \in \mathbf{U}$ and let $\mu = -\Delta u$. Then*

$$u(a) = \sum_{x \in X_0} g_x(a) \mu(x)$$

for $a \in X_0$.

Proof. Let $a \in X_0$. Using Lemmas 2.2 and 2.3 we have

$$\begin{aligned} \sum_{x \in X_0} g_a(x)\mu(x) &= -\sum_{x \in X_0} g_a(x)\Delta u(x) = -\sum_{x \in X} g_a(x)\Delta u(x) = -\sum_{x \in X} u(x)\Delta g_a(x) \\ &= -\sum_{x \in X_0} u(x)\Delta g_a(x) = \sum_{x \in X_0} u(x)\varepsilon_a(x) = u(a). \end{aligned}$$ □

We show Theorem 1.2.

Proof of Theorem 1.2. Let $\mu = -\Delta u$ and $\nu = -\Delta v$. Then $u(a) = \sum_x g_a(x)\mu(x)$ and $v(a) = \sum_x g_a(x)\nu(x)$ by Lemma 2.5. We have

$$1 = \sum_{a \in X_0} u(a)^2 = \sum_{a \in X_0} \left(\sum_{x \in X_0} g_a(x)\mu(x) \right)^2 = \sum_{a,b \in X_0} q_a(b)\mu(a)\mu(b).$$

Under this condition, Lemma 2.4 shows

$$\sum_{x\in X_0} u(x)v(x) = \sum_{x\in X_0} v(x) \sum_{a\in X_0} g_x(a)\mu(a) = \sum_{a\in X_0} \left(\sum_{x\in X_0} g_a(x)v(x) \right) \mu(a)$$
$$\geq \min_{a\in X_0} \frac{\sum_{x\in X_0} g_a(x)v(x)}{\sqrt{q_a(a)}}.$$

Next, we have

$$1 = \sum_{b\in X_0} v(b)^2 = \sum_{b\in X_0} \left(\sum_{x\in X_0} g_b(x)\nu(x) \right)^2 = \sum_{x,z\in X_0} q_x(z)\nu(x)\nu(z).$$

Under this condition, Lemma 2.4 shows

$$\sum_{x\in X_0} g_a(x)v(x) = \sum_{x\in X_0} g_a(x) \sum_{b\in X_0} g_x(b)\nu(b) = \sum_{b\in X_0} q_a(b)\nu(b) \geq \min_{b\in X_0} \frac{q_a(b)}{\sqrt{q_b(b)}}$$

for each $a \in X_0$. Therefore

$$\sum_{x\in X_0} u(x)v(x) \geq \min_{a,b\in X_0} \frac{q_a(b)}{\sqrt{q_a(a)}\sqrt{q_b(b)}}$$

as required. □

3. The Finite Linear Network with Two Boundary Nodes

Let n be an integer with $n \geq 2$. Let $\mathcal{N} = \{X, Y, K, r\}$ be a finite network such that $X = \{x_0, x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_n\}$, and $K(x_{i-1}, y_i) = -1$, $K(x_i, y_i) = 1$ for $1 \leq i \leq n$, and $K(x, y) = 0$ for any other pair (x, y). We take $r(y) = 1$ on Y. Let $X_0 = \{x_1, \ldots, x_{n-1}\}$. In this case, the Laplacian Δu is given by

$$\Delta u(x_k) = u(x_{k-1}) + u(x_{k+1}) - 2u(x_k) \quad \text{for } 0 < k < n;$$
$$\Delta u(x_0) = u(x_1) - u(x_0);$$
$$\Delta u(x_n) = u(x_{n-1}) - u(x_n).$$

The Green's function $g_{x_m}(x_k)$ of $\mathcal{N}$ with pole at x_m $(0 < m < n)$ is the function defined by the conditions:

$$g_{x_m}(x_0) = g_{x_m}(x_n) = 0, \qquad \Delta g_{x_m}(x_k) = -\epsilon_{x_m}(x_k) \quad \text{for } 0 < k < n.$$

Proposition 3.1. *The Green's function $g_{x_m}(x_k)$ is given by $g_{x_m}(x_0) = g_{x_m}(x_n) = 0$,*

$$g_{x_m}(x_k) = \left(1 - \frac{m}{n}\right) k \quad \text{for } 0 < k \leq m,$$
$$g_{x_m}(x_k) = \left(1 - \frac{k}{n}\right) m \quad \text{for } m + 1 \leq k < n.$$

Proof. By Lemma 2.3 it suffices to show that the function given here satisfies the above equations. In case $0 < k < m$ or $m < k < n$, we see easily $\Delta g_{x_m}(x_k) = 0$. In case $k = m$, we have

$$\begin{aligned}\Delta g_{x_m}(x_m) &= g_{x_m}(x_{m-1}) + g_{x_m}(x_{m+1}) - 2g_{x_m}(x_m) \\ &= \left(1 - \frac{m}{n}\right)(m-1) + \left(1 - \frac{m+1}{n}\right) m - 2\left(1 - \frac{m}{n}\right) m = -1. \qquad \square\end{aligned}$$

For $0 < l, m < n$, let us determine the biharmonic Green's function $q_{x_m}(x_l)$ defined by

$$q_{x_m}(x_l) = \sum_{k=1}^{n} g_{x_m}(x_k) g_{x_l}(x_k).$$

Note that $q_{x_m}(x_l) = q_{x_l}(x_m)$.

Theorem 3.2. *Let $0 < l \le m < n$. Then*

$$q_{x_l}(x_m) = \frac{l(n-m)(2mn + 1 - m^2 - l^2)}{6n}.$$

Proof. Let us put

$$A_l = \sum_{k=1}^{l} k^2, \qquad B_l = \sum_{k=1}^{l} k.$$

By definition, we have

$$q_{x_l}(x_m) = \sum_{k=1}^{l} g_{x_l}(x_k) g_{x_m}(x_k) + \sum_{k=l+1}^{m} g_{x_l}(x_k) g_{x_m}(x_k) + \sum_{k=m+1}^{n-1} g_{x_l}(x_k) g_{x_m}(x_k).$$

Proposition 3.1 shows that

$$\begin{aligned}\sum_{k=1}^{l} g_{x_l}(x_k) g_{x_m}(x_k) &= \sum_{k=1}^{l} \left(1 - \frac{l}{n}\right) k \left(1 - \frac{m}{n}\right) k = \frac{(n-l)(n-m)}{n^2} A_l; \\ \sum_{k=l+1}^{m} g_{x_l}(x_k) g_{x_m}(x_k) &= \sum_{k=l+1}^{m} \left(1 - \frac{k}{n}\right) l \left(1 - \frac{m}{n}\right) k \\ &= -\frac{l(n-m)}{n^2} \sum_{k=l+1}^{m} k^2 + \frac{l(n-m)}{n} \sum_{k=l+1}^{m} k \\ &= -\frac{l(n-m)}{n^2}(A_m - A_l) + \frac{l(n-m)}{n}(B_m - B_l); \\ \sum_{k=m+1}^{n-1} g_{x_l}(x_k) g_{x_m}(x_k) &= \sum_{k=m+1}^{n-1} \left(1 - \frac{k}{n}\right) l \left(1 - \frac{k}{n}\right) m = \frac{lm}{n^2} \sum_{k=m+1}^{n-1} (n-k)^2 \\ &= \frac{lm}{n^2} \sum_{k=1}^{n-m-1} k^2 = \frac{lm}{n^2} A_{n-m-1}.\end{aligned}$$

Thus we have

$$q_{x_l}(x_m) = \frac{n-m}{n}A_l - \frac{l(n-m)}{n^2}A_m + \frac{lm}{n^2}A_{n-m-1} + \frac{l(n-m)}{n}(B_m - B_l).$$

Using $A_k = k(k+1)(2k+1)/6$ and $B_k = k(k+1)/2$, we have

$$\begin{aligned}
\frac{n-m}{n}A_l &= \frac{l(n-m)}{6n}(l+1)(2l+1);\\
-\frac{l(n-m)}{n^2}A_m &= -\frac{lm(n-m)}{6n^2}(m+1)(2m+1);\\
\frac{lm}{n^2}A_{n-m-1} &= \frac{lm(n-m)}{6n^2}(n-m-1)(2n-2m-1)\\
&= \frac{lm(n-m)}{6n^2}\Big(2n^2 - 4mn - 3n + (m+1)(2m+1)\Big);\\
\frac{l(n-m)}{n}(B_m - B_l) &= \frac{l(n-m)}{2n}(m(m+1) - l(l+1))\\
&= \frac{l(n-m)(m-l)}{2n}(m+l+1).
\end{aligned}$$

Therefore

$$\begin{aligned}
q_{x_l}(x_m) &= \frac{l(n-m)}{6n}\Big((l+1)(2l+1) + m(2n-4m-3) + 3(m-l)(m+l+1)\Big)\\
&= \frac{l(n-m)}{6n}(2mn + 1 - m^2 - l^2),
\end{aligned}$$

which is the desired formula. □

We shall find the minimum of

$$A(l,m) = \frac{q_{x_l}(x_m)^2}{q_{x_l}(x_l)q_{x_m}(x_m)} \tag{3.3}$$

for $0 < l, m < n$.

Lemma 3.4. *For a fixed l, $A(l,m)$ decreases as m increases in $l \le m < n$. In particular, $A(l,m) \ge A(l,n-1)$ holds for $l \le m < n$.*

Proof. Theorem 3.2 shows that for $0 < l \le m < n$

$$A(l,m) = \frac{l^2}{6nq_{x_l}(x_l)}\frac{(n-m)(2nm+1-m^2-l^2)^2}{m(2nm+1-2m^2)}.$$

Let

$$f(t) = \frac{(n-t)(2nt+1-t^2-l^2)^2}{t(2nt+1-2t^2)}.$$

It suffices to show that $f(t)$ is a decreasing function in $l < t < n$. We have

$$\begin{aligned}\frac{d}{dt}\log f(t) &= -\frac{1}{n-t} + \frac{2(2n-2t)}{2nt+1-t^2-l^2} - \frac{1}{t} - \frac{2n-4t}{2nt+1-2t^2} \\ &= \left(\frac{2n}{2nt+1-2t^2} - \frac{1}{n-t} - \frac{1}{t}\right) + \left(\frac{4(n-t)}{2nt+1-t^2-l^2} - \frac{4(n-t)}{2nt+1-2t^2}\right) \\ &= \left(\frac{2n}{2t(n-t)+1} - \frac{n}{t(n-t)}\right) + \left(\frac{4(n-t)}{2nt+1-t^2-l^2} - \frac{4(n-t)}{2nt+1-2t^2}\right).\end{aligned}$$

Since $2t(n-t)+1 > 2t(n-t)$ and $2nt+1-t^2-l^2 > 2nt+1-2t^2$, we know that the right-hand side of the above is negative, which means that $f(t)$ is decreasing. □

Lemma 3.5. *$A(l, n-1)$ increases as l increases in $0 < l < n$. In particular, $A(l, n-1) \geq A(1, n-1)$ for $0 < l < n$.*

Proof. Theorem 3.2 shows that for $0 < l < n$

$$q_{x_l}(x_{n-1}) = \frac{l(n-(n-1))(2n(n-1)+1-(n-1)^2-l^2)}{6n} = \frac{l(n^2-l^2)}{6n},$$

so that

$$A(l, n-1) = \frac{1}{6nq_{x_{n-1}}(x_{n-1})} \frac{l(n-l)(n+l)^2}{2nl+1-2l^2}.$$

Let

$$g(t) = \frac{t(n-t)(n+t)^2}{2nt+1-2t^2}.$$

It suffices to show that $g(t)$ is an increasing function in $1 < t < n-1$. We have

$$\begin{aligned}\frac{d}{dt}\log g(t) &= \frac{1}{t} - \frac{1}{n-t} + \frac{2}{n+t} - \frac{2n-4t}{2nt+1-2t^2} \\ &= \left(\frac{1}{t} + \frac{1}{n-t} - \frac{2n}{2nt+1-2t^2}\right) + \left(\frac{4t}{2nt+1-2t^2} + \frac{2}{n+t} - \frac{2}{n-t}\right) \\ &= \left(\frac{n}{t(n-t)} - \frac{2n}{2t(n-t)+1}\right) + \left(\frac{4t}{2nt+1-2t^2} - \frac{4t}{n^2-t^2}\right).\end{aligned}$$

Since $2t(n-t) < 2t(n-t)+1$ and

$$(2nt+1-2t^2) - (n^2-t^2) = -(n-t)^2 + 1 \leq 0,$$

we know that the right-hand side of the above is positive, which means that $g(t)$ is increasing. □

We have

Theorem 3.6. *For $0 < l, m < n$, the following inequality holds:*

$$\frac{q_{x_l}(x_m)^2}{q_{x_l}(x_l)g_{x_m}(x_m)} \geq \frac{q_{x_1}(x_{n-1})^2}{q_{x_1}(x_1)q_{x_{n-1}}(x_{n-1})} = \frac{(n+1)^2}{(2n-1)^2}.$$

Proof. Since $q_{x_l}(x_m) = q_{x_m}(x_l)$, we may assume that $0 < l \le m < n$. The first inequality is shown by Lemmas 3.4 and 3.5. Theorem 3.2 shows that

$$\begin{aligned} q_{x_1}(x_{n-1}) &= \frac{2n(n-1)+1-(n-1)^2-1}{6n} = \frac{n^2-1}{6n}; \\ q_{x_1}(x_1) &= \frac{(n-1)(2n-1)}{6n}; \\ q_{x_{n-1}}(x_{n-1}) &= \frac{(n-1)(2n-1)}{6n}, \end{aligned}$$

which lead to the last equality. □

4. The Finite Linear Network with One Boundary Node

Let n be an integer with $n \ge 2$. Let $\mathcal{N} = \{X, Y, K, r\}$ be a finite network such that $X = \{x_0, x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_n\}$, and $K(x_{i-1}, y_i) = -1$, $K(x_i, y_i) = 1$ for $1 \le i \le n$, and $K(x, y) = 0$ for any other pair (x, y). We take $r(y) = 1$ on Y. Let $X_0 = \{x_1, \ldots, x_n\}$. The Green's function $g_{x_m}(x_k)$ of $\mathcal{N}$ with pole at x_m $(0 < m \le n)$ is the function defined by the conditions:

$$g_{x_m}(x_0) = 0, \qquad \Delta g_{x_m}(x_k) = -\epsilon_{x_m}(x_k) \quad \text{for } 0 < k \le n.$$

We can show the following similar to Proposition 3.1.

Proposition 4.1. *The Green's function $g_{x_m}(x_k)$ is given by $g_{x_m}(x_0) = 0$,*

$$\begin{aligned} g_{x_m}(x_k) &= k \quad \text{for } 0 < k \le m, \\ g_{x_m}(x_k) &= m \quad \text{for } m+1 \le k \le n. \end{aligned}$$

Theorem 4.2. *Let $0 < l \le m \le n$. Then*

$$q_{x_l}(x_m) = \frac{l(6mn - 3m^2 + 3m - l^2 + 1)}{6}.$$

Proof. By definition, we have

$$\begin{aligned} q_{x_l}(x_m) &= \sum_{k=1}^{l} g_{x_l}(x_k) g_{x_m}(x_k) + \sum_{k=l+1}^{m} g_{x_l}(x_k) g_{x_m}(x_k) + \sum_{k=m+1}^{n} g_{x_l}(x_k) g_{x_m}(x_k) \\ &= \sum_{k=1}^{l} k^2 + \sum_{k=l+1}^{m} lk + \sum_{k=m+1}^{n} lm \\ &= \frac{1}{6} l(l+1)(2l+1) + \frac{1}{2} l \Big(m(m+1) - l(l+1) \Big) + lm(n-m) \\ &= \frac{l(6mn - 3m^2 + 3m - l^2 + 1)}{6} \end{aligned}$$

as desired. □

Let $A(l, m)$ be the value as defined in (3.3).

Lemma 4.3. *For a fixed l, $A(l, m)$ decreases as m increases in $l \le m \le n$. In particular, $A(l, m) \ge A(l, n)$ holds for $l \le m \le n$.*

Proof. Theorem 4.2 shows that for $0 < l \leq m \leq n$

$$A(l,m) = \frac{l^2}{6q_{x_l}(x_l)} \frac{(6mn - 3m^2 + 3m - l^2 + 1)^2}{m(6mn - 4m^2 + 3m + 1)}.$$

Let

$$f(t) = \frac{(6nt - 3t^2 + 3t - l^2 + 1)^2}{t(6nt - 4t^2 + 3t + 1)}.$$

It suffices to show that $f(t)$ is a decreasing function in $l < t < n$. We have

$$\begin{aligned}
\frac{d}{dt} \log f(t) &= \frac{2(6n - 6t + 3)}{6nt - 3t^2 + 3t - l^2 + 1} - \frac{1}{t} - \frac{6n - 8t + 3}{6nt - 4t^2 + 3t + 1} \\
&= \left(\frac{6(2n - 2t + 1)}{6nt - 3t^2 + 3t - l^2 + 1} - \frac{6(2n - 2t + 1)}{6nt - 4t^2 + 3t + 1} \right) \\
&\qquad + \left(\frac{6n - 4t + 3}{6nt - 4t^2 + 3t + 1} - \frac{6n - 4t + 3}{t(6n - 4t + 3)} \right).
\end{aligned}$$

Since $6nt - 3t^2 + 3t - l^2 + 1 > 6nt - 4t^2 + 3t + 1$ and $6nt - 4t^2 + 3t + 1 > t(6n - 4t + 3)$, we know that the right-hand side of the above is negative, which means that $f(t)$ is decreasing. □

Lemma 4.4. *$A(l,n)$ increases as l increases in $0 < l \leq n$. In particular, $A(l,n) \geq A(1,n)$ for $0 < l \leq n$.*

Proof. Theorem 4.2 shows that for $0 < l \leq n$

$$A(l,n) = \frac{1}{6q_{x_n}(x_n)} \frac{l(3n^2 - l^2 + 3n + 1)^2}{6ln - 4l^2 + 3l + 1}.$$

Let

$$g(t) = \frac{t(3n^2 - t^2 + 3n + 1)^2}{6nt - 4t^2 + 3t + 1}.$$

It suffices to show that $g(t)$ is an increasing function in $1 < t < n$. We have

$$\begin{aligned}
\frac{d}{dt} \log g(t) &= \frac{1}{t} - \frac{4t}{3n^2 - t^2 + 3n + 1} - \frac{6n - 8t + 3}{6nt - 4t^2 + 3t + 1} \\
&= \left(\frac{4t}{6nt - 4t^2 + 3t + 1} - \frac{4t}{3n^2 - t^2 + 3n + 1} \right) \\
&\qquad + \left(\frac{6n - 4t + 3}{t(6n - 4t + 3)} - \frac{6n - 4t + 3}{6nt - 4t^2 + 3t + 1} \right).
\end{aligned}$$

Since

$$\begin{aligned}
(6nt - 4t^2 + 3t + 1) - (3n^2 - t^2 + 3n + 1) &= -3n^2 + 6nt - 3t^2 - 3n + 3t \\
&= -3(n - t)(n - t + 1) < 0
\end{aligned}$$

and $t(6n - 4t + 3) - (6nt - 4t^2 + 3t + 1) < 0$, we know that the right-hand side of the above is positive, which means that $g(t)$ is increasing. □

We have

Theorem 4.5. *For $0 < l, m \le n$, the following inequality holds:*

$$\frac{q_{x_l}(x_m)^2}{q_{x_l}(x_l)g_{x_m}(x_m)} \ge \frac{q_{x_1}(x_n)^2}{q_{x_1}(x_1)q_{x_n}(x_n)} = \frac{3(n+1)}{2(2n+1)}.$$

Proof. Since $q_{x_l}(x_m) = q_{x_m}(x_l)$, we may assume that $0 < l \le m \le n$. The first inequality is shown by Lemmas 4.3 and 4.4. Theorem 4.2 shows that

$$\begin{aligned} q_{x_1}(x_n) &= \frac{3n^2+3n}{6} = \frac{n(n+1)}{2}; \\ q_{x_1}(x_1) &= n; \\ q_{x_n}(x_n) &= \frac{n(2n^2+3n+1)}{6} = \frac{n(2n+1)(n+1)}{6}, \end{aligned}$$

which lead to the last equality. □

5. An Application

As stated in the introduction, we study the converse of Schwarz's inequality. A sequence $\{u_k; 0 \le k \le n\}$ is called concave if $2u_k \ge u_{k-1} + u_{k+1}$ for every $k \in \{1, \ldots, n-1\}$

Theorem 5.1. *Let $\alpha_n = \sqrt{6/\big(n(n+1)(2n+1)\big)}$.*

(i) Let $\{u_k; 0 \le k \le n\}$ and $\{v_n; 0 \le k \le n\}$ be concave sequences of real-numbers such that $u_0 = v_0 = u_n = v_n = 0$ and

$$\sum_{k=1}^{n-1} u_k^2 = \sum_{k=1}^{n-1} v_k^2 = 1.$$

Then

$$\sum_{k=1}^{n-1} u_k v_k \ge \frac{n+1}{2n-1}. \tag{5.2}$$

Let $u_k = \alpha_{n-1}(n-k)$ for $k \ge 1$ and $u_0 = 0$. Let $v_k = \alpha_{n-1}k$ for $k \le n-1$ and $v_n = 0$. Then $\{u_k\}$ and $\{v_k\}$ satisfy the above conditions and hold the equality in (5.2).

(ii) Let $\{u_k; 0 \le k \le n\}$ and $\{v_n; 0 \le k \le n\}$ be concave sequences of real-numbers such that $u_0 = v_0 = 0$ and

$$\sum_{k=1}^{n} u_k^2 = \sum_{k=1}^{n} v_k^2 = 1.$$

Then

$$\sum_{k=1}^{n} u_k v_k \ge \sqrt{\frac{3(n+1)}{2(2n+1)}}. \tag{5.3}$$

Let $u_k = 1/\sqrt{n}$ for $k \ge 1$ and $u_0 = 0$. Let $v_k = \alpha_n k$ for $k \le n$. Then $\{u_k\}$ and $\{v_k\}$ satisfy the above conditions and hold the equality in (5.3).

Proof. To prove (i) we define two functions u and v by $u(x_k) = u_k$ and $v(x_k) = v_k$. Also we define $\mathcal{N} = \{X, Y, K, r\}$ and $X_0 = \{x_1, \dots, x_{n-1}\}$ as those in Section 3. Then $u, v \in \mathbf{U}$. By Theorems 1.2 and 3.6, we have

$$\sum_{k=1}^{n} u_k v_k \geq \frac{q_{x_1}(x_{n-1})}{\sqrt{q_{x_1}(x_1)}\sqrt{q_{x_{n-1}}(x_{n-1})}} = \frac{n+1}{2n-1}.$$

It is easy to see the latter part.

Similarly Theorems 1.2 and 4.5 show (ii). □

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References

[1] Victor Anandam, *Harmonic functions and potentials on finite or infinite networks*, Lecture Notes of the Unione Matematica Italiana, vol. 12, Springer, Heidelberg, 2011.

[2] Richard Bellman, *Converses of Schwarz's inequality*, Duke Math. J. **23** (1956), 429–434.

[3] Hisayasu Kurata and Maretsugu Yamasaki, *Discrete Green potentials with finite energy*, Hokkaido Math. J. **47** (2018), no. 3, 607–624.

[4] Constantin P. Niculescu, *Converses of the Cauchy-Schwarz inequality in the C^*-framework*, An. Univ. Craiova Ser. Mat. Inform. **26** (1999), 22–28.

[5] Don Zagier, *A converse to Cauchy's inequality*, Amer. Math. Monthly **102** (1995), no. 10, 919–920.

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