DISSERTATION

ON POSITIVELY GRADED UNIQUE FACTORIZATION RINGS AND UNIQUE FACTORIZATION MODULES

IWAN ERNANTO

DEPARTEMEN OF MATHEMATICAL SCIENCES
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
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IWAN ERNANTO

DEPARTEMENT OF MATHEMATICAL SCIENCES
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
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ABSTRACT

ON POSITIVELY GRADED UNIQUE FACTORIZATION RINGS AND UNIQUE FACTORIZATION MODULES

By

IWAN ERNANTO

Let R be a prime ring that is Noetherian, and let Q be its quotient ring. Consider a (fractional) ideal A in Q. Define the left R-ideal $(R : A)_l = \{q \in Q \mid qA \subseteq R\}$, and the right R-ideal $(R : A)_r = \{q \in Q \mid Aq \subseteq R\}$. We define a v-operation: $A_v = (R : (R : A)_r)_l \supseteq A$ and if $A = A_v$ then A is called a right v-ideal. Similarly, ${}_vA = (R : (R : A)_l)_r$ and A is called a left v-ideal if $A =_v A$. If ${}_vA = A = A_v$, then A is just called a v-ideal in Q. Further,define left order $O_l(A) = \{q \in Q \mid qA \subseteq A\}$ and right order $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$ of A. In 1991, Abbasi et.al. defined a unique factorization ring (UFR for short) by using v-ideal, that is, a ring R is called a UFR if any prime ideal P with $P = P_v$ or $P =_v P$ is principal, that is, $P = pR = Rp$ for some $p \in P$.

Let $R = \oplus_{n\in\mathbb{Z}_0} R_n$ be a positively graded ring which is a sub-ring of the strongly graded ring $S = \oplus_{n\in\mathbb{Z}} R_n$, where R_0 is a Noetherian prime ring. In this dissertation, it is demonstrated that R qualifies as a unique factorization ring if and only if R_0 is a $\mathbb{Z}_0$-invariant unique factorization ring, and R_1 is a principal (R_0, R_0) bi-module. We give examples of $\mathbb{Z}_0$-invariant unique factorization rings which are not unique factorization rings.

Let M be a torsion-free module over an integral domain D with its quotient field K. In 2022, Nurwigantara et al. introduced the concept of a completely integrally closed module (CICM for short) for investigating arithmetic module theory. A module M is designated as a CICM if, for every non-zero submodule N of M, $O_K(N) = \{k \in K \mid kN \subseteq N\} = D$. Conversely, Wijayanti et al. introduced the notion of a v-submodule. In this context, a fractional submodule N in KM is termed a v-submodule if it satisfies $N = N_v$, where $N_v = (N^-)^+$. Here, $N^- = \{k \in K \mid kN \subseteq N\}$, and $\mathfrak{n}^+ = \{m \in KM \mid \mathfrak{n}m \subseteq M\}$ for a fractional M-ideal $\mathfrak{n}$ in K. Further, in 2022, Wahyuni et.al. defined a unique factorization module (UFM for short) by a submodule approach. A module M is called a UFM if M is completely integrally closed, every v-submodule of M is principal, and M satisfies the ascending chain condition on v-submodules of M. In this dissertation, we prove that if D is a unique factorization domain and M is a completely integrally closed module with the ascending chain condition on v-submodules, then M is a unique factorization module (UFM) if and only if every prime v-submodule P of M is principal, that is, $P = pM$ for some $p \in D$.

Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a strongly graded module over a strongly graded ring $D = \oplus_{n\in\mathbb{Z}} D_n$ and $L = \oplus_{n\in\mathbb{Z}0} M_n$ be a positively graded module over a positively graded domain $R = \oplus_{n\in\mathbb{Z}0} D_n$. In this dissertation, we investigated whether the properties found in UFR can be developed in UFM. Some results that can be obtained include: if M_0 is a UFM over D_0 and D is a UFD, then M is a UFM over D. Moreover, we provide a necessary and sufficient condition for a positively graded module L to be a UFM over a positively graded R.

This dissertation is organized as follows. In Chapter I, we provide the historical research of this research. In Chapter II, we provide some preliminaries regarding graded rings and graded modules. In Chapter III, we provide some results regarding to UFRs. In Chapter IV, we provide some results regarding to UFMs, particularly related to strongly graded modules and positively graded modules. In Chapter V, we end this dissertation with some results on the generalized Dedekind module and future research plans.

Keywords: positively graded ring, positively graded module, unique factorization ring, unique factorization module, generalized Dedekind module.

CHAPTER I

Introduction

1.1. Background

This dissertation represents an extension of the work presented in [27] and [41]. Consider a Noetherian prime ring R with its quotient ring Q. For a (fractional) ideal A in Q, we define the left R-ideal $(R : A)_l = \{q \in Q \mid qA \subseteq R\}$ and the right R-ideal $(R : A)_r = \{q \in Q \mid Aq \subseteq R\}$. Introducing a v-operation, we define $A_v = (R : (R : A)_r)_l \supseteq A$, where A is termed a right v-ideal if $A = A_v$. Similarly, ${}_vA = (R : (R : A)_l)_r$ defines a left v-ideal for A if $A =_v A$. When ${}_vA = A = A_v$, A is simply referred to as a v-ideal in Q. Furthermore, left and right orders of A are denoted by $O_l(A) = \{q \in Q \mid qA \subseteq A\}$ and $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, respectively. In [15], a unique factorization ring (UFR) is defined using v-ideals, where a ring R is classified as a UFR if every prime ideal P with $P = P_v$ or $P =_v P$ is principal, i.e., $P = pR = Rp$ for some $p \in P$.

Let $R = \oplus_{n\in\mathbb{Z}_0} R_n$ be a positively graded ring, which is a sub-ring of the strongly graded ring $S = \oplus_{n\in\mathbb{Z}} R_n$, where R_0 is a Noetherian prime ring. In [27], the authors established a necessary and sufficient condition for R to qualify as a maximal order, denoted by $O_l(A) = R = O_r(A)$ for any non-zero ideal A of R. Here, $O_l(A) = \{q \in Q \mid qA \subseteq A\}$ and $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, where Q represents the quotient field of R. Additionally, in [28], the authors provided insights into the structure of v-invertible ideals of R. In this dissertation, particularly in Chapter III, we prove that R attains the status of a unique factorization ring (in the sense of [15]) if and only if R_0 is a $\mathbb{Z}_0$-invariant unique factorization ring and R_1 is a principal (R_0, R_0) bi-module.

Let M be a finitely generated torsion-free module over an integrally closed domain D with its quotient field K. The module M is naturally embedded in KM, a finite-dimensional vector space over K. In [28], the authors introduced key concepts and notation for the study of arithmetic module theory. Consider a fractional D-ideal $\mathfrak{a}$ in K and a fractional D-submodule N in KM (refer to [28] for the definition of fractional D-submodules). They defined $\mathfrak{a}^+ = \{m' \in KM \mid \mathfrak{a}m' \subseteq M\}$ as a fractional D-submodule, and $N^- = \{k \in K \mid kN \subseteq M\}$ as a fractional ideal. Additionally, $N_v = (N^-)^+ \supseteq N$ is defined. A submodule N is called a v-*submodule* if $N_v = N$. If $M \supseteq N$, then N is referred to as an *integral* submodule of M. The domain D is called a generalized Dedekind domain (G-Dedekind domain) if every v-ideal of D is invertible, and D satisfies the ascending chain condition on v-ideals ([12] and [36]).

Moreover, in [41], the authors introduced the concept of a unique factorization module (UFM) using a submodule approach. A module M is designated as a UFM if it is completely integrally closed (CIC), meaning that $O_K(N) = \{k \in K \mid kN \subseteq N\} = D$ for every non-zero submodule N of M, where K is the quotient field of D. Additionally, every v-submodule of M must be principal, and M must adhere to the ascending chain condition on v-submodules. In this dissertation, specifically in Chapter IV, we establish that if D is a unique factorization domain (UFD) and M is a CIC module satisfying the ascending chain condition on v-submodules, then M qualifies as a UFM if and only if every prime v-submodule P of M is principal, denoted as $P = pM$ for some $p \in R$.

Consider $M = \oplus_{n\in\mathbb{Z}} M_n$, a strongly graded module over the strongly graded ring $D = \oplus_{n\in\mathbb{Z}} D_n$, and $L = \oplus_{n\in\mathbb{Z}_0} M_n$, a positively graded module over the positively graded domain $R = \oplus_{n\in\mathbb{Z}_0} D_n$. In this dissertation, we explore the extension of properties observed in unique factorization rings (UFRs) to unique factorization modules (UFMs). Some notable results include the following: if M_0 is a UFM over D_0 and D is a unique factorization domain (UFD), then M qualifies as a UFM over D. Additionally, we establish a necessary and sufficient condition for a positively graded module L to be a UFM over a positively graded ring R.

1.2. Limitation of Problems

Note that in the definition of positively graded rings, we always assume that R_0 is a Noetherian prime ring. Furthermore, in the definition of strongly and positively graded module, we assume that M_0 is a finitely generated torsion-free module.

1.3. Formulation of Problems

Based on the background and limitations above, the problems can be formulated as follows:

(1) to find the characterizations of positively graded rings regarding unique factorization rings;

(2) to find the characterizations of strongly graded modules regarding unique factorization modules;

(3) to find the characterizations of positively graded modules regarding unique factorization modules;

1.4. Research Method

The first thing done in the method is the basic properties of multiplicative ideal theory about fractional ideals, including invertible ideals and v-invertible ideals. Then, the properties of completely integrally closed domains, Dedekind domains, G-Dedekind domains, and maximal order are studied. Then, the theoretical study is continued by learning the basic properties of fractional submodules, completely integrally closed modules, and unique factorization modules. Next, we study strongly and positively graded ring types of $\mathbb{Z}$, regarding maximal order, generalized Dedekind rings, and unique factorization rings. After that, we study the unique factorization module from the point of view of the submodule. After that, we generalized the result in positively graded rings to the positively graded module.

CHAPTER II

Preliminaries

2.1. Graded Rings and Graded Modules

Definition 2.1.1 *Let R be a ring, and G be a commutative group. A ring R is called a G-graded ring, or simply a graded ring, if it can be expressed as $R = \oplus_{g\in G} R_g$, where each R_g is an additive subgroup of R, and the product $R_g R_h$ is contained in R_{gh} for all $g, h \in G$. Furthermore, if $R_g R_h = R_{gh}$ holds for all $g, h \in G$, then the ring R is specifically referred to as a strongly graded ring.*

The set $R^h = \cup_{g\in G} R_g$ is denoted as the set of all homogeneous elements of A. Each additive subgroup R_g is referred to as the g-component of R, and the non-zero elements belonging to R_g are called homogeneous elements of degree g.

Proposition 2.1.2 *Let $R = \oplus_{g\in G} R_g$ be a graded ring type G. Then*

(1) 1_R is a homogenous of degree e, where e is the identity element of G;

(2) R_e is a subring of R;

(3) Ecah R_g is a R_e-bimodule; item For an invertible element $r \in R_g$, its inverse, r^{-1} is a homogenous of degree g^{-1}, that is $r^{-1} \in R_{g^{-1}}$ where g^{-1} is the inverse of g.

Definition 2.1.3 *Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A subring S of R is called a graded subring if $S = \oplus_{g\in G} S_g$ where $S_g = S \cap R_g$. Moreover, an ideal I of R is called a graded ideal if $I = \oplus_{g\in G} I_g$ where $I_g = I \cap R_g$.*

Example 2.1.4 Let $G = (\mathbb{Z}_2, +)$ and $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$. Suppose that $R_{\overline{0}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{Z} \right\}$ and $R_{\overline{1}} = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \mid b, c \in \mathbb{Z} \right\}$. Then R is a $\mathbb{Z}_2$-graded ring. Moreover, if $S = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{Z} \right\}$ then S is a graded subring of R with $S_{\bar{0}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{Z} \right\}$ and $S_{\bar{1}} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{Z} \right\}$.

Example 2.1.5 Let $G = (\mathbb{Z}_2, +)$ and $S = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{Z} \right\}$. From Example 2.1.4, it is known that S is a $\mathbb{Z}_2$-graded ring. Let $I = \left\{ \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \mid b, d \in \mathbb{Z} \right\}$. Then I is a graded ideal of S with $I_{\bar{0}} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \mid d \in \mathbb{Z} \right\}$ and $I_{\bar{1}} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{Z} \right\}$.

Definition 2.1.6 *Consider a graded ring R and an R-module M. We define M as a graded R-module if there exists a family of additive subgroups $\{M_g\}_{g \in G}$ of M such that M can be expressed as the direct sum $\oplus_{g \in G} M_g$, that is, $M = \oplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ holds for all $g, h \in G$. Additionally, a module M is called a strongly graded module if $R_g M_h = M_{gh}$ for all $g, h \in G$.*

Definition 2.1.7 *Consider a graded R-module $M = \oplus_{g \in G} M_g$ and let N be a submodule of M. A submodule N is referred to as a graded (or homogeneous) submodule of M if it can be expressed as $N = \oplus_{g \in G} N_g$, where $N_g = N \cap M_g$.*

In the rest of this dissertation, we always consider the commutative group G as a group of integers $\mathbb{Z}$ and we just consider the strongly graded ring and module type of $\mathbb{Z}$.

2.2. Positively Graded Rings which are Maximal Orders

Consider $R = \oplus_{n \in \mathbb{Z}_0} R_n$, a positively graded ring which is a subring of the strongly graded ring $S = \oplus_{n \in \mathbb{Z}} R_n$. In this context, R_0 represents a prime Goldie ring, and it comes with its quotient ring Q_0.

We initiate this section with the subsequent proposition.

Proposition 2.2.1 *(Proposition 2.1 of [27]) The ring R is Noetherian if and only if R_0 is Noetherian.*

In this section, it is assumed that the positively graded ring R is Noetherian, along with its quotient ring Q, unless explicitly mentioned otherwise. The subsequent lemma is derived analogously to the case of strongly graded rings (refer to Corollary 1.2 of [21]).

Lemma 2.2.2 *(Lemma 2.1 of [28]) Let $\mathcal{C}_0$ denote the set of all regular elements in R_0. The following statements hold:*

(1) $\mathcal{C}_0$ forms an Ore set of R, and $Q_0^g = \oplus_{n \in \mathbb{Z}_0} Q_0 R_n$ represents the graded quotient ring of R at $\mathcal{C}_0$, where $Q_0 R_n = R_n Q_0$ for any $n \in \mathbb{Z}_0$.

(2) $Q_0^g = Q_0[X, \sigma]$, identified as a skew polynomial ring, where X stands as a regular element in R_1 with $XQ_0 = R_1 Q_0 = Q_0 R_1 = Q_0 X$. The automorphism σ operates on R_0, and Q_0^g is characterized as a principal ideal ring.

Definition 2.2.3 *(Definition 2.1 of [27])*

(1) Let A_0 be an (R_0, R_0)-bimodule of Q_0. Then A_0 is called $\mathbb{Z}_0$-invariant if $R_n A_0 = A_0 R_n$ holds for every $n \in \mathbb{Z}_0$.

(2) An ideal A of R is called a $\mathbb{Z}_0$-invariant if the condition $R_n A = A R_n$ holds for all $n \in \mathbb{Z}_0$.

Lemma 2.2.4 *(Lemma 2.2 of [27]) Let A_0 be a $\mathbb{Z}_0$-invariant R_0-ideal in Q_0. Then $A = A_0 R$ forms an R-ideal in Q. If A_0 is an ideal of R_0, the converse also holds.*

Consider a prime Goldie ring R with its quotient ring Q. For a (fractional) right (left) R-ideal $I(J)$, define $(R : I)_l = \{q \in Q \mid qI \subseteq R\}$ as a left R-ideal in Q, and $(R : J)_r = \{q \in Q \mid Jq \subseteq R\}$ as a right R-ideal in Q. Introduce a v-operation: $I_v = (R : (R : I)_l)_r \supseteq I$, and label I as a right v-ideal if $I = I_v$. Similarly, ${}_vJ = (R : (R : J)_r)_l$, and J is termed a left v-ideal if $J =_v J$. For an R-ideal A in Q, designate A as a v-ideal if ${}_vA = A = A_v$. Additionally, define

$O_l(A) = \{q \in Q \mid qA \subseteq A\}$ as a left order of A, and $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$ as a right order of A.

Definition 2.2.5 *(Definition 2.2 of [27]) Let R be a prime Goldie ring with its quotient ring Q. A v-ideal A in Q is labelled as v-invertible if it fulfills the condition ${}_v((R : A)_l A) = R = (A(R : A)_r)_v$.*

Lemma 2.2.6 *(Lemma 2.3 of [27]) Let R be a prime Goldie ring with its quotient ring Q and A be an R-ideal in Q.*

(1) When $O_l(A) = R = O_r(A)$, it follows that $(R : A)_l = A^{-1} = (R : A)_r$, where $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$, and A^{-1} is an R-ideal in Q.

(2) If A is v-invertible, then both $O_l(A) = R$ and $O_r(A) = R$ hold.

Proof. By the proof of Lemma 2.3 of [27], we have the following:

(1) Let $q \in (R : A)_l$. This implies $q \in Q$ and satisfies $qA \subseteq R$. As A is an R-ideal, we have $AqA \subseteq A$, implying $q \in A^{-1}$. Thus, $(R : A)_l \subseteq A^{-1}$. Conversely, assume $q \in A^{-1}$, meaning $AqA \subseteq A$ and implying $qA \subseteq O_r(A) = R$. Therefore, $q \in (R : A)_l$. Similarly, $A^{-1} = (R : A)_l$, and it is evident that A^{-1} is also an R-ideal in Q.

(2) It is clear that $R \subseteq O_l(A)$. For $q \in O_l(A)$, which implies $qA \subseteq A$, we find $q \in qR = q(A(R : A)_r)_v = (qA(R : A)_r)_v \subseteq (A(R : A)_r)_v \subseteq R$. Therefore, $O_l(A) = R$. Similarly, $O_r(A) = R$.

■

Next, we will describe all prime ideals of R.

Proposition 2.2.7 *(Proposition 2.2 of [27]) Let P be a prime ideal of R such that $P_0 = P \cap R_0 \neq (0)$ and is $\mathbb{Z}_0$-invariant. Then*

(1) $P_1 = P_0 R$ is a prime ideal.

(2) If P_1 is v-invertible and $P = P_v$, then $P = P_1$.

Lemma 2.2.8 *(Lemma 2.4 of [27]) Let P be a prime ideal of R. Then*

(1) If $P \not\supseteq R_1$, then P and $P_0 = P \cap R_0$ are both $\mathbb{Z}_0$-invariant.

(2) If P contains R_1 and $P = P_v$, then $P = \oplus_{n \geq 1} R_n$ and is an invertible ideal.

Lemma 2.2.9 *(Lemma 2.5 of [27]) Let I_0 be a right R_0-ideal in Q_0 and J_0 be a left R_0-ideal in Q_0. Then*

(1) $(R : I_0R)_l = R(R_0 : I_0)_l$ and $(R : RJ_0)_r = (R_0 : J_0)_rR$.

(2) $(I_0R)_v = (I_0)_vR$ and $_v(RJ_0) = R(_vJ_0)$.

(3) Let A_0 be a $\mathbb{Z}_0$-invariant R_0-ideal in Q_0. Then $O_l(A_0R) = RO_l(A_0)$ and $O_r(A_0R) = O_r(A_0)R$.

Proof. By the proof of Lemma 2.5 of [27], we have the following:

(1) Clearly, $R(R_0 : I_0)_l \subseteq (R : I_0R)_l$. Let $q \in (R : I_0R)_l$, that is, $qI_0R \subseteq R$ and $qI_0Q_0^g \subseteq Q_0^g$. Therefore $q \in Q_0^g$ since $I_0Q_0^g = Q_0^g$. Express $q = q_n + \cdots + q_0$, where $q_i \in Q_0R_i = R_iQ_0$. Then $R \supseteq qI_0$ implies $q_iI_0 \subseteq R_i$ and $R_{-i}q_iI_0 \subseteq R_0$, that is, $R_{-i}q_i \subseteq (R_0 : I_0)_l$. Thus $q_i \in R_i(R_0 : I_0)_l \subseteq R(R_0 : I_0)_l$. Hence $(R : I_0R)_l = R(R_0 : I_0)_l$. Similarly we have $(R : RJ_0)_r = (R_0 : J_0)_rR$.

(2) By (1) we have

$$\begin{aligned}(I_0R)_v &= (R : (R : I_0R)_l)_r = (R : R(R_0 : I_0)_l)_r \\ &= (R_0 : (R_0 : I_0)_l)_rR = (I_0)_vR.\end{aligned}$$

Similarly $_v(RJ_0) = R(_vJ_0)$.

(3) The proof follows a similar approach as the proof of (1).

■

Definition 2.2.10 *(Definition 2.3 of [27]) R_0 is called a $\mathbb{Z}_0$-invariant maximal order in Q_0 if $O_l(A_0) = R_0 = O_r(A_0)$ holds for every $\mathbb{Z}_0$-invariant ideal A_0 of R_0.*

Lemma 2.2.11 *Let A_0 and B_0 be $\mathbb{Z}_0$-invariant R_0-ideals in Q_0. Then*

(1) $(R_0 : A_0)_l, (R_0 : A_0)_r, O_l(A_0)$ and $O_r(A_0)$ are all $\mathbb{Z}_0$-invariant.

(2) A_0B_0 and $A_0 \cap B_0$ are $\mathbb{Z}_0$-invariant R_0-ideals in Q_0.

(3) Assume taht R_0 is a $\mathbb{Z}_0$-invariant maximal order in Q_0. Then, for any $\mathbb{Z}_0$-invariant R_0-ideal A_0 in Q_0, it holds that $O_l(A_0) = R_0 = O_r(A_0)$.

Proof.

(1) We prove that $(R_0 : A_0)_l$ is a $\mathbb{Z}_0$-invariant and $(R_0 : A_0)_r, O_l(A_0)$ and $O_r(A_0)$ can be proved in similar way. Let $q \in (R_0 : A_0)_l$, that is $q \in Q_0$ and it satisfies $qA_0 \subseteq R_0$. Since A_0 is a $\mathbb{Z}_0$-invariant, then $R_{-n}qR_nA_0 = R_{-n}qA_0R_n \subseteq R_{-n}R_0R_n = R_0$ and implies $R_{-n}qR_n \subseteq (R_0 : A_0)_l$ for all n. Hence $(R_0 : A_0)_l$ is a $\mathbb{Z}_0$-invariant.

(2) Clearly that A_0B_0 is a $\mathbb{Z}_0$-invariant. To prove $A_0 \cap B_0$ is a $\mathbb{Z}_0$-invariant, let $q \in A_0 \cap B_0$. Then $R_{-n}qR_n \subseteq R_{-n}A_0R_n = A_0$ and $R_{-n}qR_n \subseteq R_{-n}B_0R_n = B_0$ which implies $R_{-n}qR_n \subseteq A_0 \cap B_0$ and so $R_{-n}(A_0 \cap B_0)R_n \subseteq A_0 \cap B_0$ for all $n \in \mathbb{Z}_0$. Hence $A_0 \cap B_0$ is a $\mathbb{Z}_0$-invariant.

(3) Assume A_0 is a $\mathbb{Z}_0$-invariant R_0-ideal in Q_0. There exists an element $c_0 \in \mathcal{C}_0$ such that $c_0A_0 \subseteq R_0$. Consequently, $C_0 = (R_0 : A_0)_l \cap R_0$ forms a non-zero $\mathbb{Z}_0$-invariant ideal of R_0 by using properties (1) and (2) with $C_0A_0 \subseteq R_0$. This implies $R_0 = O_r(C_0A_0) \supseteq O_r(A_0) \supseteq R_0$, leading to $R_0 = O_r(A_0)$. Similarly, $R_0 = O_l(A_0)$.

■

Proposition 2.2.12 *(Proposition 2.3 of [27]) Suppose R_0 is a $\mathbb{Z}_0$-invariant maximal order in Q_0. Then:*

(1) For any $\mathbb{Z}_0$-invariant v-ideal A_0 in Q_0, it is true that $(A_0)_v = {}_v(A_0)$.

(2) The set $D(R_0)$ of all $\mathbb{Z}_0$-invariant v-ideals in Q_0 is a commutative group under the multiplication "$\circ$": $A_0 \circ B_0 = (A_0B_0)_v$, where $A_0, B_0 \in D(R_0)$ and the

generators are maximal $\mathbb{Z}_0$-invariant v-ideals of R_0 (ideals maximal amongst the $\mathbb{Z}_0$-invariant v-ideals).

Lemma 2.2.13 *(Lemma 2.7 of [27]) Assume that R_0 is a $\mathbb{Z}_0$-invariant maximal order in Q_0, and let A be an ideal of R such that $A = A_v$ and $A_0 = A \cap R_0 \neq (0)$. Consequently, $A = A_0R$, and A_0 is identified as a $\mathbb{Z}_0$-invariant v-invertible ideal. Specifically, A is v-invertible.*

Lemma 2.2.14 *(Lemma 2.8 of [27]) Assume R_0 is a $\mathbb{Z}_0$-invariant maximal order in Q_0, and consider an ideal A of R such that $A = A_v$ and $A \cap R_0 = (0)$. Then, A is v-invertible.*

The following theorem is the necessary and sufficient condition for positively graded ring R to be a maximal order.

Theorem 2.2.15 *(Theorem 2.1 of [27]) Let R_0 be a Noetherian prime ring, and $R = \oplus_{n\in\mathbb{Z}_0} R_n$ be a positively graded ring. The ring R is a maximal order in Q if and only if R_0 is a $\mathbb{Z}_0$-invariant maximal order in Q_0.*

Proof. By the proof in Theorem 2.1 of [27], the following results are obtained.

Assume R is a maximal order. Consider A_0 as a $\mathbb{Z}_0$-invariant ideal of R_0, and let $A = A_0R$. By Proposition 2.1.1 of [20] and Lemma 2.2.9, it is deduced that $R = O_l(A) = RO_l(A_0)$, implying $R_0 = O_l(A_0)$. Similarly, $R_0 = O_r(A_0)$. Consequently, R_0 is identified as a $\mathbb{Z}_0$-invariant maximal order.

Conversely, assume R_0 is a $\mathbb{Z}_0$-invariant maximal order. Consider a non-zero ideal A of R. Given $R \subseteq O_l(A) \subseteq O_l(A_v)$, assume $A = A_v$ to prove $O_l(A) = R$. If $A_0 = A \cap R_0 \neq (0)$, then $A = A_0R$ with A_0 being $\mathbb{Z}_0$-invariant (as per Lemma 2.2.13). Thus, $O_l(A) = RO_l(A_0) = R$ using Lemma 2.2.9 and the assumption. In the case where $A_0 = (0)$, according to Lemma 2.2.14, it is shown that A is v-invertible. Consequently, $O_l(A) = R$ by Lemma 2.2.6. Similarly, $O_r(A) = R$. Therefore, by Proposition 2.1.1 in [20], R is recognized as a maximal order.

■

2.3. Positively Graded Rings which are Generalized Dedekind Rings

Let $R = \oplus_{n \in \mathbb{Z}_0} R_n$ be a positively graded ring, which is a subring of the strongly graded ring $S = \oplus_{n \in \mathbb{Z}} R_n$. In this context, R_0 represents a prime Goldie ring with its quotient ring Q_0. Throughout this section, we assume that the positively graded ring R is Noetherian, along with its quotient ring Q, unless explicitly mentioned otherwise. We initiate this section with the subsequent lemma.

Lemma 2.3.1 *(Lemma 3.1 of [27]) Consider the following definitions:*

$$\textit{Spec } (Q_0^g) = \{P' \mid P' \textit{ is prime ideal of } Q_0^g\},$$
$$\textit{Spec}_0(R) = \{P \mid P \textit{ is prime ideal of } R \textit{ and } P \cap R_0 = (0)\}.$$

(1) A one-to-one correspondence exists between $\mathrm{Spec}(Q_0^g)$ *and* $\mathrm{Spec}_0(R)$*:*

$$\mathrm{Spec}_0(R) \longrightarrow \mathrm{Spec}(Q_0^g), P \mapsto P' = PQ_0^g;$$
$$\mathrm{Spec}(Q_0^g) \longrightarrow \mathrm{Spec}_0(R), P' \mapsto P = P' \cap R.$$

In particular, each prime ideal P of R is a v-ideal.

(2) For an element $w \in Q_0^g$, it is labeled as a prime element when wQ_0^g qualifies as a prime ideal in Q_0^g. Consequently,

$$\mathrm{Spec}(Q_0^g) = \{P_1' = \oplus_{n \geq 1} Q_0 R_n, P' = wQ_0^g \mid w \textit{ is a central prime element in } Q_0^g\}.$$

Assume that R is a maximal order. The set $D(R)$, encompassing all v-ideals in Q, forms an Abelian group under the multiplication operation " $\circ$ ", defined as $A \circ B = (AB)_v$, for any $A, B \in D(R)$. The generators of $D(R)$ are identified as the maximal v-ideals of R (refer to Theorem 2.1.2 in [20]).

Proposition 2.3.2 *(Proposition 3.1 of [27]) Suppose R is a maximal order in Q. Then, a maximal v-invertible ideal P of R can take one of the following forms:*

(1) $P = P_0 R$, where P_0 is a maximal $\mathbb{Z}_0$-invariant v-invertible ideal of R_0;

(2) $P_1 = \oplus_{n\geq 1} R_n$*; and*

(3) $P = P' \cap R$*, where* $P' \in SpecQ_0^g$ *such that* $P' = wQ_0^g$ *for some central prime element* w *in* Q_0^g*.*

In particular, if $P = P' \cap Q_0^g$ *with* $P' = wQ_0^g$*, then* $P = wA_0R$*, where* A_0 *is a* $\mathbb{Z}_0$*-invariant* v*-invertible ideal in* Q_0*.*

From Proposition 2.3.2, we derive the subsequent theorem delineating v-invertible ideals in Q_0:

Theorem 2.3.3 *(Theorem 3.1 of [27]) Suppose* R_0 *is a Noetherian prime ring, and* $R = \oplus_{n\in\mathbb{Z}_0} R_n$ *is a maximal order. Then, any* v*-invertible ideal can be expressed as* $P_1^l w_1^{l_1} \dots w_k^{l_k} B_0 R$*, where* $P_1 = \oplus_{n\geq 1} R_n$*,* B_0 *is a* $\mathbb{Z}_0$*-invariant* v*-invertible ideal in* Q_0*,* w_i *are central prime elements in* Q_0^g*, and* $l, l_i \in \mathbb{Z}$ *(*$1 \leq i \leq k$*).*

In [10], the concept of a G-Dedekind prime ring is introduced, demonstrating that if R is a G-Dedekind prime ring with the PI condition, then both the polynomial ring $R[X]$ and the Rees ring $R[Xt]$ are G-Dedekind prime rings. In the absence of the PI condition, prior findings in [11] indicate that if R is a G-Dedekind prime ring, then so is $R[X]$. However, the converse has not been explored yet. It is noteworthy that both polynomial rings and Rees rings are positively graded rings.

Definition 2.3.4 *(Definition 3.1 of [27])*

(1) A prime Goldie ring R *is referred to as a generalized Dedekind prime ring (abbreviated as G-Dedekind prime ring) if it satisfies the following conditions:*

(i) R *is a maximal order;*

(ii) Every v*-ideal in* R *is invertible.*

(2) R_0 *is denoted as a* $\mathbb{Z}_0$*-invariant G-Dedekind prime ring if it satisfies the following conditions:*

(i) R_0 *is a* $\mathbb{Z}_0$*-invariant maximal order in* Q_0*;*

(ii) Every $\mathbb{Z}_0$-invariant v-ideal of R_0 is invertible.

Consider a $\mathbb{Z}_0$-invariant R_0-ideal, denoted as B_0, in the ring Q_0. It is straightforward to observe that B_0 is invertible if and only if $B = B_0R$ is also invertible in Q. As a result, the following theorems emerge as direct consequences of Theorems 2.2.15 and 2.3.3.

Theorem 2.3.5 *(Theorem 3.2 of [27]) Consider a Noetherian prime ring R_0 and a positively graded $R = \oplus_{n\in\mathbb{Z}_0} R_n$. The ring R is a G-Dedekind prime ring if and only if R_0 is a $\mathbb{Z}_0$-invariant G-Dedekind prime ring.*

Theorem 2.3.6 *(Theorem 3.3 of [27]) Let R_0 be a Noetherian prime ring, and consider the G-Dedekind prime ring $R = \oplus_{n\in\mathbb{Z}_0} R_n$. For any invertible R-ideal in Q, it can be expressed as $P_1^l w_1^{l_1} \dots w_k^{l_k} B_0 R$, where $P_1 = \oplus_{n\geq 1} R_n$, B_0 is a $\mathbb{Z}_0$-invariant invertible R_0-ideal in Q_0, w_i are central prime elements in Q_0^g, and $l, l_i \in \mathbb{Z}$ with $1 \leq i \leq k$.*

CHAPTER III

Positively Graded Rings which are Unique Factorization Rings

3.1. Unique Factorization Rings

Let $R = \oplus_{n\in\mathbb{Z}_0} R_n$ be a positively graded ring, which is a subring of the strongly graded ring $S = \oplus_{n\in\mathbb{Z}} R_n$. Here, R_0 is a Noetherian prime ring with its quotient ring Q_0. In this section, we initially provide alternative characterizations of unique factorization rings (UFRs) in terms of maximal orders (see Proposition 3.1.1). This characterization is instrumental in establishing the main result (Theorem 3.1.5), which asserts that a positively graded ring R is a UFR if and only if R_0 is a $\mathbb{Z}_0$-invariant UFR, and R_1 is a principal (R_0, R_0) bi-module, denoted by the existence of $p_1 \in R_1$ such that $R_1 = p_1 R_0 = R_0 p_1$.

In this section, let R denote a Noetherian prime ring with its quotient ring Q. It's worth recalling that R is considered a maximal order in Q if, for any non-zero ideal A of R, the conditions $O_l(A) = R = O_r(A)$ hold, as established by Proposition 2.1.1 in [20]. We commence with the subsequent proposition. .

Proposition 3.1.1 *(Proposition 1 of [23]) Let R represent a Noetherian prime ring with its quotient ring Q. The following conditions are mutually equivalent:*

(1) R is a unique factorization ring (UFR).

(2) R is a maximal order, and every v-ideal of R is principal.

(3) R is a maximal order, and every prime v-ideal of R is principal.

Proof. By the proof of Proposition 1 of [23] we have the following.

(1) $\Longrightarrow$ (2): Let $\mathcal{S} = \{A : \text{ideal of } R \mid A = A_v\}$ and P is a maximal member in $\mathcal{S}$. Then P is a prime ideal by ([12], Lemma 2.1) and so, by definition, $P = pR = Rp$ for some $p \in P$. Suppose that there is an $A \in \mathcal{S}$ such that A is not principal and we may assume that A is maximal with this property. Then there exists a prime

ideal $P \supset A$ such that $P = pR = Rp$. It follows that $R = P^{-1}P \supseteq P^{-1}A \supseteq A$ and $(P^{-1}A)_v = P^{-1}A$ by ([12], Lemma 2.1). If $P^{-1}A = A$, then $P^{-1}AR_P = AR_P$ (note that P is localizable and R_P, the localization of R at P, is a local Dedekind prime ring ([34], Proposition 1.7 and Proposition 1.9). So $P^{-1} \subseteq O_l(AR_P) = R_P$ and $R = PP^{-1} \subseteq PR_P$, a contradiction. Hence $P^{-1}A \supset A$ and so, by the choice of A, $P^{-1}A = bR = Rb$ for some $b \in P^{-1}A$ and $A = pbR = pRb = Rpb$, a contradiction. Hence if $A = A_v$, then A is principal. The symmetric argument shows that A is principal if ${}_vA = A$. To prove that R is a maximal order, let A be an ideal of R. Then $R \subseteq O_l(A) \subseteq O_l(A_v) = R$ since A_v is principal and so $R = O_l(A)$. Similarly $R = O_r(A)$. Hence R is a maximal order and it follows from the discussions above and Lemma 2.2.6 that each v-ideal of R is principal.

(2) $\Longrightarrow$ (3): This is a special case.

(3) $\Longrightarrow$ (1): Let P be a prime ideal with $P = P_v$ or $P = {}_vP$. Then P is a v-ideal by Lemma 2.2.6. Thus P is principal and hence R is a UFR. ■

Remark 3.1.2 *(Remark 1 of [23]) In [2], UFRs are defined as follows: every prime ideal contains a principal prime ideal. Interestingly, it is observed that UFRs in the sense of [2] align with UFRs in the sense of [15], but the converse is not necessarily true (refer to [15] for counter-examples).*

Let $\mathcal{C}_0$ denote the set of all regular elements in R_0. It is established that $\mathcal{C}_0$ forms an Ore set of R, and the graded quotient ring of R, denoted as Q_0^g, is defined as $\oplus_{n\in\mathbb{Z}_0} Q_0R_n$, where $Q_0R_n = R_nQ_0$. This graded quotient ring is represented as $Q_0^g = Q_0[X, \sigma]$, a skew polynomial ring over Q_0, with σ being an automorphism of Q_0 and X being a regular element in R_1 (see Lemma 2.2.2).

It is worth recalling that an R_0-ideal A_0 in Q_0 is called a $\mathbb{Z}_0$-*invariant* if $R_nA_0 = A_0R_n$ for all $n \in \mathbb{Z}_0$ ([27]).

Definition 3.1.3 *(Definition 1 of [23]) R_0 is called a $\mathbb{Z}_0$-invariant UFR if*

(1) R_0 is a $\mathbb{Z}_0$-invariant maximal order in Q_0, that is, for any $\mathbb{Z}_0$ -invariant ideal A_0 of R_0, $O_l(A_0) = R_0 = O_r(A_0)$.

(2) Each $\mathbb{Z}_0$-invariant v-ideal of R_0 is principal.

Lemma 3.1.4 *(Lemma 2 of [23]) Assume that R_0 is a $\mathbb{Z}_0$-invariant unique factorization ring (UFR). It follows that any $\mathbb{Z}_0$-invariant v-ideal in Q_0 is necessarily a principal ideal.*

Proof. By the proof of Lemma 2 of [23] we have the following.

Consider a $\mathbb{Z}_0$-invariant v-ideal A_0 in Q_0. According to Proposition 2.2.12, A_0 can be expressed as $A_0 = (P_{01}^{l_1} \dots P_{0k}^{l_k})_v$, where P_{0i} represents maximal $\mathbb{Z}_0$-invariant v-ideals of R_0, and $l_i \in \mathbb{Z}$ for $1 \leq i \leq k$. Since P_{0i} are principal, it follows from ([12], Lemma 2.1 (3)) that A_0 is also a principal ideal. ■

Theorem 3.1.5 *(Theorem 1 of [23]) A positively graded ring $R = \oplus_{n \in \mathbb{Z}_0} R_n$ is a unique factorization ring (UFR) if and only if:*

(1) R_0 is a $\mathbb{Z}_0$-invariant unique factorization ring (UFR).

(2) R_1 is a principal (R_0, R_0) bi-module, meaning there exists $p_1 \in R_1$ such that $R_1 = p_1 R_0 = R_0 p_1$.

Proof. By the proof of Theorem 1 of [23] we have the following.

($\Rightarrow$) (1) Suppose that R is a UFR. Then R is a maximal order in Q by Proposition 3.1.1. Thus R_0 is a $\mathbb{Z}_0$-invariant maximal order by Theorem 2.2.15. Let A_0 be a $\mathbb{Z}_0$-invariant v-ideal of R_0 and let $A = A_0 R$, which is a v-ideal of R by Lemma 2.2.7 and Lemma 2.2.9. So $A = xR = Rx$ for some $x = x_0 + \cdots + x_n \in A$ and $x_i \in R_i$. For any $a_0 \in A_0$, $a_0 = xr$ for some $r = r_0 + \cdots + r_k \in R$ with $r_i \in R_i$ and so $a_0 = x_0 r_0$ + (the higher degree part). Thus $a_0 = x_0 r_0 \in x_0 R_0$ follows, that is, $A_0 \subseteq x_0 R_0$. To prove the converse inclusion, let $r_0 \in R_0$. Then $A_0 R \ni x r_0 = \sum_{i=1}^{l} a_i t_i$ for some $a_i \in A_0$ and $t_i = \sum t_{i_j} (t_{i_j} \in R_j)$. It follows that $x_0 r_0 + x_1 r_0 + \cdots + x_n r_0 = x r_0 = (a_1 t_{1_0} + \cdots + a_l t_{l_0})$ + (the higher degree part). Thus $x_0 r_0 = a_1 t_{1_0} + \cdots + a_l t_{l_0} \in A_0$ and $x_0 R_0 \subseteq A_0$. Hence $A_0 = x_0 R_0$. Similarly $A_0 = R_0 x_0$. Therefore R_0 is a $\mathbb{Z}_0$-invariant UFR.

(2) $P_1 = R_1 R = \oplus_{n \geq 1} R_n$ is a prime invertible ideal by Lemma 2.2.8. So P_1 is principal, that is, $P_1 = pR = Rp$ for some $p = p_1 + p_2 + \cdots + p_n (p_i \in R_i)$. It is clear

that $p_1R_0 \subseteq R_1$. Conversely let $r_1 \in R_1$, then $r_1 = ps$ for some $s = s_0 + \cdots + s_l$, where $s_i \in R_i$ and $r_1 = p_1s_0+$ (the higher degree part). So $r_1 = p_1s_0 \in p_1R_0$, that is, $R_1 \subseteq p_1R_0$. Hence $R_1 = p_1R_0$ and similarly $R_1 = R_0p_1$.

($\Leftarrow$) Suppose that R satisfies the conditions (1) and (2). Then R is a maximal order by (1) and Theorem 2.2.15. Let P be a prime v-ideal of R. If $P_0 = P \cap R_0 \neq (0)$, then $P = P_0R$ and P_0 is a $\mathbb{Z}_0$-invariant v-ideal in R_0 by Lemma 2.2.13. So $P_0 = R_0p_0 = p_0R_0$ for some $p_0 \in P_0$ and $P = p_0R = Rp_0$ follows. If $P_0 = P \cap R_0 = (0)$, then , by Proposition 2.3.2, either $P = \oplus_{n \geq 1} R_n = R_1R$ or $P = P' \cap R$, where $P' = wQ_0^g$ for a central prime element $w \in Q_0^g$. If $P = R_1R$, then P is principal by (2). In the latter case $P = wA_0R$, where A_0 is a $\mathbb{Z}_0$-invariant v-ideal in Q_0 by Theorem 2.3.5 and Theorem 2.3.6 and so A_0 is principal by Lemma 3.1.4. Thus P is principal and hence R is a UFR by Proposition 3.1.1. ■

CHAPTER IV

Module over a Unique Factorization Domain

4.1. Unique Factorization Modules

Let M be a torsion-free module over an integral domain D with the field of fractions K. Consider a non-zero submodule N of KM, which is a fractional submodule in KM if there exists a non-zero element $r \in D$ such that $rN \subseteq M$ and $KN = KM$. Similarly, for a non-zero submodule $\mathfrak{a}$ of K, it is called a fractional M-ideal in K if there exists a non-zero element $m \in M$ such that $\mathfrak{a}m \subseteq M$.

Let $F(M)$ denote the collection of all fractional D-submodules in KM, and $F_M(D)$ be the set comprising all fractional M-ideals in K. Assume $N \in F(M)$ and $\mathfrak{a} \in F_M(D)$. We define $N^- = \{k \in K \mid kN \subseteq M\}$ and $\mathfrak{a}^+ = \{m \in KM \mid \mathfrak{a}m \subseteq M\}$. It is straightforward to observe that $N^- \in F_M(D)$ and $\mathfrak{a}^+ \in F(M)$.

For $N \in F(M)$ and $\mathfrak{a} \in F_M(D)$, we define $N_v = (N^-)^+$ and $\mathfrak{a}_{v1} = (\mathfrak{a}^+)^-$. Consequently, $N_v \in F(M)$ and satisfies $N_v \supseteq N$. Similarly, $\mathfrak{a}_{v1} \in F_M(D)$ and satisfies $\mathfrak{a}_{v1} \supseteq \mathfrak{a}$. When $N = N_v$, we classify N as a fractional v-submodule in KM. Moreover, $\mathfrak{a}$ is called a v_1-ideal if $\mathfrak{a} = \mathfrak{a}_{v1}$.

In [41], the concept of a unique factorization module was introduced using a submodule approach. The authors provided the definition and characterization of unique factorization modules, as outlined below.

Definition 4.1.1 (Definition 2 of [41]) *A torsion-free module M over an integral domain D is called* a unique factorization module *(UFM for short) if*

(1) M is completely integrally closed (CIC for short), that is, $O_K(N) = \{k \in K \mid kN \subseteq N\} = D$ for every non-zero submodule N of M, where K is the quotient field of D;

(2) every v-submodule N of M is principal, that is, $N = pM$ for some $p \in D$;

(3) M satisfies the ascending chain condition on v-submodules of M.

Theorem 4.1.2 (Theorem 1 of [41]) *Suppose $O_K(M) = D$. The following conditions are equivalent:*

(1) M is a unique factorization module (UFM).

(2) M is a v-multiplication module, and D is a unique factorization domain (UFD).

(3) (a) D is a UFD.

(b) For every prime element p of D, pM is a maximal v-submodule.

(c) For every v-submodule N of M, $\mathfrak{n} = (N : M) \neq \{0\}$, where $(N : M) = \{r \in D \mid rM \subseteq N\}$.

(4) Every v-submodule of M is principal, and D is a UFD.

Lemma 4.1.3 *(Lemma 2.1 of [24]) For a finitely generated torsion-free module M over an integrally closed domain D, it holds that $O_K(M) = \{k \in K \mid kM \subseteq M\} = D$.*

Proof. Let $M = Dm_1 + \ldots + Dm_t$, where $m_i \in M$ for all $i \in \{1, \ldots, t\}$. It is clear that $D \subseteq O_K(M)$. Let $k \in O_K(M)$, that is, $k \in K$ and $kM \subseteq M$. Then $km_i \in M$ for all $i \in \{1, \ldots, t\}$. We write

$$\begin{aligned}
km_1 &= d_{1_1}m_1 + \ldots + d_{1_t}m_t;\\
km_2 &= d_{2_1}m_1 + \ldots + d_{2_t}m_t;\\
&\vdots\\
km_i &= d_{i_1}m_1 + \ldots + d_{i_t}m_t;\\
&\vdots\\
km_k &= d_{k_1}m_1 + \ldots + d_{k_t}m_t;
\end{aligned}$$

where $d_{i_j} \in D$ for all $i, j \in \{1, \ldots, t\}$. Then

$$k \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_i \\ \vdots \\ m_t \end{bmatrix} = \begin{bmatrix} d_{1_1} & \ldots & d_{1_t} \\ d_{2_1} & \ldots & d_{2_t} \\ \vdots & \ldots & \vdots \\ d_{i_1} & \ldots & d_{i_t} \\ \vdots & \ldots & \vdots \\ d_{k_1} & \ldots & d_{t_t} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_i \\ \vdots \\ m_t \end{bmatrix}$$

$$\begin{bmatrix} k - d_{1_1} & \ldots & -d_{1_t} & \\ -d_{2_1} & \ldots & -d_{2_t} & \\ \vdots & \vdots & \ldots & \vdots \\ -d_{i_1} & \ldots & -d_{i_t} & \\ \vdots & \ldots & \vdots & \\ -d_{k_1} & \ldots & k - d_{t_t} & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_i \\ \vdots \\ m_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then

$$\det\left(\begin{bmatrix} k - d_{1_1} & \ldots & -d_{1_t} & \\ -d_{2_1} & \ldots & -d_{2_t} & \\ \vdots & \vdots & \ldots & \vdots \\ -d_{i_1} & \ldots & -d_{i_t} & \\ \vdots & \ldots & \vdots & \\ -d_{k_1} & \ldots & k - d_{t_t} & \end{bmatrix} \right) = 0$$

$$k^t + C_{n-1}k^{t-1} + \ldots + C_1 k + C_0 = 0$$

where $C_i \in D$ for all $i \in \{1, \ldots, t-1\}$. Then there is $f(x) = x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0 \in D[x]$ such that $f(k) = 0$. Thus $k \in D$ since D is an integrally closed domain. Hence $O_K(M) = D$. ■

Throughout this dissertation, M *is a finitely generated torsion-free* D*-module that adheres to the ascending chain condition on* v*-submodules of* M.

Lemma 4.1.4 *(Lemma 2.4 of [26]) Let* P *be a maximal* v*-submodule of* M*. It fo-*

llows that P is a prime submodule of M.

Proof. Let $r \in D$ and $m \in M$ such that $rm \in P$. If $m \notin P$, then $P \subset Dm + P \subseteq (Dm+P)_v \subseteq M$, implying $(Dm+P)_v = M$. Consequently, $P \supseteq (Drm+rP)_v = (r(Dm+P))_v = r(Dm+P)_v = rM$. Thus, P is a prime submodule of M. ■

Theorem 4.1.5 *(Theorem 2.5 of [26]) Assume D is a UFD and M is a completely integrally closed module that fulfills the ascending chain condition on v-submodules of M. Then the module M is a unique factorization module if and only if each prime v-submodule of M is principal.*

Proof. If M is a UFM, then every prime v-submodule of M is principal, as per Theorem 4.1.2. Conversely, assuming the contrary, let's suppose that M is not a UFM. Take N as a non-principal v-submodule of M with maximal satisfying this property. This is feasible since M satisfies the ascending chain condition on v-submodules. Choose a maximal v-submodule P of M containing N; therefore, $P = pM$ for some non-zero $p \in D$ by Lemma 4.1.4. As $N \subset P \subset M$, we have $N \subseteq p^{-1}N \subset M$, implying $(p^{-1}N)_v = p^{-1}N_v = p^{-1}N$. Now, either $N = p^{-1}N$ or $p^{-1}N$ is principal due to the maximality of N. If $p^{-1}N$ is principal, then $p^{-1}N = tM$ for some $t \in D$, leading to $N = ptM$, which is a contradiction. Therefore, $N = p^{-1}N$, implying $p^{-1} \in O_K(N) = D$. Consequently, $P = pM \supseteq p(p^{-1}M) = M$, which is again a contradiction. Hence, every v-submodule N of M is principal, confirming that M is a UFM. ■

In a Unique Factorization Domain (UFD), the notions of a principal ideal, a v-ideal, and an invertible ideal are equivalent.

Remark 4.1.6 *(Remark 2.6 of [26]) Let D be a unique factorization domain, and let A be a v-ideal of D. Then, the following statements are held:*

(1) D is a unique factorization module over D.

(2) A is a unique factorization module over D.

(3) If M is a finitely generated projective module over D, then M is a unique

factorization module. Specifically, any finite direct sum of D is also a unique factorization module.

Proof.

(1) It is clear.

(2) Note that since A is a v-ideal of D and is principal, A is isomorphic to D as a D-module. Therefore, by (1), A is a Unique Factorization Module (UFM).

(3) By Theorem 3.1 of [29], it is known that M is a v-multiplication module. Consequently, by Theorem 4.1.2, M is a UFM, because D is a Unique Factorization Domain (UFD).

■

4.2. Strongly Graded Modules which are Unique Factorization Modules

In this section, consider the strongly graded domain $D = \oplus_{n\in\mathbb{Z}} D_n$. According to Theorem 2.1 of [40], D is a G-Dedekind domain if and only if D_0 is a G-Dedekind domain. Let K_0 and K be the quotient fields of D_0 and D respectively. Assume $M = \oplus_{n\in\mathbb{Z}} M_n$ is a strongly graded module over D, with M_0 being a finitely generated torsion-free D_0-module. Additionally, assume that M satisfies the ascending chain condition on v-submodules of M. In this section, we aim to establish that if M_0 is a UFM over D_0, then M is a UFM over D.

In a UFD, the notions of a principal ideal, a v-ideal, and an invertible ideal are equivalent. This section commences with the subsequent proposition.

Proposition 4.2.1 *(Proposition 3.1 of [26]) If D_0 is a UFD, then the strongly graded ring $D = \oplus_{n\in\mathbb{Z}} D_n$ is also a UFD.*

Proof. Suppose D_0 is a UFD, that is, D is a maximal order, and every prime v-ideal P_0 of D_0 is principal (refer to Proposition 1 in [24]). According to Theorem 1 in [22], D is a maximal order. Consider a non-zero prime v-ideal P of D. If $P_0 = P \cap D_0 \neq (0)$, then $P = P_0 D$ and P_0 is a v-ideal of D_0. This implies

$P_0 = p_0 D_0 = D_0 p_0$ for some $p_0 \in P_0$, and consequently, $P = p_0 D = D p_0$. In the case where $P_0 = P \cap D_0 = (0)$, then $P = w A_0^{-1} B_0 D$ for invertible ideals A_0, B_0 of D_0. This situation implies P is principal since D_0 is a UFD. Hence, P is principal, and consequently, D is a UFD following Proposition 1 in [23]. ■

Remember that a module M over a CIC domain D is a UFM if and only if each prime v-submodule P of M is principal, that is, $P = pM$ for some element $p \in D$ (refer to Theorem 4.1.5.

Note that M is a finitely generated torsion-free D-module since M_0 is a finitely generated torsion-free D_0-module. Furthermore, M_0 is CIC if and only if M is CIC by Theorem 3.1 of [24].

In the rest of this section, we assume that M_0 *is a UFM*. Then D_0 *is a UFD* (see Theorem 4.1.1).

Next, we study the structure of a v-submodule P of M with $P \cap M_0 \neq (0)$.

Lemma 4.2.2 (Lemma 5.1 of [24]) *Let N_0 be a fractional D_0-submodule of M_0 with $N_0 \subseteq M_0$ and $N = DN_0$. Then*

(1) $N^- = D(N_0)^-$, and

(2) $N_v = D(N_0)_v$.

Proof. By the proof of Lemma 5.1 of [24].

(1) Note that $D(N_0)^- N = D(N_0)^- D N_0 = D(N_0)^- N_0 \subseteq DM_0 = M$. Then we have $D(N_0)^- \subseteq N^-$.
Conversely, let $q \in N^-$, that is, $q \in K$ and $qN \subseteq M$. Then $qK^g M = qK^g N = K^g qN \subseteq K^g M$ and so $q \in K^g$. Write $q = q_n + q_{n-1} +$ (the lower degree parts) where $q_i \in K_0 D_i$ for all i. Since $qN \subseteq M$, we have that $qN_0 \subseteq M$ and $q_i N_0 \subseteq M_i$ for all i. Then $D_{-i} q_i N_0 \subseteq D_{-i} M_i = M_0$ and so $D_{-i} q_i \subseteq (N_0)^-$ which implies that $q_i \in D_i (N_0)^-$. Hence $q = q_n + q_{n-1} +$ (the lower degree parts) $\in D(N_0)^-$.

(2) Note that $M_0 \supseteq ((N_0)_v)^- (N_0)_v = (N_0)^- (N_0)_v$ by Lemma 2.4 (3) of [28]. Then $M = DM_0 \supseteq D(N_0)^- (N_0)_v = N^- (N_0)_v$ which implies that $(N_0)_v \subseteq$

$(N^-)^+ = N_v$ and so $D(N_0)_v \subseteq N_v$.

Conversely, let $m \in N_v$, that is, $m \in KM$ and $M \supseteq N^- m$. Then $K_0M \supseteq K_0N^-m = K_0D(N_0)^-m = K_0Dm$ and so $m \in K_0M$. Write $m = m_n + m_{n-1}+$(the lower degree parts) where $m_i \in K_0M_i$ for all i. Since $(N_0)^-(m_n + m_{n-1} +$ (the lower degree parts)$) = (N_0)^- m \subseteq DN_0^- m \subseteq M$, we have $N_0^- m_i \subseteq M_i$ and so $N_0^- D_{-i}m_i = D_{-i}N_0^- m_i \subseteq M_0$ for all i. Moreover $D_{-i}m_i \subseteq (N_0)_v$ and so $m_i \in D_i(N_0)_v$ for all i. Thus $m = m_n + m_{n-1} +$ (the lower degree parts) $\in D_n(N_0)_v + \ldots + D_0(N_0)_v \subseteq D(N_0)_v$. Hence $N_v = D(N_0)_v$

■

Lemma 4.2.3 *(Lemma 3.3 of [23]) Let P be a prime D-submodule of M with $P_0 = P \cap M_0 \neq (0)$. Then*

(1) P_0 is a prime submodule of M_0, and

(2) $P' = DP_0$ is a prime submodule of M.

(3) If P is a prime v-submodule, then P_0 is a prime v-submodule of M_0, and $P = DP_0$.

Proof.

(1) Suppose that $r_0m_0 \in P_0$ and $m_0 \notin P_0$ where $r_0 \in D_0$ and $m_0 \in M_0$. Then $m_0 \notin P$ and $r_0m_0 \in P_0 \subseteq P$. Thus $P \supseteq r_0M \supseteq r_0M_0$ and $r_0M_0 \subseteq P \cap M_0 = P_0$. Hence P_0 is a prime submodule.

(2) Without lost of generality, we may assume that $r = r_n + r_{n-1} + \ldots + r_0 \in D$ and $m = m_l + \ldots + m_0 \in M$. Suppose that $rm \in P'$ and $m \notin P'$. We may assume that $m_l \notin P'$ and we prove (2) by induction on $n = \deg(r)$. Then $D_{-l}m_l \nsubseteq P_0$ since $m_l \notin D_lP_0$. If $r = r_0$, then $rm = r_0m_l + \ldots + r_0m_0 \in P'$ and $r_0m_l \in D_lP_0 = P' \cap M_l$. Then $r_0D_{-l}m_l = D_{-l}r_0m_l \subseteq P_0$ and $D_{-l}m_l \nsubseteq P_0$. Thus by (1), $r_0M_0 \subseteq P_0$ and $r_0M_t \subseteq D_tP_0$ for all $t \in \mathbb{Z}$, which implies that $r_0M \subseteq P'$.

Since $rm = r_n m_l + \ldots + r_0 m_0 \in P' = DP_0$, we have $r_n m_l \in D_{n+l} P_0$. Then $D_{-n} r_n D_{-l} m_l \subseteq P_0$ and $D_{-l} m_l \nsubseteq P_0$, which implies that $D_{-n} r_n M_0 \subseteq P_0$ and so $r_n M_0 \subseteq D_n P_0$. Thus $r_n M_t \subseteq D_{n+t} P_0$ for all $t \in \mathbb{Z}$ which implies that $r_n M \subseteq DP_0 = P'$.

In particular $r_n m \in P'$ and $(r - r_n)m \in P'$. By induction on n, $(r - r_n)M \subseteq P'$ and $rM \subseteq P'$. Hence P' is a prime submodule of M.

(3) Let $P' = DP_0 \subseteq M$. Consider that $P = P_v \supseteq (P')_v = (DP_0)_v = D(P_0)_v$ by Lemma 4.2.2. Thus $P_0 = P \cap M_0 \supseteq D(P_0)_v \cap M_0 = (P_0)_v$. Hence $P_0 = (P_0)_v$ and so P_0 is a prime v-submodule by (1).

Note that $P' = DP_0 = Dp_0 M_0$ for some non-zero $p_0 \in D_0$ because M_0 is a UFM. Since Dp_0 is an invertible ideal, then $(P')^- = (Dp_0)^{-1} = Dp_0^{-1} \supseteq P^-$, which implies $D \supseteq Dp_0 P^-$ and $P' = Dp_0 M_0 = Dp_0 M \supseteq Dp_0 P^- P$. If $P \supset P'$ then $Dp_0 P^- M \subseteq P' \subseteq Dp_0 M$ since P' is a prime submodule by (2). Then $P^- M \subseteq M$ and so $P^- = D$ since M is a CIC. Thus $P = P_v = (P^-)^+ = (D)^+ = M$, a contradiction. Hence $P = DP_0$.

■

In the rest of this section, we assume that M *satisfies the ascending chain conditions on v-submodules of M.*

Proposition 4.2.4 *(Proposition 3.4 of [26]) Let N be a v-submodule of M with $N_0 = N \cap M_0 \neq (0)$. Then*

(1) N_0 is a v-submodule of M_0, and there exists an ideal $\mathfrak{n}_0$ of D_0 such that $N_0 = \mathfrak{n}_0 M_0$.

(2) $N = D\mathfrak{n}_0 M$, and $D\mathfrak{n}_0 = (N : M)$.

Proof. By the proof of Proposition 3.4 of [26], we have the following:

(1) By applying Theorem 4.1.2, similar to the previous lemma, it is established that N_0 is a v-submodule of M_0. Moreover, $N_0 = \mathfrak{n}_0 M_0$ for some ideal $\mathfrak{n}_0$ of D_0, as M_0 is a UFM over D_0.

(2) Assume there exists a v-submodule N such that $N \neq D\mathfrak{n}_0 M$ where $\mathfrak{n}_0$ is an ideal of D_0. Without loss of generality, let N be maximal with this property as M satisfies the ascending chain condition on v-submodules. Thereby, a maximal v-submodule P with $P \supseteq N$ and $P = D\mathfrak{p}_0 M$, where $\mathfrak{p}_0$ is a maximal ideal of D_0, is obtained. It implies $M \supseteq (D\mathfrak{p}_0)^{-1}N \supseteq N$. If $(D\mathfrak{p}_0)^{-1}N = N$, then $(D\mathfrak{p}_0)^{-1} \subseteq D$, leading to a contradiction since M is CIC. Therefore, $(D\mathfrak{p}_0)^{-1}N \supset N$, and it follows from Lemma 3.2 of [28] that $((D\mathfrak{p}_0)^{-1}N)_v = (D\mathfrak{p}_0)^{-1}N$. By the choice of N, $(D\mathfrak{p}_0)^{-1}N = D\mathfrak{t}_0 M$ for some ideal $\mathfrak{t}_0$ of D_0. Consequently, $N = D\mathfrak{p}_0\mathfrak{t}_0 M$, resulting in a contradiction. Thus, $N = D\mathfrak{n}_0 M$ for some ideal $\mathfrak{n}_0$ of D_0. The last statement is easily derived since $D\mathfrak{n}_0$ is invertible.

■

Next we study the structure of a prime v-submodule P of M such that $P \cap M_0 = (0)$. Since $K^g = \oplus_{n\in\mathbb{Z}} K_0 D_n = K_0 D$ is a principal ideal domain by [22] and $K_0 M$ is a finitely generated torsion-free K^g-module, we have that a v-submodule P_1 of $K_0 M$ is prime if only if $P_1 = \mathfrak{p}_1 K_0 M$, where $\mathfrak{p}_1$ is a maximal ideal of K^g such that $\mathfrak{p}_1 = (P_1 : K_0 M)$ by Theorem 3.3 of [28].

Note that if D_0 is a UFD and $\mathfrak{p}$ is a prime v-ideal of D, then $\mathfrak{p} = \mathfrak{p}_0 D$ for some prime v-ideal $\mathfrak{p}_0$ of D_0 or $\mathfrak{p} = \mathfrak{p}_1 \cap D$ for some prime ideal $\mathfrak{p}_1$ of $K_0 D$ by Lemma 2.6 of [40], and moreover $\mathfrak{p} = pD$ for some $p \in D$ by Proposition 4.2.1.

The following lemma is a graded version of Lemma 4.5 of [28].

Lemma 4.2.5 *(Lemma 3.5 of [26]) Let N be a D-submodule of M. Then*

(1) $(K_0 N : K_0 M) = K_0\mathfrak{n}$, *where* $\mathfrak{n} = (N : M)$ *and* $K_0 N^- = (K_0 N)^-$.

(2) $(K_0 N)_v = K_0 N_v$.

Proof. By the proof of Lemma 3.5 of [26], we have the following:

(1) Let $\mathfrak{n} = (N : M)$, that is, $\mathfrak{n}M \subseteq N$. Then $K_0 N \supseteq K_0\mathfrak{n}M = K_0\mathfrak{n}K_0 M$ which implies that $K_0\mathfrak{n} \supseteq (K_0 N : K_0 M)$.

Conversely, assume that $r \in (K_0N : K_0M)$, that is, $r \in K_0D$ with $rK_0M \subseteq K_0N$. We write $M = Dm_1 + \ldots + Dm_l$ where $m_i \in M$ for all $i = 1, 2, \ldots, l$. For all i, $rm_i \in rM \subseteq rK_0M \subseteq K_0N$, then we can write $rm_i = \sum_{j=1}^{t} k_{0_{ij}} n_{ij}$ where $k_{0_{ij}} \in K_0$ and $n_{ij} \in N$. Then there is $s \in D_0$ such that $sk_{0_{ij}} \in D_0$ for all i, j and so $srm_i \in D_0N = N$ for all i. Then $srM \subseteq N$ and $sr \in (N : M) = \mathfrak{n}$ which implies that $r \in s^{-1}\mathfrak{n} \subseteq K_0\mathfrak{n}$. Thus $K_0\mathfrak{n} = (K_0N : K_0M)$.

To prove $K_0N^- = (K_0N)^-$, first we consider that $K_0N^-K_0N = K_0N^-N \subseteq K_0M$ and we have $K_0N^- \subseteq (K_0N)^-$. Conversely, let $x \in (K_0N)^-$, that is $x \in K$ and $xK_0N \subseteq K_0M$. Since D is a Noetherian domain, we have N is finitely generated. Then there exist $r \in D_0$ such that $rxN \subseteq M$ which implies that $rx \in N^-$ and so $x \in r^{-1}N^- \subseteq K_0N^-$. Hence $K_0N^- = (K_0N)^-$.

(2) Let $m' \in (K_0N)_v = ((K_0N)^-)^+ = (K_0N^-)^+$, that is $K_0M \supseteq K_0N^-m' \supseteq N^-m'$. Then there is $r \in D_0$ such that $N^-rm' = rN^-m' \subseteq M$. Thus $rm' \in (N^-)^+ = N_v$ and so $m' \in r^{-1}N_v \subseteq K_0N_v$.

Conversely, let $m' \in K_0N_v$. We write $m' = \sum_{i=1}^{t} k_{0_i} m_i$ where $k_{0_i} \in K_0$ and $m_i \in N_v$ for all $i = 1, 2, \ldots t$. Then for all $i = 1, 2, \ldots, t$, we have $N^-m_i \subseteq M$ and so $K_0N^-m' = K_0N^-\left(\sum_{i=1}^{t} k_{0_i} m_i\right) \subseteq N^-(K_0m_1 + \ldots + K_0m_t) \subseteq K_0M$. Then $m' \in (K_0N^-)^+ = ((K_0N)^-)^+ = (K_0N)_v$. Hence $(K_0N)_v = K_0N_v$.

■

The subsequent lemma serves as a graded counterpart to Lemma 4.6 in [28]. The proof is provided due to the necessity of the v_1-operation to establish the final properties (refer to [28], [35] for comprehensive details concerning v-submodules and v_1-operation).

Lemma 4.2.6 *(Lemma 3.6 of [26]) Let M_0 be a UFM over D_0, and let $P_1 = \mathfrak{p}_1K_0M$ be a prime v-submodule of K_0M, where $\mathfrak{p}_1$ is a maximal ideal of K_0D. Define $P = P_1 \cap M$ and $\mathfrak{p} = \mathfrak{p}_1 \cap D$. Then the following statements hold:*

(1) P is a prime submodule of M, and $\mathfrak{p} = (P : M)$.

(2) $K_0P = P_1$*, and* $P \cap M_0 = (0)$*.*

(3) $P = \mathfrak{p}M$*, and* P *is a maximal* v*-submodule of* M*.*

Proof. By the proof of Lemma 3.6 of [26], we have the following:

(1) Let $r \in D$ and $m \in M$ such that $rm \in P$ and $m \notin P$. Since $m \notin P_1$ and P_1 is prime, we have $rM \subseteq rK_0M \subseteq P_1$ and so $rM \subseteq P$. Hence P is a prime submodule of M.

Since $\mathfrak{p}M \subseteq \mathfrak{p}K_0M = P_1$, we have $\mathfrak{p}M \subseteq P$, so $\mathfrak{p} \subseteq (P : M)$. Conversely let $r \in (P : M)$, that is $r \in D$ and $rM \subseteq P$. Then $rK_0M \subseteq K_0P \subseteq P_1$, so $r \in (P_1 : K_0M) = \mathfrak{p}_1$. Thus $r \in \mathfrak{p}_1 \cap D = \mathfrak{p}$. Hence $\mathfrak{p} = (P : M)$.

(2) Let $m' \in P_1$ and we write $m' = \sum_{i=1}^{n} t_i m_i$ where $t_i \in \mathfrak{p}_1$ and $m'_i \in K_0M$. Then there are $\alpha, \beta \in D_0$ such that $\alpha t_i \in \mathfrak{p}$ and $\beta m'_i \in M$ and so $\alpha\beta m' \in \mathfrak{p}M \subseteq P$. Thus $m' \in (\alpha\beta)^{-1}P \subseteq K_0P$. Hence $K_0P = P_1$.

Note that $\mathfrak{p}_1 = \langle t \rangle = tK_0D$ for some prime element $t \in K_0D$ with $\deg(t) \geq 1$. If $P \cap M_0 \neq \{0\}$ and let $0 \neq m \in P \cap M_0$. Then $m = tm'$ for some $m' \in K_0M$, since $K_0P = P_1 = tK_0M$. Write $t = t_n + t_{n-1} + \ldots + t_0$ $(t_i \in K_0D_i,$ with $t_n \neq 0)$ and $m' = m_l + \ldots + m_0$ $(m_j \in K_0M_j)$. Then we get $t_n m_l = 0$, so $m_l = 0$ and so on. Then we have $m = 0$, a contradiction. Hence $P \cap M_0 = \{0\}$.

(3) By Lemma 4.2.5 and (2) we have $P_1 = (P_1)_v = (K_0P)_v = K_0P_v$, so P is a v-submodule of M. Since M is a v-Noetherian D-module there are finite elements $m_i \in P$ such that $P = (Dm_1 + \ldots + Dm_k)_v$. Note that $K_0P = K_0(Dm_1 + \ldots + Dm_k)_v = (K_0Dm_1 + \ldots + K_0Dm_k)_v$ by Lemma 4.2.5. Further since $K_0P = P_1 = K_0\mathfrak{p}K_0M = \mathfrak{p}K_0M$, for m_i there are finite $p_{ij} \in \mathfrak{p}$ and $l_{ij} \in K_0M$ such that $m_i = \sum_j p_{ij} l_{ij}$. Then there is a non-zero $c \in D_0$ with $cl_{ij} \in M$ for all l_{ij} so that $cm_i \in \mathfrak{p}M$. Put $\mathfrak{a} = \{r_0 \in D_0 \mid r_0P \subseteq \mathfrak{p}M\}$, an ideal of D_0 with $\mathfrak{a}P \subseteq \mathfrak{p}M$. If $\mathfrak{a} = D_0$, then $P = \mathfrak{p}M$ and we are done. If $\mathfrak{a} \subset D_0$, by Lemma 3.2 of [35], $\mathfrak{a}_{v_1}P \subseteq (\mathfrak{a}_{v_1}P)_v = (\mathfrak{a}P)_v \subseteq (\mathfrak{p}M)_v = \mathfrak{p}M_v = \mathfrak{p}M$ because $\mathfrak{p}$ is an invertible ideal. By the definition of $\mathfrak{a}$, we have $\mathfrak{a}_{v_1} \subseteq \mathfrak{a}$, which implies $\mathfrak{a}_{v_1} = \mathfrak{a}$, that is, $\mathfrak{a}$ is a v_1-ideal of D_0. Since $\mathfrak{a}$ is a v_1-ideal of D_0, then $\mathfrak{a}^+$ is a v-submodule of M_0 by Lemma 2.3 of [29],

which implies $\mathfrak{a}^+ = r_0 M_0$ for some $r_0 \in D_0$ because M_0 is a UFM. Then $\mathfrak{a} = \mathfrak{a}_{v_1} = (\mathfrak{a}^+)^- = (r_0 M_0)^- = r_0^{-1} D_0$ and so $\mathfrak{a}$ is an invertible ideal. Note that $\mathfrak{p}^{-1}\mathfrak{a}P \subseteq M$ and $K_0\mathfrak{p}^{-1}\mathfrak{a}P = K_0\mathfrak{p}^{-1}\mathfrak{p}_1 K_0 M = K_0 M$, since $K_0 D\mathfrak{p} = \mathfrak{p}_1$. It follows that $\mathfrak{p}^{-1}\mathfrak{a}P \cap M \neq \{0\}$ and $(\mathfrak{p}^{-1}\mathfrak{a}P)_v = \mathfrak{p}^{-1}\mathfrak{a}P_v = \mathfrak{p}^{-1}\mathfrak{a}P$ by Lemma 3.2 of [28] since $\mathfrak{p}^{-1}\mathfrak{a}$ is an invertible D-ideal in K^g. Then by Proposition 4.2.4, $\mathfrak{p}^{-1}\mathfrak{a}P = \mathfrak{n}DM$ for some ideal $\mathfrak{n}$ of D_0 and $P = \mathfrak{p}\mathfrak{a}^{-1}\mathfrak{n}DM$. It follows that $\mathfrak{p} = (P : M) = \mathfrak{p}\mathfrak{a}^{-1}\mathfrak{n}D$ and that $D = \mathfrak{a}^{-1}\mathfrak{n}D$. Hence $P = \mathfrak{p}M$.

To prove that P is a maximal v-submodule of M, let N be a maximal v-submodule of M containing P. Then $K_0 N$ is a v-submodule of $K_0 M$ containing $K_0 P = P_1$ by Lemma 4.2.5 (2), so $K_0 N = P_1$ by the assumption. Thus $P = P_1 \cap M \supseteq N$ and $N = P$ follows. Hence P is a maximal v-submodule of M.

■

Lemma 4.2.7 *(Lemma 3.7 [26]) Suppose M_0 is a UFM over D_0, and let P be a prime v-submodule of M with $P \cap M_0 = (0)$. Then, there exists a maximal v-submodule P_1 of $K_0 M$ such that $P = P_1 \cap M$.*

Proof. By the proof of Lemma 3.6 of [26], we have the following:

Let $\mathfrak{p} = (P : M)$. Then $\mathfrak{p}$ is a prime v-ideal of D, making it a non-zero minimal prime ideal. This implies that $\mathfrak{p}$ takes one of two forms: either $\mathfrak{p} = \mathfrak{p}_0 D$ for some prime ideal $\mathfrak{p}_0$ of D_0, or $\mathfrak{p} = \mathfrak{p}_1 \cap D$ for some prime ideal $\mathfrak{p}_1$ of $K_0 D$ as per Theorem 2.1 and Lemma 2.6 in [40]. In the first case, $P \supseteq \mathfrak{p}_0 DM \supseteq \mathfrak{p}_0 M_0 \neq (0)$, leading to a contradiction.

Therefore, $\mathfrak{p} = \mathfrak{p}_1 \cap D$ with $K_0\mathfrak{p} = \mathfrak{p}_1$. As $P \cap M_0 = (0)$, $K_0 M \supset K_0 P = (K_0 P)_v$ by Lemma 4.2.5. This implies the existence of a maximal v-submodule P_1 of $K_0 M$ such that $P_1 \supseteq K_0 P$. By Lemma 4.2.5, $(P_1 : K_0 M) \supseteq (K_0 P : K_0 M) = K_0(P : M) = K_0\mathfrak{p} = \mathfrak{p}_1$. Since $(P_1 : K_0 M)$ is a prime ideal of $K_0 D$, we get $\mathfrak{p}_1 = (P_1 : K_0 M)$. Consequently, $P_1 = \mathfrak{p}_1 K_0 M$ and $P_1 \cap M \supseteq P$. Through Lemma 4.2.6, we find $P_1 \cap M = \mathfrak{p}M \subseteq P$, ultimately leading to $P = P_1 \cap M$ and $P = \mathfrak{p}M$. ■

Proposition 4.2.8 *(Proposition 3.8 of [26]) Let P be a prime v-submodule of M with $P \cap M_0 = (0)$. Then, there exists a prime v-ideal $\mathfrak{p}$ of D such that $P = \mathfrak{p}M$, where $\mathfrak{p} \cap D_0 = (0)$.*

From Lemma 4.2.3 and Proposition 4.2.8, the following theorem is obtained.

Theorem 4.2.9 *(Theorem 3.9 of [26]) Let $D = \oplus_{n\in\mathbb{Z}} D_n$ be a strongly graded domain, and $M = \oplus_{n\in\mathbb{Z}} M_n$ be a strongly graded module over D. Assume that M satisfies the ascending chain condition on v-submodules of M. If M_0 is a unique factorization module (UFM) over D_0, then M is also a UFM over D.*

Proof. By the proof of Theorem 3.9 of [26], we have the following:

Given that D_0 is a unique factorization domain, Proposition 4.2.1 ensures that D is also a UFD, and consequently, every prime v-ideal of D is principal. The assertion that D is a maximal order, supported by Proposition 1 of [23], implies that D_0 is a maximal order, as established by Theorem 1 of [22]. Consequently, M is a CIC due to Theorem 3.1 of [23]. To demonstrate that M is a unique factorization module, consider a prime v-submodule P of M. Let $P_0 = P \cap M_0$.

1. Consider the case where $P_0 \neq (0)$. In this case, $P = DP_0$, and according to Lemma 4.2.3, P_0 qualifies as a prime v-submodule of D_0. As M_0 is a UFM, we can deduce that $P_0 = p_0 M_0$ for a certain $p_0 \in D_0$, leading to $P = DP_0 = Dp_0M_0 = p_0DM_0 = p_0M$.

2. Now, consider the case where $P_0 = (0)$. In this case, $P = \mathfrak{p}M$ for some prime v-ideal $\mathfrak{p}$ of D with $\mathfrak{p} \cap D_0 = \{0\}$, as per Proposition 4.2.8. Since D is a UFD, $\mathfrak{p} = pD$ for some $p \in D$, and consequently, $P = \mathfrak{p}M = pDM = pM$ for a certain $p \in D$.

Hence, every prime v-submodule of M is principal, and thus, by Theorem 4.1.5, M is a UFM. ■

As an application of Theorem 4.2.9, we have the following examples.

Example 4.2.10 (Example 3.10 of [26]) If M is a unique factorization module over an integral domain D, then the Laurent polynomial module $M[x, x^{-1}]$ is also a UFM over the Laurent polynomial ring $D[x, x^{-1}]$.

Example 4.2.11 (Example 3.11 of [26]) Let T be any unique factorization domain, and consider two non-zero v-ideals A and B in T. Let K denote the quotient field of T. Then, define the module

$$M = \oplus_{n\in\mathbb{Z}} AB^n x^n = \ldots + AB^{-2}x^{-2} + AB^{-1}x^{-1} + A + ABx + AB^2x^2 + \ldots.$$

This module is a unique factorization module over $D = \oplus_{n\in\mathbb{Z}} B^n x^n = \ldots + B^{-2}x^{-2}B^{-1}x^{-1} + T + Bx + B^2x^2 + \ldots$, which is a subring of $K[x, x^{-1}]$, a Laurent polynomial ring over K.

4.3. Positively Graded Modules which are Unique Factorization Modules

Let $R = \oplus_{n\in\mathbb{Z}_0} D_n$ be a positively graded domain, which is a subdomain of the strongly graded domain $D = \oplus_{n\in\mathbb{Z}} D_n$. The fact that R is Noetherian holds if and only if D_0 is Noetherian, as stated in Proposition 2.1 of [27]. In this section, we aim to demonstrate that $L = \oplus_{n\in\mathbb{Z}_0} M_n$, a positively graded module over R, is a unique factorization module (UFM) if and only if M_0 is a UFM over D_0, under the condition that D_0 is a Noetherian domain.

In the rest of this section, let $R = \oplus_{n\in\mathbb{Z}_0} D_n$ and $L = \oplus_{n\in\mathbb{Z}_0} M_n$, where D_0 is a Noetherian domain and M_0 is a finitely generated torsion-free D_0-module.

In [23], it is established that R is a UFR if and only if D_0 is a UFR, and D_1 is a principal D_0-module. This section commences with the following proposition, which corresponds to the commutative case of Theorem 1 in [23].

Proposition 4.3.1 *(Proposition 4.1 of [26]) A positively graded domain $R = \oplus_{n\in\mathbb{Z}_0} D_n$ is a UFD if and only if:*

(1) D_0 is a UFD, and

(2) D_1 is a principal D_0-module, meaning there exists $p_1 \in D_1$ such that $D_1 =$

D_0p_1.

Note that L is a finitely generated torsion-free R-module since M_0 is a finitely generated torsion-free D_0-module ([24], Lemma 4.4). Additionally, M_0 being CIC is equivalent to L being CIC, as per Theorem 4.1 of [24].

The subsequent lemma corresponds to a module version of Lemma 2.5 (2) in [28] and can be demonstrated similarly to Lemma 5.1 in [24].

Lemma 4.3.2 *(Lemma 4.2 of [26]) Suppose N_0 is a fractional D_0-submodule of M_0 such that N_0 is contained in M_0, and let $N = RN_0$. The following properties hold:*

(1) $N^- = R(N_0)^-$,

(2) $N_v = R(N_0)_v$.

The subsequent lemma is an adaptation of Lemma 4.2 and Lemma 4.3 found in [28].

Lemma 4.3.3 *(Lemma 4.3 of [26]) Let M_0 be a UFM over D_0 and P be a prime R-submodule of L with $P_0 = P \cap M_0 \neq (0)$. Then*

(1) P_0 is a prime submodule of M_0.

(2) $P' = RP_0$ is a prime submodule of L.

(3) If P is a prime v-submodule, then P_0 is a prime v-submodule of M_0, and $P = RP_0$.

Proof. By the proof of Lemma 4.3 of [26], we have the following:

The proof of (1) and (2) are similar to the proof of Lemma 4.2 (1) and (2) of [29].

(3) Let $P' = RP_0 \subseteq L$. Note that $P = P_v \supseteq (P')_v = (RP_0)_v = R(P_0)_v$ by Lemma 4.3.2. Thus $P_0 = P \cap M_0 \supseteq R(P_0)_v \cap M_0 = (P_0)_v$. Hence $P_0 = (P_0)_v$, and so P_0 is a prime v-submodule by (1). Note that $P' = RP_0 = Rp_0M_0$ for some non-zero $p_0 \in D_0$ because M_0 is a UFM. Since Rp_0 is an invertible

ideal, $(P')^- = (Rp_0)^{-1} = Rp_0^{-1} \supseteq P^-$, which implies $R \supseteq Rp_0P^-$ and $P' = Rp_0M_0 = Rp_0L \supseteq Rp_0P^-P$. If $P \supset P'$ then $Rp_0P^-L \subseteq P' = Rp_0L$ since P' is a prime submodule by Lemma (2). Then $P^-L \subseteq L$ and so $P^- = D$ because $O_Q(L) = D$. Thus $P = P_v = (P^-)^+ = (D)^+ = L$, a contradiction. Hence $P = RP_0$.

■

The following proposition is a graded version of Proposition 4.4 of [28].

Proposition 4.3.4 *(Proposition 4.4 of [26]) Let M_0 be a UFM over D_0, and let N be a submodule of L with $N_0 = N \cap M_0 \neq (0)$. Then the following conditions hold:*

(1) N_0 is a submodule of M_0, and N_0 can be expressed as $\mathfrak{n}_0M_0$ for some ideal $\mathfrak{n}_0$ of D_0.

(2) $N = R\mathfrak{n}_0L$, and $R\mathfrak{n}_0 = (N : L)$.

Proof. By the proof of Proposition 4.4 of [26], we have the following:

(1) Similarly to the previous lemma, we conclude that N_0 is a submodule of M_0. Moreover, it holds that $N_0 = \mathfrak{n}_0M_0$ for some ideal $\mathfrak{n}_0$ of D_0, as implied by Theorem 4.1.2, considering the fact that M_0 is a UFM over D_0.

(2) Suppose there is a v-submodule N such that $N \neq R\mathfrak{n}_0L$ where $\mathfrak{n}_0$ is an ideal of D_0. We may assume that N is maximal with this property because M is Noetherian. Then there is a maximal v-submodule P with $P \supseteq N$ and $P = R\mathfrak{p}_0L$, where $\mathfrak{p}_0$ is a maximal ideal of D_0. It follows that $L \supseteq (R\mathfrak{p}_0)^{-1}N \supseteq N$. If $(R\mathfrak{p}_0)^{-1}N = N$, then $(R\mathfrak{p}_0)^{-1} \subseteq R$, a contradiction because L is a CIC. Thus $(R\mathfrak{p}_0)^{-1}N \supset N$ and it follows from Lemma 3.2 of [28] that $((R\mathfrak{p}_0)^{-1}N)_v = (R\mathfrak{p}_0)^{-1}N$. By the choice of N, $(R\mathfrak{p}_0)^{-1}N = R\mathfrak{t}_0L$ for some ideal $\mathfrak{t}_0$ of D_0. Hence $N = R\mathfrak{p}_0\mathfrak{t}_0L$, a contradiction. Hence $N = R\mathfrak{n}_0L$ for some ideal $\mathfrak{n}_0$ of D_0. The last statement easily follows since $R\mathfrak{n}_0$ is invertible.

■

Next we study the structure of a prime v-submodule P of L such that $P \cap M_0 = \{0\}$. Since $Q^g = \oplus_{n\in\mathbb{Z}_0} K_0 D_n = K_0 R$ is a principal ideal domain by Lemma 2.1 of [27] and $K_0 L$ is a finitely generated torsion-free Q^g-module, we have that a v-submodule P_1 of $K_0 L$ is prime if only if $P_1 = \mathfrak{p}_1 K_0 L$, where $\mathfrak{p}_1$ is a maximal ideal of Q^g such that $\mathfrak{p}_1 = (P_1 : K_0 L)$ by Theorem 3.3 of [28].

The following lemma is a graded version of Lemma 4.5 of [28].

Lemma 4.3.5 *(Lemma 4.5 of [26]) Let N be an R-submodule of L. Then, it follows that*

(1) $(K_0 N : K_0 L) = K_0 \mathfrak{n}$, *where* $\mathfrak{n} = (N : L)$, *and* $K_0 N^- = (K_0 N)^-$.

(2) $(K_0 N)_v = K_0 N_v$.

Proof. By the proof of Lemma 4.5 of [26], we have the following:

(1) The proof follows a similar structure to the proof of Lemma 4.5 (1) in [28].

(2) Let $m' \in (K_0 N)_v = ((K_0 N)^-)^+ = (K_0 N^-)^+$, that is, $K_0 L \supseteq K_0 N^- m' \supseteq N^- m'$. Then, there exists $r \in D_0$ such that $N^- r m' = r N^- m' \subseteq L$. This implies $rm' \in (N^-)^+ = N_v$, and so $m' \in r^{-1} N_v \subseteq K_0 N_v$.
Conversely, let $m' \in K_0 N_v$. Write $m' = \sum_{i=1}^{t} k_{0_i} m_i$ where $k_{0_i} \in K_0$ and $m_i \in N_v$ for all $i = 1, 2, \ldots t$. For each $i = 1, 2, \ldots, t$, $N^- m_i \subseteq L$, and so $K_0 N^- m' = K_0 N^- \left(\sum_{i=1}^{t} k_{0_i} m_i \right) \subseteq N^-(K_0 m_1 + \ldots + K_0 m_t) \subseteq K_0 L$. Therefore, $m' \in (K_0 N^-)^+ = ((K_0 N)^-)^+ = (K_0 N)_v$. Hence, $(K_0 N)_v = K_0 N_v$.

■

The subsequent lemma corresponds to a graded adaptation of lemma 4.6 from [28]. We present the proof since the final properties necessitate the use of the v_1-operation (refer to [28], [29], [35] for detailed explanations on v-submodules and v_1-operation).

Lemma 4.3.6 *(Lemma 4.6 of [26]) Let M_0 be a UFM over D_0, and consider $P_1 = \mathfrak{p}_1 K_0 L$, a prime v-submodule of $K_0 L$. Here, $\mathfrak{p}_1$ is a maximal ideal of $K_0 R$, $P = P_1 \cap L$, and $\mathfrak{p} = \mathfrak{p}_1 \cap R$. The following statements hold:*

(1) P is a prime submodule of L, and $\mathfrak{p} = (P : L)$.

(2) $K_0P = P_1$, and $P \cap M_0 = (0)$.

(3) $P = \mathfrak{p}L$, and P is a maximal v-submodule of L.

Proof. By the proof of Lemma 4.6 of [26], we have the following:

(1) Let $r \in R$ and $m \in L$ such that $rm \in P$ and $m \notin P$. Since $m \notin P_1$ and P_1 is prime, we have $rL \subseteq rK_0L \subseteq P_1$ and so $rL \subseteq P$. Hence P is a prime submodule of L.

Since $\mathfrak{p}L \subseteq \mathfrak{p}K_0L = P_1$, we have $\mathfrak{p}L \subseteq P$, so $\mathfrak{p} \subseteq (P : L)$. Conversely let $r \in (P : L)$, that is $r \in R$ and $rL \subseteq P$. Then $rK_0L \subseteq K_0P \subseteq P_1$, so $r \in (P_1 : K_0L) = \mathfrak{p}_1$. Thus $r \in \mathfrak{p}_1 \cap R = \mathfrak{p}$. Hence $\mathfrak{p} = (P : L)$.

(2) Let $m' \in P_1$ and we write $m' = \sum_{i=1}^{n} t_i m_i$ where $t_i \in \mathfrak{p}_1$ and $m_i' \in K_0L$. Then there are $\alpha, \beta \in D_0$ such that $\alpha t_i \in \mathfrak{p}$ and $\beta m_i' \in L$ and so $\alpha\beta m' \in \mathfrak{p}L \subseteq P$. Thus $m' \in (\alpha\beta)^{-1}P \subseteq K_0P$. Hence $K_0P = P_1$.

Note that $\mathfrak{p}_1 = \langle t \rangle = tK_0R$ for some prime element $t \in K_0R$ with $\deg(t) \geq 1$. If $P \cap M_0 \neq \{0\}$ and let $0 \neq m \in P \cap M_0$. Then $m = tm'$ for some $m' \in K_0L$, since $K_0P = P_1 = tK_0L$. Write $t = t_n + t_{n-1} + \ldots + t_0$ $(t_i \in K_0D_i,$ with $t_n \neq 0)$ and $m' = m_l + \ldots + m_0$ $(m_j \in K_0M_j)$. Then we get $t_n m_l = 0$, so $m_l = 0$ and so on. Then we have $m = 0$, a contradiction. Hence $P \cap M_0 = (0)$.

(3) The proof is similar to Lemma 4.2.6 (3).

■

Lemma 4.3.7 *(Lemma 4.7 of [26]) Let M_0 be a UFM over D_0 and P be a prime v-submodule of L such that $P \cap M_0 = (0)$. Then $P = \oplus_{n\geq 1} M_n = D_1L$ or there is a maximal v-submodule P_1 of K_0L such that $P = P_1 \cap L$.*

Proof. By the proof of Lemma 4.7 of [26], we have the following. Let $\mathfrak{p} = (P : L)$. Then $\mathfrak{p}$ is a prime v-ideal of R, so $\mathfrak{p}$ is a non-zero minimal prime ideal. Thus, $\mathfrak{p}$ is in one of the following forms: $\mathfrak{p} = \mathfrak{p}_0R$ for some prime ideal $\mathfrak{p}_0$ of D_0, $\mathfrak{p} = \oplus_{n\geq 1} D_n$, or $\mathfrak{p} = \mathfrak{p}_1 \cap R$ for some prime ideal $\mathfrak{p}_1$ of K_0R by Proposition 3.1 of [27].

In the first case, $P \supseteq \mathfrak{p}_0 RL \supseteq \mathfrak{p}_0 M_0 \neq (0)$, leading to a contradiction.

In the second case, if $P \supseteq (\oplus_{n\geq 1} D_n)L = RD_1L = D_1L = \oplus_{n\geq 1} M_n$. If $P \supset \oplus_{n\geq 1} M_n$, there is a non-zero submodule T_0 of M_0 such that $P = T_0 + \oplus_{n\geq 1} M_n$. Then $P \cap M_0 \supseteq T_0 \neq \{0\}$, a contradiction. Hence $P = \oplus_{n\geq 1} M_n$.

In the last case, $\mathfrak{p} = \mathfrak{p}_1 \cap R$ with $K_0\mathfrak{p} = \mathfrak{p}_1$. Since $P \cap M_0 = (0)$, $K_0L \supset K_0P = (K_0P)_v$ by Lemma 4.3.5. Thus, there is a maximal v-submodule P_1 of K_0L such that $P_1 \supseteq K_0P$. By Lemma 4.3.5, $(P_1 : K_0L) \supseteq (K_0P : K_0L) = K_0(P : L) = K_0\mathfrak{p} = \mathfrak{p}_1$. Since $(P_1 : K_0L)$ is a prime ideal of K_0R, $\mathfrak{p}_1 = (P_1 : K_0L)$. Hence $P_1 = \mathfrak{p}_1 K_0 L$ and $P_1 \cap L \supseteq P$. By Lemma 4.3.6, $P_1 \cap L = \mathfrak{p}L \subseteq P$, and hence $P = P_1 \cap L$ and $P = \mathfrak{p}L$. Therefore, by the last two cases, $P = \oplus_{n\geq 1} M_n$ or there is a maximal v-submodule P_1 of K_0L such that $P = P_1 \cap L$. ■

Consider the case where $R = \oplus_{n\in\mathbb{Z}_0} D_n$ is a Noetherian UFD. In this case, $R = D_0[p_1]$ for some element $p_1 \in D_1$, as established by Theorem 1 of [23]. Consequently, $M = M_0[p_1]$, forming a polynomial module. The necessary condition of Theorem 4.3.8 has already been proven in [41]. However, we provide an alternative proof using the v_1-operator.

Theorem 4.3.8 *(Theorem 4.8 of [26]) Assume $R = \oplus_{n\in\mathbb{Z}_0} D_n$ is a Noetherian UFD, and $L = \oplus_{n\in\mathbb{Z}_0} M_n$ is a positively graded module over R. Then L is a UFM if and only if M_0 is a UFM.*

Proof. By the proof of Theorem 4.8 of [26], we have the following:

($\Rightarrow$) Assume that $L = \oplus_{n\in\mathbb{Z}_0} M_n$ is a UFM over R. As a consequence, L is CIC, implying that M_0 inherits the CIC property according to Theorem 4.1 of [24]. Let P_0 be a non-zero prime v-submodule of M_0. By Lemma 4.3.2, $P = RP_0$ is a v-submodule of L. Additionally, based on Lemma 4.3.3 (2), RP_0 qualifies as a prime submodule of L. Consequently, since L is a UFM, RP_0 is a principal prime v-submodule. This implies $RP_0 = rL$ for some $r \in R$. Since $(0) \neq P_0 \subset RP_0 = rL$, it follows that $r \in D_0$, leading to $P_0 = rM_0$. Therefore, P_0 is a principal submodule, establishing that M_0 is a UFM by Theorem 4.1.5.

($\Leftarrow$) Suppose that M_0 is a UFM over D_0. Since R is a UFD, it is evident that

R is a maximal order by Proposition 1 of [23]. This implies that D_0 is a maximal order by Theorem 2.1 of [27], and consequently, L is CIC by Theorem 4.1 of [24]. Given that D_0 is a UFD and D_1 is a principal D_0-module, a result of R being a UFD, we aim to prove that L is a UFM. Let P be a prime v-submodule of L, and let $P_0 = P \cap M_0$. According to Lemma 4.3.3 (3), P_0 is a prime v-submodule.

1. Case $P_0 \neq (0)$: In this case, $P = RP_0$ by Lemma 4.3.3 (3). Since M_0 is a UFM, $P_0 = p_0 M_0$ for some $p_0 \in D_0$, leading to $P = RP_0 = Rp_0M_0 = p_0RM_0 = p_0L$.

2. Case $P_0 = (0)$: In this case, $P = \oplus_{n\geq 1} M_n = D_1L$ or $P = \mathfrak{p}L$ for some v-ideal $\mathfrak{p}$ of R by Lemma 4.3.7. If $P = \oplus_{n\geq 1} M_n = D_1L$, then $P = d_1D_0L = d_1L$ for some $d_1 \in D_1$ since D_1 is a principal D_0-module. If $P = \mathfrak{p}L$, then $P = \mathfrak{p}L = pRL = pL$ for some $p \in R$ since R is a UFD.

Hence, every prime v-submodule of L is principal, establishing that L is a UFM by Theorem 4.1.5. ■

We end this section with examples of a positively graded module which is a UFM.

Example 4.3.9 (Example 4.9 of [26]) Let $R = \oplus_{n\in\mathbb{Z}_0} D_n$ be a positively graded domain, where D_0 is a Noetherian UFD and D_1 is a principal D_0-module. Consider a positively graded module $M = R \oplus R \oplus \ldots \oplus R$ over R, and let P be a graded submodule of M such that $M = P \oplus T$ for some graded submodule T. The claim is that P is a UFM.

Proof. Observe that P is a projective module, making it a generalized Dedekind module. Additionally, according to Theorem 3.1 of [28], P is a v-multiplication module. Considering that P is a v-multiplication module and R is a UFD, we conclude that P is a UFM, as per Theorem 4.1.2. ■

Lemma 4.3.10 *(Lemma 4.10 of [26]) Let D be a domain, B be an invertible ideal of D and A be a non-zero ideal of D. Let $R = D + Bx + B^2x^2 + \ldots \subseteq D[x]$, where*

$D[x]$ is a polynomial ring over D and $L = A + ABx + AB^2x^2 + \ldots = AR$. Then L is a positively graded module over the positively graded domain R.

From Remark 4.1.6 and Lemma 4.3.10, we obtain the following example.

Example 4.3.11 (Example 4.11 of [26]) Let D be any Noetherian UFD, and let A, B be two non-zero v-ideals of D. Then $L = A + ABx + AB^2x^2 + \ldots$ is a UFM over $R = D + Bx + B^2x^2 + \ldots$.

Proof. Consider R, a UFD as a consequence of D being a UFD and Bx acting as a principal D-module. Since A is a non-zero v-ideal in D by Remark 4.1.6, it is established as a UFM. Applying Theorem 4.3.8 yields the conclusion that L is a UFM over R. ■

CHAPTER V

Generalized Dedekind Modules and Further Work

5.1. Generalized Dedekind Modules

A very important object of study related to Krull rings and Krull modules, the generalized Dedekind ring (G-Dedekind rings for short) and the generalized Dedekind modules (G-Dedekind module) have been defined and extensively studied. They are defined as follows:

In [29], the authors say that D is a generalized Dedekind domain if it satisfies the following condition:

(i) every every v-ideal $\mathfrak{a}$ of D is invertible, that is $(D : \mathfrak{a})\mathfrak{a} = D$, where $(D : \mathfrak{a}) = \{k \in K \mid k\mathfrak{a} \subseteq D\}$;

(ii) D satisfies the ascending chain condition on v-ideals of D.

Furthermore, serving as an extension of the concept of a generalized Dedekind domain, the authors in [28] introduced the notion of a generalized Dedekind module.

Definition 5.1.1 *(Definition 3.1 of [28]) Consider a finitely generated torsion-free module M over an integrally closed domain D with its quotient field K. A module M is called a generalized Dedekind module (G-Dedekind module for brevity) if it satisfies the following conditions:*

(i) Every v-submodule N of M is invertible, denoted as $N^{-}N = M$, where $N^{-} = \{k \in K \mid kN \subseteq M\}$;

(ii) M satisfies the ascending chain condition on v-submodules of M.

Moreover, in [29], the authors say that M is a Krull module if it satisfies the following condition:

(i) every every v-submodule N of M is v-invertible, that is $(N^{-}N)_v = M$, where $N^{-} = \{k \in K \mid kN \subseteq M\}$;

(ii) M satisfies the ascending chain condition on v-submodules of M.

Concerning these, the following result holds.

Proposition 5.1.2 *(Proposition 2.9 of [25]) Consider a G-Dedekind domain D and a finitely generated torsion-free D-module M. Assuming that M is a v-multiplication module, it follows that M qualifies as a G-Dedekind module.*

Proof. By the proof of Proposition 2.9 of [25] we have the following.

Let N be a v-submodule of M. Then $N = \mathfrak{n}M$ by the assumption, where $\mathfrak{n} = (N : M)$ and $\mathfrak{n}$ is a v-ideal by [5, Lemma 2.4]. Hence N is invertible since $\mathfrak{n}$ is invertible.

Let N_i be v-submodules of M such that $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \ldots$ and write $N_i = \mathfrak{n}_i M$ for all i, where $\mathfrak{n}_i = (N_i : M)$ which are invertible. $\mathfrak{n}_i M \subseteq \mathfrak{n}_{i+1} M$ implies $\mathfrak{n}_{i+1}^{-1}\mathfrak{n}_i M \subseteq M$ and so $\mathfrak{n}_{i+1}^{-1}\mathfrak{n}_i \subseteq D$ by the determinant argument, that is, $\mathfrak{n}_i \subseteq \mathfrak{n}_{i+1}$. Thus there is an i such that $\mathfrak{n}_i = \mathfrak{n}_{i+1}$ and hence $N_i = N_{i+1}$. Therefore M is a G-Dedekind module. ■

In general, if M is not a v-multiplication module, then M does not need to be a generalized Dedekind module.

Proposition 5.1.3 *(Proposition 2.7 (1) of [25]) Let D be a Noetherian G-Dedekind domain and $\mathfrak{a}_i$ be proper prime ideals of D such that $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n$, $(\mathfrak{a}_n)_v = D$ and $\mathfrak{p}$ be a minimal prime ideal of D.*

Put $M = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n$. If $\mathfrak{a}_n \supset \mathfrak{p}$ then the following are hold:

(1) $\{P_i \mid 0 \leq i \leq n\}$ is the set of v-submodules of M containing $\mathfrak{p}M$. In particular, P_n is a maximal v-submodul of M.

(2) P_i are not v-multiplication submodules for all i $(1 \leq i \leq n)$.

(3) P_i are not invertible for each i $(2 \leq i \leq n)$. So M is not a G-Dedekind module

From the Proposition 5.1.3, we have the following example.

Example 5.1.4 (Example 2.8 of [25]) Lat D_0 be a Noetherian G-Dedekind domain and $\mathfrak{a}_0$ be a maximal ideal of D_0. Put $D = D_0[x_1, x_2, \cdots, x_n]$ which is the polynomials ring over D_0 in indeterminate $x_1, x_2, \cdots, x_n$, $\mathfrak{a}_1 = \mathfrak{a}_0 + x_1 D + \cdots + x_n D$, $\mathfrak{a}_i = \mathfrak{a}_0[x_1, \cdots, x_{i-1}] + x_i D + \cdots + x_n D$ for each i $(2 \leq i \leq n)$ and $\mathfrak{p}_i = x_i D$ for all i $(1 \leq i \leq n)$. Then

(1) $\mathfrak{p}_i$ are all minimal prime ideals of D such that $\mathfrak{a}_n \supset \mathfrak{p}_n$ and $\mathfrak{a}_i \supset \mathfrak{p}_i$ and $\mathfrak{a}_{i+1} \not\supset \mathfrak{p}_i$ for each i $(1 \leq i < n)$.

(2) $\mathfrak{a}_i$ are all prime ideals of D such that $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n$ and $(\mathfrak{a}_n)_v = D$

On the other hand, related to the generalized Dedekind module and strongly graded module we have the following theorem.

Theorem 5.1.5 *(Theorem 6.1 of [24]) Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a strongly graded module over strongly graded domain $D = \oplus_{n\in\mathbb{Z}} D_n$. If M_0 is a G-Dedekind module, then M is a G-Dedekind D-module.*

5.2. Further Work

The research will continue with the following approach.

1. Investigate whether the converse of Proposition 4.2.1 holds.
2. Examine whether the converse of Theorem 4.2.9 is applicable.
3. Investigate whether the converse of Proposition 5.1.2 holds.
4. Investigate whether the converse of Theorem 5.1.5 holds.
5. Identifying the necessary and sufficient condition under which a strongly graded module $M = \oplus_{n\in\mathbb{Z}} M_n$ and a positively graded module $L = \oplus_{n\in\mathbb{Z}_0} M_n$ can be classified as Krull modules.

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