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ON LOCAL WELL-POSEDNESS FOR H^s-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. This paper concerns the Cauchy problem for the nonlinear Schrödinger equation with power nonlinearity. Time local well-posedness in $H^s(\mathbb{R}^N)$ is proved in the case where the nonlinear term is critical from the scaling point of view, and has limited regularity so that the nonlinear term does not belong to $C^s(\mathbb{R}^2;\mathbb{R}^2)$.

1. INTRODUCTION

In this paper we consider the following Cauchy problem for the nonlinear Schrödinger equation with power nonlinearity:

$$i\partial_t u + \Delta u = f(u) \equiv \lambda |u|^\alpha u, \quad u(0) = \varphi, \tag{1.1}$$

where u is a complex-valued function defined on the spacetime $\mathbb{R}^{1+N}$ with $N \geq 3$, Δ is the Laplace operator on $\mathbb{R}^N$, $\lambda \in \mathbb{C}$, and $\alpha > 0$. We introduce the free propagator $U(t) = \exp(it\Delta)$ and convert (1.1) into at least formally equivalent integral equation

$$u(t) = \Phi(u)(t) \equiv U(t)\varphi - i\int_0^t U(t-\tau)f(u(\tau))\,d\tau. \tag{1.2}$$

We need the Sobolev space $H^s(\mathbb{R}^N)$ and the Besov space $B^s_{r,2}(\mathbb{R}^N)$, where $s \in \mathbb{R}$ and $1 < r < \infty$. We are interested in the well-posedness of (1.1) in $H^s(\mathbb{R}^N)$. From the scaling point of view, the critical exponent for α in $H^s(\mathbb{R}^N)$ is

$$\alpha = \alpha^*(s) \equiv \frac{4}{N-2s}.$$

Roughly speaking, (1.1) is expected to be time locally well-posed in $H^s(\mathbb{R}^N)$ if $\alpha \leq \alpha^*(s)$ ($\alpha < \infty$ if $s \geq N/2$). We mainly study the critical case, so throughout the paper we always assume $0 < s < N/2$.

There is a large amount of work on the well-posedness of (1.1). To describe the preceding results precisely, we should distinguish the cases (i) $\alpha > s-1$ and (ii) $\alpha \leq s-1$. We note that the complex-valued function $f(z)$ belongs to the class $C^{\alpha+1}(\mathbb{R}^2;\mathbb{R}^2)$, which is understood as $C^{\alpha,1}(\mathbb{R}^2;\mathbb{R}^2)$ if α is an integer. Therefore, if $\alpha > s-1$, then the nonlinear term $f(u)$ is s-times

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differentiable in x, so that we can directly multiply $(1-\Delta)^{s/2}$ by the equation and estimate u in $C([0,T];H^s(\mathbb{R}^N))$ or Bochner type spaces $L^q(0,T;B^s_{r,2}(\mathbb{R}^N))$. On the other hand, if $\alpha \le s-1$, then $f(u)$ is not smooth enough to be estimated in Sobolev/Besov spaces of order s. To overcome this difficulty, we should reduce the total number of derivatives by evaluating $\partial_t u$ instead of Δu, since one time derivative is homogeneous to two space derivatives for the Schrödinger equation. Once we obtain the estimate of $\partial_t u$, then using the equation itself we can recover spatial regularity.

We first summarize the previous results concerning the case (i) $\alpha > s-1$. It is known that if $s-1 < \alpha \le \alpha^*(s)$, then (1.1) is time locally well-posed in $H^s(\mathbb{R}^N)$. See Ginibre–Velo [9, 10], Tsutsumi [22] and Kato [11, 12] for subcritical cases $\alpha < \alpha^*(s)$ with $s = 0, 1$; for the case $s-1 < \alpha \le \alpha^*(s)$ with $0 \le s < N/2$, where the critical case is included, see Cazenave–Weissler [5] and Kato [13]. Precisely speaking, more strict condition $\alpha > [s]$ is assumed in [5, 13], but if we use a nonlinear estimate derived by Ginibre–Ozawa–Velo [8], we can bring down the lower bound to $\alpha > s-1$.

We next consider the case (ii) $\alpha \le s-1$. Let $\alpha_*(s) \equiv \max\{0; (s-2)/2; s-3\}$. We have $\alpha_*(s) < s-1$ for $s > 1$. It is known that if $\alpha_*(s) < \alpha < \alpha^*(s)$ with $1 < s < N/2$, namely if the nonlinear term is subcritical, then (1.1) is time locally well-posed in $H^s(\mathbb{R}^N)$; see Tsutsumi [21], Kato [11, 12] and Cazenave–Weissler [5] for $s = 2$, and Pecher [17], Fang–Han [7], Uchizono–Wada [23, 24], and Wada [25] for $1 < s < N/2$. Unlike the case (i), the well-posedness for the critical case has not been well-studied. If $s = 2$ and $\alpha = \alpha^*(2) = 4/(N-4)$ with $N \ge 8$, Cazenave–Fang–Han [4] showed that (1.1) is time locally well-posed in $H^2(\mathbb{R}^N)$. Nakamura–Wada [15, 16] showed that if $1 < s < 4$ and $\alpha_*(s) < \alpha = \alpha^*(s)$, then (1.1) is time globally well-posed in $H^s(\mathbb{R}^N)$ for small data (the case $s = 2$ had been solved in [5]). Nevertheless, to the best of our knowledge, there is no prior work except [4] concerning the critical case without restriction on the size of data.

This paper aims to prove the time local well-posedness for large data under the condition that $\alpha = \alpha^*(s) \le s-1$. To state the main result in this paper, we set

$$\alpha_0(s) = \begin{cases} 1, & 2 < s \le 3, \\ s-2, & 3 < s \le 4, \\ 2, & 4 < s \le 5, \\ s-3, & 5 < s. \end{cases}$$

Clearly, $\alpha_*(s) < \alpha_0(s) < s-1$ for $2 < s < 5$, and $\alpha_*(s) = \alpha_0(s)$ for $s \ge 5$. We shall prove the following:

Theorem 1.1. *Let $2 < s < N/2$ and let $\alpha_0(s) < \alpha = \alpha^*(s)$. For any $\varphi \in H^s(\mathbb{R}^N)$, there exists $T > 0$ such that the following hold:*

(i) *The equation (1.2) has a unique solution $u \in C(I;H^s(\mathbb{R}^N))$, where $I = [0,T]$. Furthermore, $u \in L^q(I;B^s_{r,2}(\mathbb{R}^N))$ for any admissible pair (q,r).*

(ii) *Let* $\{\varphi_k\}_{k=1}^{\infty} \subset H^s(\mathbb{R}^N)$ *satisfy* $\varphi_k \to \varphi$ *in* $H^s(\mathbb{R}^N)$. *Then, for sufficiently large* k, (1.2) *with* φ *replaced with* φ_k *has a unique solution* $u_k \in C(I; H^s(\mathbb{R}^N))$. *Furthermore,* $u_k \to u$ *in* $C(I; H^s(\mathbb{R}^N))$.

There exists a number s satisfying $2 < s < N/2$ and $\alpha^*(s) \le s-1$ only when $N \ge 8$. In this case, Theorem 1.1 gives a new result. On the other hand, if $5 \le N \le 7$, the result above has already been shown in [5,6], including continuous dependence with respect to the data.

This paper is organized as follows. In §2, we first give notation used in this paper. Next we summarize linear and nonlinear estimates used in the proof of Theorem 1.1. In §3, we give the proof of Theorem 1.1, which is done by a series of propositions. We divide the proof into the case $s-2 < \alpha \le s-1$ (Propositions 3.1 and 3.2), and the case $s-3 < \alpha \le s-2$ (Propositions 3.3 and 3.4), since in the latter case we need the second derivative $\partial_t^2 u$. In the critical case, it is difficult to show that the nonlinear term $f(u)$ is sufficiently small in $L^q(I; B^{s-2}_{r,2})$ so that the contraction mapping principle works. To this end, we estimate $f(u) - U(t)\varphi$ instead, by the use of Lemmas 2.3 and 2.4.

2. Preliminaries

We begin this section with giving notation used in this paper. For $1 \le r \le \infty$, we set $r' = r/(r-1)$. We denote by $L^r(\mathbb{R}^N)$ the usual Lebesgue spaces. Let $s \in \mathbb{R}$ and $1 \le r, m \le \infty$. We define the Sobolev space $H^s(\mathbb{R}^N)$ by

$$H^s(\mathbb{R}^N) = \{u \in \mathscr{S}'(\mathbb{R}^N) : \|u\|_{H^s} \equiv \|(1-\Delta)^{s/2}u\|_{L^2} < \infty\}.$$

We also define the Besov space $B^s_{r,m}(\mathbb{R}^N)$. For this purpose, we need Littlewood–Paley decomposition. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a spherically symmetric function satisfying $0 \le \chi(\xi) \le 1$ with $\chi(\xi) = 1$ for $|\xi| \le 1$, and with $\chi(\xi) = 0$ for $|\xi| \ge 2$. We set $\eta_k(\xi) = \chi(\xi/2^k) - \chi(\xi/2^{k-1})$. Then we have $\operatorname{supp} \eta_k \subset \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$, and $\chi(\xi) + \sum_{k=1}^\infty \eta_k(\xi) = 1$ for all $\xi \in \mathbb{R}^N$. We set

$$B^s_{r,m}(\mathbb{R}^N) = \{u \in \mathscr{S}'(\mathbb{R}^N) : \|u\|_{B^s_{r,m}} < \infty\},$$

where

$$\|u\|_{B^s_{r,m}} \equiv \|\chi(D)u\|_{L^r} + \left(\sum_{k=1}^\infty 2^{skm}\|\eta_k(D)u\|_{L^r}^m\right)^{1/m},$$

with trivial modification if $m = \infty$. Here, $\chi(D) = \mathscr{F}^{-1}\chi(\xi)\mathscr{F}$, and $\mathscr{F}$ denotes the Fourier transform. In this paper, we always take $m = 2$, so we omit the third index and write $B^s_r(\mathbb{R}^N) = B^s_{r,2}(\mathbb{R}^N)$ for short. If $r = 2$, then $B^s_2(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. For the detail, see [1,2,19,20]. For an interval $I \subset \mathbb{R}$ and a Banach space X, we denote by $C^k(I; X)$ the space of C^k-functions from I to X, and by $L^q(I; X)$ the space of measurable functions u from I to X with $\|u\|_{L^q(X)} \equiv \|u\|_{L^q(I;X)} < \infty$. Similarly, we denote by $W^{k,q}(I; X)$ the space of functions u from I to X which are weakly differentiable up to k-times with $\|u\|_{W^{k,q}(X)} \equiv \max_{0 \le j \le k} \|\partial_t^j u\|_{L^q(X)} < \infty$.

We next prepare several indices that are used throughout the paper. Let $\alpha = \alpha^*(s)$. We set

$$\gamma = \alpha + 2, \qquad \rho = \frac{\alpha+2}{1+\alpha s/N}, \qquad \frac{1}{\rho^*} = \frac{1}{2} - \frac{1}{N}, \tag{2.1}$$

$$\frac{1}{\mu} = \frac{1}{2} - \frac{s}{N}, \qquad \frac{1}{\nu} = \frac{1}{\rho} - \frac{s}{N} = \frac{N-2s}{N(\alpha+2)}. \tag{2.2}$$

Since $N \geq 3$, $\alpha > 0$ and $0 < s < N/2$, we see $2 < \gamma, \rho, \rho^*, \mu, \nu < \infty$. These exponents are defined so that $1/\rho' = \alpha/\nu + 1/\rho$ and that $2/\gamma = N/2 - N/\rho = 2/(\alpha+2)$; namely (γ, ρ) is an admissible pair defined below. The pair $(2, \rho^*)$ is also an admissible pair, which is called the endpoint. Furthermore, for $j = 1, 2$ with $s - 2j > 0$, we also set

$$\frac{1}{\rho_j} = \frac{1}{\rho} - \frac{2j}{N}, \qquad \frac{1}{\rho_j^*} = \frac{1}{2} - \frac{(2j+1)}{N}, \qquad \frac{1}{\kappa_j} = \frac{1}{2} - \frac{2j}{N}. \tag{2.3}$$

provided that the right-hand sides are positive. It follows from the Sobolev embedding theorem [2, Theorem 6.5.1] that

$$H^s(\mathbb{R}^N) \subset B^{s-2j}_{\kappa_j}(\mathbb{R}^N) \subset L^\mu(\mathbb{R}^N) \quad \text{and} \quad B^s_\rho(\mathbb{R}^N) \subset B^{s-2j}_{\rho_j}(\mathbb{R}^N) \subset L^\nu(\mathbb{R}^N).$$

Definition 2.1. Let $N \geq 3$. A pair of numbers (q, r) is said to be admissible if $(q, r) \in [2, \infty] \times [2, 2N/(N-2)]$ and if $2/q = N/2 - N/r$.

Lemma 2.1. *Let $s \in \mathbb{R}$. Let (q, r) and (q_0, r_0) be two admissible pairs. For $\varphi \in H^s(\mathbb{R}^N)$ and $f \in L^{q_0'}(\mathbb{R}; B^s_{r_0'}(\mathbb{R}^N))$, we set*

$$u(t) = U(t)\varphi - i\int_0^t U(t-\tau)f(\tau)\,d\tau.$$

Then, the following estimate holds:

$$\|u\|_{L^q(B^s_r)} \lesssim \|\varphi\|_{H^s} + \|f\|_{L^{q_0'}(B^s_{r_0'})}. \tag{2.4}$$

Furthermore, we have $u \in C(\mathbb{R}; H^s(\mathbb{R}^N))$.

Proof. See [5, 14, 18, 26] □

Definition 2.2. Let $\alpha > 0$. We say that a function $g : \mathbb{C} \to \mathbb{C}$ belongs to the class $A(\alpha+1)$ if $g \in C^m(\mathbb{R}^2, \mathbb{R}^2)$ for any nonnegative integer $m < \alpha + 1$, if $g(0) = g'(0) = \cdots = g^{(m)}(0) = 0$, and if

$$|g^{(m)}(z_1) - g^{(m)}(z_2)| \leq C \begin{cases} (|z_1| + |z_2|)^{\alpha-m}|z_1 - z_2|, & m < \alpha, \\ |z_1 - z_2|^{\alpha+1-m}, & \alpha \leq m < \alpha + 1. \end{cases}$$

Remark. (i) $g(z) = |z|^\alpha z \in A(\alpha+1)$ for $\alpha > 0$;

(ii) if $g(z) \in A(\alpha+1)$ with $\alpha > 1$, then $g'(z) \in A(\alpha)$.

Lemma 2.2. *Let $\alpha > 0$, $g \in A(\alpha+1)$, $0 < \sigma < \alpha+1$ and $1 < r, r_0, r_1, r_2, r_3 < \infty$. Then the following estimates hold for all u, v, w such that the norms in the right-hand sides are finite:*

(i) *If $1/r = \alpha/r_0 + 1/r_1$, then*

$$\|g(u)\|_{B^\sigma_r} \le C\|u\|^\alpha_{r_0}\|u\|_{B^\sigma_{r_1}}. \tag{2.5}$$

(ii) *If $\max\{1;\sigma\} < \alpha$ and $1/r = (\alpha-1)/r_0 + 1/r_1 + 1/r_2 - \sigma/N$ with $2N/(N+2\sigma) \le r_j < N/\sigma$, $1 \le j \le 2$, then*

$$\|g'(u)v\|_{B^\sigma_r} \lesssim \|u\|^{\alpha-1}_{L^{r_0}}\|u\|_{B^\sigma_{r_1}}\|v\|_{B^\sigma_{r_2}}. \tag{2.6}$$

(iii) *If $\max\{1;\sigma\} < \alpha-1$ and $1/r = (\alpha-2)/r_0 + 1/r_1 + 1/r_2 + 1/r_3 - 2\sigma/N$ with $2N/(N+2\sigma) \le r_j < N/\sigma$, $1 \le j \le 3$, then*

$$\|g''(u)vw\|_{B^\sigma_r} \lesssim \|u\|^{\alpha-2}_{L^{r_0}}\|u\|_{B^\sigma_{r_1}}\|v\|_{B^\sigma_{r_2}}\|w\|_{B^\sigma_{r_3}}. \tag{2.7}$$

Proof. For the proof of (2.5), see e.g. [8, Lemma 3.4]. To prove (2.6), we set $1/m_j = 1/r_j - \sigma/N$, $1 \le j \le 2$. By assumption, we see $2 \le m_j < \infty$. From the Sobolev inequality, we have the inclusions $B^\sigma_{r_j} \subset L^{m_j}$. Let $1/m^* = 1/r - 1/m_2$ and $1/r^* = 1/r - 1/r_2$. Since $1/m^* = (\alpha-1)/r_0 + 1/r_1$ and $1/r^* = (\alpha-1)/r_0 + 1/m_1$, it follows from (2.5) together with the Leibniz rule that

$$\begin{aligned}\|g'(u)v\|_{B^\sigma_r} &\lesssim \|g'(u)\|_{B^\sigma_{m^*}}\|v\|_{m_2} + \|g'(u)\|_{r^*}\|v\|_{B^\sigma_{r_2}}\\ &\lesssim \|u\|^{\alpha-1}_{L^{r_0}}\|u\|_{B^\sigma_{r_1}}\|v\|_{L^{m_2}} + \|u\|^{\alpha-1}_{L^{r_0}}\|u\|_{L^{m_1}}\|v\|_{B^\sigma_{r_2}} \lesssim \|u\|^{\alpha-1}_{L^{r_0}}\|u\|_{B^\sigma_{r_1}}\|v\|_{B^\sigma_{r_2}}.\end{aligned}$$

Thus we have proved (2.6). We can prove (2.7) in the same way. □

Lemma 2.3. *Let $1 < q \le r < \infty$ and let $1 < r_0, r_1 < \infty$ with $q/r = (q-1)/r_0 + 1/r_1$. Let $I = [0,T]$ and let $v \in L^q(I; L^{r_0}(\mathbb{R}^N)) \cap W^{1,q}(I; L^{r_1}(\mathbb{R}^N))$ satisfy $v(0) = 0$. Then the following estimate holds:*

$$\|v\|^q_{L^\infty(L^r)} \lesssim \|v\|^{q-1}_{L^q(L^{r_0})}\|\dot v\|_{L^q(L^{r_1})}. \tag{2.8}$$

Moreover, if $\sigma \in \mathbb{R}$, $q = 2$ and $v \in L^2(I; B^\sigma_{r_0}(\mathbb{R}^N)) \cap W^{1,2}(I; B^\sigma_{r_1}(\mathbb{R}^N))$ with $v(0) = 0$, then the following estimate holds:

$$\|v\|^2_{L^\infty(B^\sigma_r)} \lesssim \|v\|_{L^2(B^\sigma_{r_0})}\|\dot v\|_{L^2(B^\sigma_{r_1})}. \tag{2.9}$$

Proof. We have the identity $\partial_t|v|^q = q|v|^{q-2}\,\mathrm{Re}(\bar v\dot v)$. Integrating the both sides on t and using $v(0) = 0$, we obtain

$$|v(t)|^q \le q\int_0^T |v(t)|^{q-1}|\dot v(t)|\,dt.$$

Therefore, it follows from the Minkowski and Hölder inequalities that

$$\|v(t)\|^q_{L^r} \le q\int_0^T \|v(t)\|^{q-1}_{L^{r_0}}\|\dot v(t)\|_{L^{r_1}}\,dt \le q\|v\|^{q-1}_{L^q(L^{r_0})}\|\dot v\|_{L^q(L^{r_1})}. \tag{2.10}$$

Taking the supremum on I, we obtain (2.8). We shall next prove (2.9). Recall that χ and η_k are Littlewood–Paley functions. Letting $q = 2$ in (2.10) and replacing v with $\eta_k(D)v$, we obtain

$$\|\eta_k(D)v(t)\|_{L^r}^2 \leq 2\|\eta_k(D)v\|_{L^2(L^{r_0})}\|\eta_k(D)\dot{v}\|_{L^2(L^{r_1})}$$

for every $t \in I$. We also have the analogous inequality for $\chi(D)v$. Taking the summation and applying the Schwarz inequality in k, we obtain the desired estimate. □

Lemma 2.4. *Let $2 < s < N/2$ and $\alpha = \alpha^*(s)$. Let $1 \leq q \leq \infty$ and $1 < r < N/2$. Let $1/r_1 = 1/r - 2/N$. Then the following estimates hold for all u, v, w such that the norms in the right-hand sides are finite:*

(i) *If $0 < \sigma < \alpha + 1$, then we have*

$$\|f(u)\|_{L^q(B_r^\sigma)} \lesssim \|u\|_{L^\infty(L^\mu)}^\alpha \|u\|_{L^q(B_{r_1}^\sigma)}. \tag{2.11}$$

(ii) *If $0 < \sigma \leq s - 2$, $\max\{1; \sigma\} < \alpha$, and $2N/(N + 2\sigma + 4) \leq r < N/(\sigma + 2)$, then we have*

$$\|f(u) - f(v)\|_{L^q(B_r^\sigma)} \lesssim (\|u\|_{L^\infty(L^\mu)} \vee \|v\|_{L^\infty(L^\mu)})^{\alpha-1} \times (\|u\|_{L^q(B_{r_1}^\sigma)} \vee \|v\|_{L^q(B_{r_1}^\sigma)})\|u - v\|_{L^\infty(B_{\kappa_1}^{s-2})}. \tag{2.12}$$

(iii) *If $s > 4$, $\max\{1; s - 4\} < \alpha$, and $2N/(N + 2s) \leq r < N/s$, then we have*

$$\|f'(u)v\|_{L^q(B_r^{s-4})} \lesssim \|u\|_{L^\infty(L^\mu)}^{\alpha-1}\|u\|_{L^q(B_{r_1}^{s-2})}\|v\|_{L^\infty(B_{\kappa_1}^{s-4})}, \tag{2.13}$$

$$\|f(u) - f(v)\|_{L^q(B_r^{s-4})} \lesssim (\|u\|_{L^\infty(L^\mu)} \vee \|v\|_{L^\infty(L^\mu)})^{\alpha-1} \times (\|u\|_{L^q(B_{r_1}^{s-2})} \vee \|v\|_{L^q(B_{r_1}^{s-2})})\|u - v\|_{L^\infty(B_{\kappa_1}^{s-4})}. \tag{2.14}$$

(iv) *If $s > 4$, $\max\{1; s-4\} < \alpha - 1$, and $2N/(N+2s-4) \leq r < N/(s-2)$, then we have*

$$\|\{f'(u) - f'(v)\}w\|_{L^q(B_r^{s-4})} \lesssim (\|u\|_{L^\infty(L^\mu)} \vee \|v\|_{L^\infty(L^\mu)})^{\alpha-2}(\|u\|_{L^\infty(B_{\kappa_2}^{s-4})} \vee \|v\|_{L^\infty(B_{\kappa_2}^{s-4})}) \times \|u - v\|_{L^\infty(B_{\kappa_2}^{s-4})}\|w\|_{L^q(B_{r_1}^{s-4})}. \tag{2.15}$$

Proof. Since $1/r = \alpha/\mu + 1/r_1$, the estimate (2.11) follows from Lemma 2.2 (i). The inequality (2.12) is proved by Lemma 2.2 together with the mean value theorem. We write

$$f(u) - f(v) = \int_0^1 f'(u_\theta)(u - v)\, d\theta$$

with $u_\theta = \theta u + (1 - \theta)v$. Therefore, we have

$$\|f(u) - f(v)\|_{L^q(B_r^\sigma)} \leq \max_{0\leq\theta\leq 1} \|f'(u_\theta)(u - v)\|_{L^q(B_r^\sigma)}.$$

We set $1/\mu^* = 1/\mu + \sigma/N = 1/\kappa_1 - (s - \sigma - 2)/N$, so that we have the relation $1/r = (\alpha - 1)/\mu + 1/r_1 + 1/\mu^* - \sigma/N$. By assumption, we have $2N/(N + 2\sigma) \leq r_1, \mu^* < N/\sigma$.

From the Sobolev inequality, we have the inclusion $B^{s-2}_{\kappa_1}(\mathbb{R}^N) \subset B^{\sigma}_{\mu^*}(\mathbb{R}^N)$. Hence, it follows from Lemma 2.2 (ii) that

$$\begin{aligned}\|f'(u_\theta)(u-v)\|_{B^\sigma_r} &\lesssim \|u_\theta\|^{\alpha-1}_{L^\mu}\|u_\theta\|_{B^\sigma_{r_1}}\|u-v\|_{B^\sigma_{\mu^*}} \\ &\lesssim (\|u\|_{L^\mu} \vee \|v\|_{L^\mu})^{\alpha-1}(\|u\|_{B^\sigma_{r_1}} \vee \|v\|_{B^\sigma_{r_1}})\|u-v\|_{B^{s-2}_{\kappa_1}}. \end{aligned} \tag{2.16}$$

Taking the L^q-norm in t, we obtain (2.12). We shall next prove (2.13). Let $1/r_2 = 1/r - 4/N$. From the Sobolev inequality, $B^{s-2}_{r_1} \subset B^{s-4}_{r_2}$. We have the relation $1/r = (\alpha-1)/\mu + 1/r_2 + 1/\kappa_1 - (s-4)/N$. Hence, it follows from Lemma 2.2 (ii) that

$$\|f'(u)v\|_{B^{s-4}_r} \lesssim \|u\|^{\alpha-1}_{L^\mu}\|u\|_{B^{s-4}_{r_2}}\|v\|_{B^{s-4}_{\kappa_1}} \lesssim \|u\|^{\alpha-1}_{L^\mu}\|u\|_{B^{s-2}_{r_1}}\|v\|_{B^{s-4}_{\kappa_1}}.$$

Taking the L^q-norm in t, we obtain (2.13). The estimate (2.14) immediately follows from (2.13) and the mean value theorem. To prove (2.15), we write

$$\{f'(u) - f'(v)\}w = \int_0^1 f''(u_\theta)(u-v)w\,d\theta,$$

so that the left-hand side of (2.15) is bounded by $\max_{0\le\theta\le1}\|f''(u_\theta)(u-v)w\|_{L^q(B^{s-4}_r)}$. Here, u_θ is the same as above. Since $1/r = (\alpha-2)/\mu + 2/\kappa_2 + 1/r_1 - 2(s-4)/N$, it follows from Lemma 2.2 (iii) that

$$\begin{aligned}\|f''(u_\theta)(u-v)w\|_{B^{s-4}_r} &\lesssim \|u_\theta\|^{\alpha-2}_{L^\mu}\|u_\theta\|_{B^{s-4}_{\kappa_2}}\|u-v\|_{B^{s-4}_{\kappa_2}}\|w\|_{B^{s-4}_{r_1}} \\ &\lesssim (\|u\|_{L^\mu} \vee \|v\|_{L^\mu})^{\alpha-2}(\|u\|_{B^{s-4}_{\kappa_2}} \vee \|v\|_{B^{s-4}_{\kappa_2}})\|u-v\|_{B^{s-4}_{\kappa_2}}\|w\|_{B^{s-4}_{r_1}}.\end{aligned}$$

Taking the L^q-norm in t, we obtain (2.15). □

Lemma 2.5. *Let $2 < s < N/2$ and $\alpha = \alpha^*(s)$. Then the following estimates hold for all u, u_1, u_2 such that the norms in the right-hand sides are finite:*

(i) *If $0 < \sigma < \min\{\alpha+1; N/2\}$, then*

$$\|f(u)\|_{L^{\gamma'}(B^\sigma_{\rho'})} \lesssim \|u\|^\alpha_{L^\gamma(L^\nu)}\|u\|_{L^\gamma(B^\sigma_\rho)}, \tag{2.17}$$

$$\|f(u)\|_{L^2(B^\sigma_{\rho^*})} \lesssim (\|u\|_{L^\infty(L^\mu)}\|u\|_{L^\gamma(L^\nu)})^{\alpha/2}\|u\|_{L^\gamma(B^\sigma_{\rho_1})}. \tag{2.18}$$

(ii) *If $\alpha > \max\{1; s-2\}$, then*

$$\|f'(u)u_1\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \lesssim \|u\|^{\alpha-1}_{L^\gamma(L^\nu)}\|u\|_{L^\gamma(B^{s-2}_{\rho_1})}\|u_1\|_{L^\gamma(B^{s-2}_\rho)}, \tag{2.19}$$

$$\begin{aligned}\|f'(u)u_1\|_{L^2(B^{s-2}_{\rho^*})} &\lesssim (\|u\|_{L^\infty(L^\mu)}\|u\|_{L^\gamma(L^\nu)})^{(\alpha-1)/2} \\ &\quad\times(\|u\|_{L^\infty(B^{s-2}_{\kappa_1})}\|u\|_{L^\gamma(B^{s-2}_{\rho_1})})^{1/2}\|u_1\|_{L^\gamma(B^{s-2}_{\rho_1})}.\end{aligned} \tag{2.20}$$

(iii) *If* $s > 4$ *and* $\alpha > \max\{2; s-3\}$, *then*

$$\|f'(u)u_1\|_{L^{\gamma'}(B^{s-4}_{\rho'})} \lesssim \|u\|^{\alpha-1}_{L^\gamma(L^\nu)}\|u\|_{L^\gamma(B^{s-4}_{\rho_2})}\|u_1\|_{L^\gamma(B^{s-4}_{\rho})}, \tag{2.21}$$

$$\|f'(u)u_1\|_{L^2(B^{s-4}_{\rho^*})} \lesssim (\|u\|_{L^\infty(L^\mu)}\|u\|_{L^\gamma(L^\nu)})^{(\alpha-1)/2} \times (\|u\|_{L^\infty(B^{s-4}_{\kappa_2})}\|u\|_{L^\gamma(B^{s-4}_{\rho_2})})^{1/2}\|u_1\|_{L^\gamma(B^{s-4}_{\rho_1})}, \tag{2.22}$$

$$\|f'(u)u_1\|_{L^2(B^{s-4}_{\rho_1^*})} \lesssim (\|u\|_{L^\infty(L^\mu)}\|u\|_{L^\gamma(L^\nu)})^{(\alpha-1)/2} \times (\|u\|_{L^\infty(B^{s-4}_{\kappa_2})}\|u\|_{L^\gamma(B^{s-4}_{\rho_2})})^{1/2}\|u_1\|_{L^\gamma(B^{s-4}_{\rho_2})}, \tag{2.23}$$

$$\|f''(u)u_1u_2\|_{L^{\gamma'}(B^{s-4}_{\rho'})} \lesssim \|u\|^{\alpha-2}_{L^\gamma(L^\nu)}\|u\|_{L^\gamma(B^{s-4}_{\rho_2})}\|u_1\|_{L^\gamma(B^{s-4}_{\rho_1})}\|u_2\|_{L^\gamma(B^{s-4}_{\rho_1})}. \tag{2.24}$$

$$\|f''(u)u_1u_2\|_{L^2(B^{s-4}_{\rho^*})} \lesssim (\|u\|_{L^\infty(L^\mu)}\|u\|_{L^\gamma(L^\nu)})^{(\alpha-2)/2}(\|u\|_{L^\infty(B^{s-4}_{\kappa_2})}\|u\|_{L^\gamma(B^{s-4}_{\rho_2})})^{1/2} \times (\|u_1\|_{L^\infty(B^{s-4}_{\kappa_2})}\|u_1\|_{L^\gamma(B^{s-4}_{\rho_2})})^{1/2}\|u_2\|_{L^\gamma(B^{s-4}_{\rho_1})}. \tag{2.25}$$

Proof. The estimates (2.17)–(2.24) follow from Lemma 2.2 together with the Hölder inequality. For the proof, we note that the indices satisfy the relations $1/\rho' = \alpha/\nu + 1/\rho$ and $2/\rho^* = \alpha(1/\mu+1/\nu)+2/\rho_1$. For instance, we shall prove (2.20) and (2.24). We set $2/\beta_0 = 1/\mu+1/\nu$ and $2/\beta_1 = 1/\kappa_1 + 1/\rho_1$, so that $\|u\|^2_{L^{\beta_0}} \le \|u\|_{L^\mu}\|u\|_{L^\nu}$ and $\|u\|^2_{B^{s-2}_{\beta_1}} \lesssim \|u\|_{B^{s-2}_{\kappa_1}}\|u\|_{B^{s-2}_{\rho_1}}$. Since $1/\rho^* = (\alpha-1)/\beta_0 + 1/\beta_1 + 1/\rho_1 - (s-2)/N$, we obtain from Lemma 2.2 (ii)

$$\|f'(u)u_1\|_{B^{s-2}_{\rho^*}} \lesssim \|u\|^{\alpha-1}_{L^{\beta_0}}\|u\|_{B^{s-2}_{\beta_1}}\|u_1\|_{B^{s-2}_{\rho_1}} \lesssim (\|u\|_{L^\mu}\|u\|_{L^\nu})^{(\alpha-1)/2}(\|u\|_{B^{s-2}_{\kappa_1}}\|u\|_{B^{s-2}_{\rho_1}})^{1/2}\|u_1\|_{B^{s-2}_{\rho_1}}.$$

Taking L^2 norm in t and using $\gamma = \alpha+2$, we obtain (2.20). Similarly, since $1/\rho' = (\alpha-2)/\nu + 1/\rho_2 + 2/\rho_1 - 2(s-4)/N$, we obtain from Lemma 2.2 (iii)

$$\|f''(u)u_1u_2\|_{B^{s-4}_{\rho'}} \lesssim \|u\|^{\alpha-2}_{L^\nu}\|u\|_{B^{s-4}_{\rho_2}}\|u_1\|_{B^{s-4}_{\rho_1}}\|u_2\|_{B^{s-4}_{\rho_1}}.$$

Taking $L^{\gamma'}$ norm in t and using the relation $\gamma' = \gamma/(\alpha+1)$, we obtain (2.24). The other estimates can be proved analogously. □

Lemma 2.6. *Let* $2 < s < N/2$ *and* $\alpha = \alpha^*(s)$. *We define operators* F *and* F_1 *by* $F : u \mapsto f(u)$ *and* $F_1 : [u,v] \mapsto f'(u)v$ *respectively.*

(i) *Let* $\alpha > \max\{1; s-2\}$. *Then, the operator* F *is locally Lipschitz continuous from* $L^\infty(I; B^{s-2}_{\kappa_1}(\mathbb{R}^N))$ *into* $L^\infty(I; H^{s-2}(\mathbb{R}^N))$.

(ii) *Let* $s > 4$ *and* $\alpha > \max\{2; s-3\}$. *If* $\{u_k\}$ *is bounded in* $L^\infty(I; H^s(\mathbb{R}^N))$ *and* $u_k \to u$ *in* $L^\infty(I; B^{s-2}_{\kappa_1}(\mathbb{R}^N))$, *then* $f(u_k) \to f(u)$ *in* $L^\infty(I; H^{s-2}(\mathbb{R}^N))$; *especially, the operator* F *is continuous from* $L^\infty(I; H^s(\mathbb{R}^N))$ *into* $L^\infty(I; H^{s-2}(\mathbb{R}^N))$.

(iii) *Let* $s > 4$ *and* $\alpha > \max\{2; s-3\}$. *If* $[u_k, v_k] \to [u, v]$ *in* $L^\infty(I; B^{s-4}_{\kappa_2}(\mathbb{R}^N) \times B^{s-4}_{\kappa_1}(\mathbb{R}^N))$, *then* $f'(u_k)v_k \to f'(u)v$ *in* $L^\infty(I; H^{s-4}(\mathbb{R}^N))$; *especially, the operator* F_1 *is continuous from* $L^\infty(I; H^s(\mathbb{R}^N) \times H^{s-2}(\mathbb{R}^N))$ *into* $L^\infty(I; H^{s-4}(\mathbb{R}^N))$.

Proof. (i) For $u, v \in L^\infty(I; B^{s-2}_{\kappa_1})$, we obtain from Lemma 2.4 (ii) that

$$\|f(u) - f(v)\|_{L^\infty(H^{s-2})} \lesssim (\|u\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|v\|_{L^\infty(B^{s-2}_{\kappa_1})})^\alpha \|u-v\|_{L^\infty(B^{s-2}_{\kappa_1})}, \tag{2.26}$$

which means that F is locally Lipschitz continuous.

(ii) We put $M = \sup_k \|u_k\|_{L^\infty(H^s)}$. We choose $0 < \varepsilon < 1$ such that $s-2+\varepsilon < \alpha+1$. Then, it follows from Lemma 2.4 (i) together with the Sobolev inequality that $\|f(u)\|_{L^\infty(B^{s-2+\varepsilon}_{2N/(N+2\varepsilon)})} \lesssim \|u\|^{\alpha+1}_{L^\infty(H^s)} \le M^{\alpha+1}$. On the other hand, it follows from Lemma 2.4 (ii) that

$$\begin{aligned}\|f(u_k)-f(u)\|_{L^\infty(B^{s-4}_{\kappa_1})} &\lesssim (\|u_k\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|u\|_{L^\infty(B^{s-2}_{\kappa_1})})^\alpha \|u_k-u\|_{L^\infty(B^{s-2}_{\kappa_1})}\\ &\lesssim M^\alpha \|u_k-u\|_{L^\infty(B^{s-2}_{\kappa_1})} \to 0.\end{aligned}$$

Thus, by the interpolation relation

$$(B^{s-4}_{\kappa_1}(\mathbb{R}^N), B^{s-2+\varepsilon}_{2N/(N+2\varepsilon)}(\mathbb{R}^N))_{2/(\varepsilon+2),2} = H^{s-2}(\mathbb{R}^N) \tag{2.27}$$

(see [2, Theorem 6.4.5]), we obtain $\|f(u_k)-f(u)\|_{L^\infty(H^{s-2})} \to 0$, which implies the continuity of F.

(iii) We put $M_1 = \sup_k \|[u_k, v_k]\|_{L^\infty(B^{s-4}_{\kappa_2}\times B^{s-4}_{\kappa_1})}$. From Lemma 2.4 (iii)–(iv), we have

$$\begin{aligned}&\|f'(u_k)v_k - f'(u)v\|_{L^\infty(H^{s-4})}\\ &\lesssim \|f'(u_k)(v_k-v)\|_{L^\infty(H^{s-4})} + \|(f'(u_k)-f'(u))v\|_{L^\infty(H^{s-4})}\\ &\lesssim (\|u_k\|_{L^\infty(B^{s-4}_{\kappa_2})} \vee \|u\|_{L^\infty(B^{s-4}_{\kappa_2})})^{\alpha-1}\\ &\quad \times (\|u_k\|_{L^\infty(B^{s-4}_{\kappa_2})}\|v_k-v\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k-u\|_{L^\infty(B^{s-4}_{\kappa_2})}\|v\|_{L^\infty(B^{s-4}_{\kappa_1})})\\ &\lesssim M_1^\alpha(\|v_k-v\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k-u\|_{L^\infty(B^{s-4}_{\kappa_2})}) \to 0.\end{aligned}$$ □

Lemma 2.7. *Let $2 < s < N/2$ and $\alpha = \alpha^*(s)$. Let $\bar\gamma = \alpha+1$, $\bar\rho = 2N(\alpha+1)/(N+2\alpha s)$.*

(i) *Let $\alpha > \max\{1; s-2\}$. If $u \in \bigcap_{j=0}^{1} W^{j,\bar\gamma}(I; B^{s-2j}_{\bar\rho}(\mathbb{R}^N))$, then $f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$.*

(ii) *Let $s > 4$ and $\alpha > \max\{2; s-3\}$. If $u \in \bigcap_{j=0}^{2} W^{j,\bar\gamma}(I; B^{s-2j}_{\bar\rho}(\mathbb{R}^N))$, then $f(u) \in C^1(I; H^{s-4}(\mathbb{R}^N)) \cap C(I; B^{s-4}_{\kappa_1}(\mathbb{R}^N))$. Furthermore, if $u \in L^\infty(I; H^s(\mathbb{R}^N))$, then $f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$.*

Remark. We can easily check that $(\bar\gamma, \bar\rho)$ is an admissible pair.

Proof. (i) Let $1/\bar\nu = 1/\bar\rho - s/N$ and let $1/\bar\rho_j = 1/\bar\rho - 2j/N$, $j = 1, 2$ with $s-2j > 0$. Then the inclusion $B^s_{\bar\rho} \subset B^{s-2j}_{\bar\rho_j} \subset L^{\bar\nu}$ holds. We have the relation $1/2 = (\alpha-1)/\bar\nu + 1/\bar\rho_1 + 1/\bar\rho - (s-2)/N$. Applying Lemma 2.2 (ii), we see

$$\|\partial_t f(u)\|_{L^1(H^{s-2})} \lesssim \|u\|^{\alpha-1}_{L^{\bar\gamma}(L^{\bar\nu})}\|u\|_{L^{\bar\gamma}(B^{s-2}_{\bar\rho_1})}\|\dot u\|_{L^{\bar\gamma}(B^{s-2}_{\bar\rho})} \lesssim \|u\|^\alpha_{L^{\bar\gamma}(B^s_{\bar\rho})}\|\dot u\|_{L^{\bar\gamma}(B^{s-2}_{\bar\rho})}.$$

Similarly, we see $\|f(u)\|_{L^1(H^{s-2})} \lesssim \|u\|^\alpha_{L^{\bar\gamma}(B^s_{\bar\rho})}\|u\|_{L^{\bar\gamma}(B^{s-2}_{\bar\rho})}$. These estimates show that $f(u) \in W^{1,1}(I; H^{s-2}(\mathbb{R}^N))$, which implies $f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$.

(ii) From similar estimates with the index $(s-2)$ replaced by $(s-4)$, we obtain $f(u) \in W^{1,1}(I; H^{s-4}(\mathbb{R}^N))$. Furthermore, applying Lemma 2.2 (iii), we see

$$\begin{aligned}
\|\partial_t^2 f(u)\|_{L^1(H^{s-4})} &= \|f'(u)\ddot{u} + f''(u)\dot{u}\dot{u}\|_{L^1(H^{s-4})} \\
&\lesssim \|u\|_{L^{\bar{\gamma}}(L^{\bar{\nu}})}^{\alpha-1} \|u\|_{L^{\bar{\gamma}}(B^{s-4}_{\bar{\rho}_2})} \|\ddot{u}\|_{L^{\bar{\gamma}}(B^{s-4}_{\bar{\rho}})} + \|u\|_{L^{\bar{\gamma}}(L^{\bar{\nu}})}^{\alpha-2} \|u\|_{L^{\bar{\gamma}}(B^{s-4}_{\bar{\rho}_2})} \|\dot{u}\|^2_{L^{\bar{\gamma}}(B^{s-4}_{\bar{\rho}_1})} \\
&\lesssim \|u\|_{L^{\bar{\gamma}}(B^{s}_{\bar{\rho}})}^{\alpha-1} (\|u\|_{L^{\bar{\gamma}}(B^{s}_{\bar{\rho}})} \|\ddot{u}\|_{L^{\bar{\gamma}}(B^{s-4}_{\bar{\rho}})} + \|\dot{u}\|^2_{L^{\bar{\gamma}}(B^{s-2}_{\bar{\rho}})}).
\end{aligned}$$

This estimate shows that $f(u) \in W^{2,1}(I; H^{s-4}(\mathbb{R}^N))$, which implies $f(u) \in C^1(I; H^{s-4}(\mathbb{R}^N))$. On the other hand, since $\dot{B}^{s-4}_{\kappa_1}(\mathbb{R}^N)$ has the same scale as $\dot{H}^{s-2}(\mathbb{R}^N)$, we see as before that $f(u) \in W^{1,1}(I; B^{s-4}_{\kappa_1}(\mathbb{R}^N))$, although $f(u)$ might not be differentiable $(s-2)$-times. Hence we obtain $f(u) \in C(I; B^{s-4}_{\kappa_1}(\mathbb{R}^N))$. To prove the last assertion, let $u \in L^\infty(I; H^s(\mathbb{R}^N))$. Then, from Lemma 2.4 (i), we obtain $f(u) \in L^\infty(I; B^{s-2+\varepsilon}_{2N/(N+2\varepsilon)}(\mathbb{R}^N))$. Therefore, as in the proof of Lemma 2.6, by the interpolation relation (2.27), we obtain $f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$. □

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. Let $\Phi(u)$ be defined by (1.2). We look for solutions to (1.2) by finding fixed points of Φ in appropriate metric spaces. Since the nonlinear term has limited regularity, as explained in §1, we replace spatial derivatives by temporal ones to reduce the total number of derivatives. Taking the time derivative of (1.2) and integrating by parts (see [17]), we obtain

$$\partial_t \Phi(u)(t) = U(t)\dot{\varphi} - i\int_0^t U(t-\tau)\partial_\tau f(u(\tau))\,d\tau, \tag{3.1}$$

where $\dot{\varphi} = i(\Delta\varphi - f(\varphi))$ is the initial data for $\partial_t \Phi(u)$. Furthermore, if $s-3 < \alpha^*(s) \leq s-2$ with $s > 4$, then we also need the second derivative $\partial_t^2 \Phi(u)$, which satisfies

$$\partial_t^2 \Phi(u)(t) = U(t)\ddot{\varphi} - i\int_0^t U(t-\tau)\partial_\tau^2 f(u(\tau))\,d\tau \tag{3.2}$$

with $\ddot{\varphi} = i(\Delta\dot{\varphi} - f'(\varphi)\dot{\varphi})$. For each $j \geq 0$, the corresponding differential equation for $\partial_t^j \Phi(u)$ is

$$(i\partial_t + \Delta)\partial_t^j \Phi(u) = \partial_t^j f(u). \tag{3.3}$$

It follows from Lemma 2.6 that $\dot{\varphi} \in H^{s-2}(\mathbb{R}^N)$, and $\ddot{\varphi} \in H^{s-4}(\mathbb{R}^N)$. Furthermore, if $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^N)$, then $\dot{\varphi}_k \equiv i(\Delta\varphi_k - f(\varphi_k)) \to \dot{\varphi}$ in $H^{s-2}(\mathbb{R}^N)$, and $\ddot{\varphi}_k \equiv i(\Delta\dot{\varphi}_k - f'(\varphi_k)\dot{\varphi}_k) \to \ddot{\varphi}$ in $H^{s-4}(\mathbb{R}^N)$.

Let $I = [0, T]$ for some $T > 0$. We set $Z(I) = L^\gamma(I; L^\rho(\mathbb{R}^N)) \cap L^2(I; L^{\rho^*}(\mathbb{R}^N))$, and

$$X_j(I) = L^\gamma(I; B^{s-2j}_\rho(\mathbb{R}^N)) \cap L^2(I; B^{s-2j}_{\rho^*}(\mathbb{R}^N))$$

for $j = 0, 1, 2$ with $s - 2j > 0$. Here, the admissible pair (γ, ρ) and the index ρ^* are defined by (2.1). We set

$$X(I) = \{u \in X_0(I) : \partial_t u \in X_1(I)\},$$

with $\|u\|_X \equiv \|u\|_{X_0} \vee \|\dot{u}\|_{X_1}$. If $\max\{2; s-3\} < \alpha^*(s) \le s-2$ with $s > 4$, we also need the spaces

$$Y_j(I) = L^\gamma(I; B^{s-2j-2}_{\rho_1}(\mathbb{R}^N)) \cap L^2(I; B^{s-2j-2}_{\rho_1^*}(\mathbb{R}^N)),$$

$j = 0, 1$, where the indices ρ_1 and ρ_1^* are defined by (2.3). We have the inclusion $X_j(I) \subset Y_j(I)$. We set

$$Y(I) = \{u \in Y_0(I) \cap X_1(I) : \partial_t^j u \in X_j(I), j = 1, 2\}$$

with $\|u\|_Y \equiv \|u\|_{Y_0} \vee \|u\|_{X_1} \vee \|\dot{u}\|_{X_1} \vee \|\ddot{u}\|_{X_2}$.

We begin with the case $\alpha^*(s) > s-2$. We shall first show the unique existence of solutions in Proposition 3.1. The proof of the continuous dependence of solutions on data is proved in Proposition 3.2.

Proposition 3.1. *Let $N \ge 8$, $2 < s < N/2$ and let $\alpha = \alpha^*(s)$ satisfy $\max\{1; s-2\} < \alpha \le s-1$. Then, for any $\varphi \in H^s(\mathbb{R}^N)$, there exists $T > 0$ such that (1.2) has a unique solution $u \in C(I; H^s(\mathbb{R}^N))$, where $I = [0, T]$. Furthermore, $u \in L^q(I; B^s_r(\mathbb{R}^N))$ for any admissible pair (q, r).*

Proof. For $R > 0$, we define the metric space

$$B_R = \{u \in X(I) : \ u(0) = \varphi, \ \|u\|_X \le R\}$$

with metric $d(u_1, u_2) = \|u_1 - u_2\|_Z$. We note that (B_R, d) is a complete metric space. For suitable T and R to be specified later, we shall prove that the mapping Φ is a contraction mapping on B_R. We set

$$R_0 = R_0(\varphi; T) \equiv \|U(\cdot)\varphi\|_X \vee \|U(\cdot)\dot{\varphi}\|_{X_1} \vee \|f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-2}_\rho)}. \tag{3.4}$$

From Lemma 2.1, we see $U(\cdot)\varphi \in X(I)$, $U(\cdot)\dot{\varphi} \in X_1(I)$. It follows from Lemma 2.4 (i) together with the unitarity of $U(t)$ in $H^s(\mathbb{R}^N)$ that $\|f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-2}_\rho)} \lesssim \|\varphi\|^\alpha_{H^s}\|U(\cdot)\varphi\|_{L^\gamma(B^s_\rho)}$. We have $\lim_{T\to 0} R_0(\varphi; T) = 0$ by the definition of $X(I)$. For $u \in B_R$, we set $v(t) = u(t) - U(t)\varphi$. Since $v(0) = 0$, it follows from Lemma 2.3 that

$$\|v\|^2_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|v\|_{L^2(B^{s-2}_{\rho_1^*})}\|\dot{v}\|_{L^2(B^{s-2}_{\rho^*})} \lesssim \|v\|^2_X. \tag{3.5}$$

From (3.5) together with the inequalities $\|u-v\|_X \lesssim \|\varphi\|_{H^s} \wedge R_0$ and $\|u-v\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|\varphi\|_{H^s}$, we obtain

$$\|v\|_X \lesssim \|u\|_X \vee R_0, \qquad \|u\|_{L^\infty(L^\mu)} \lesssim \|u\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|u\|_X \vee \|\varphi\|_{H^s}. \tag{3.6}$$

We shall show that Φ maps B_R into itself. We apply Lemma 2.1 to (1.2) and (3.1) together with Lemma 2.5 to obtain

$$\|\Phi(u)\|_{X_1} \lesssim \|U(\cdot)\varphi\|_{X_1} + \|f(u)\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \lesssim R_0 + \|u\|^\alpha_{X_0}\|u\|_{X_1} \lesssim R_0 + R^{\alpha+1}, \tag{3.7}$$

$$\|\partial_t\Phi(u)\|_{X_1} \lesssim \|U(\cdot)\dot{\varphi}\|_{X_1} + \|f'(u)\dot{u}\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \lesssim R_0 + \|u\|^\alpha_{X_0}\|\dot{u}\|_{X_1} \lesssim R_0 + R^{\alpha+1}, \tag{3.8}$$

especially, we have used (2.17) for (3.7), and (2.19) for (3.8) respectively. Lemma 2.1 also shows that $\Phi(u) \in C^1(I; H^{s-2})$, with the estimate

$$\max_{j=0,1} \|\partial_t^j \Phi(u)\|_{L^\infty(H^{s-2})} \lesssim \|\varphi\|_{H^s} \vee \|\dot{\varphi}\|_{H^{s-2}} + R^{\alpha+1}. \tag{3.9}$$

For the estimate of $\Phi(u)$ in $X(I)$, we use the equation (3.3) with $j = 0$. Then, we obtain

$$\|\Phi(u)\|_{X_0} \sim \|(1-\Delta)\Phi(u)\|_{X_1} \lesssim \|\Phi(u)\|_{X_1} + \|\partial_t \Phi(u)\|_{X_1} + \|f(u)\|_{X_1}. \tag{3.10}$$

From the estimates (3.7) and (3.8), the first two terms of the right-hand side are bounded by $CR_0 + CR^{\alpha+1}$, so it suffices to consider the third term. In $L^2(I; B^{s-2}_{\rho^*}(\mathbb{R}^N))$, we can treat $f(u)$ directly. It follows from (2.18) that

$$\|f(u)\|_{L^2(B^{s-2}_{\rho^*})} \lesssim \|u\|^{\alpha/2}_{L^\infty(B^{s-2}_{\kappa_1})} \|u\|^{\alpha/2+1}_{L^\gamma(B^s_\rho)} \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha/2} R^{\alpha/2+1}. \tag{3.11}$$

On the other hand, in $L^\gamma(I; B^{s-2}_\rho(\mathbb{R}^N))$, similar estimate would not work. (See the remark below.) Instead, taking account of the inequality $\|f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-2}_\rho)} \le R_0$, we estimate the difference $f(u) - f(U(\cdot)\varphi)$. It follows from Lemma 2.4 (ii) and (3.5) that

$$\begin{aligned}
&\|f(u) - f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-2}_\rho)} \\
&\quad\lesssim (\|u\|_{L^\infty(L^\mu)} \vee \|U(\cdot)\varphi\|_{L^\infty(L^\mu)})^{\alpha-1} (\|u\|_{L^\gamma(B^s_\rho)} \vee \|U(\cdot)\varphi\|_{L^\gamma(B^s_\rho)}) \|v\|_{L^\infty(B^{s-2}_{\kappa_1})} \\
&\quad\lesssim (\|u\|_X \vee \|\varphi\|_{H^s})^{\alpha-1} (\|u\|_X \vee \|U(\cdot)\varphi\|_X) \|v\|_X \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha-1} (R \vee R_0)^2.
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\|f(u)\|_{X_1} \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha/2} R^{\alpha/2+1} + (R \vee \|\varphi\|_{H^s})^{\alpha-1} (R \vee R_0)^2 + R_0. \tag{3.13}$$

Collecting the estimates (3.7)–(3.10) and (3.13), we obtain

$$\|\Phi(u)\|_X \le CR_0 + C(R \vee \|\varphi\|_{H^s})^{\alpha/2} R^{\alpha/2+1} + C(R \vee \|\varphi\|_{H^s})^{\alpha-1} (R \vee R_0)^2$$

for some constant $C \ge 1$. Similarly, for $u_1, u_2 \in B_R$, we can easily show

$$\|\Phi(u_1) - \Phi(u_2)\|_Z \le CR^\alpha \|u_1 - u_2\|_Z.$$

Now, we choose R and T so small that

$$3CR_0(\varphi; T) < R < \min\left\{ \frac{1}{(3C)^{2/\alpha} \|\varphi\|_{H^s}}; \frac{1}{3C\|\varphi\|^{\alpha-1}_{H^s}}; (3C)^{-1/\alpha} \right\}. \tag{3.14}$$

Then, we can obtain $\|\Phi(u)\|_X < R$, so that Φ maps B_R into itself. We also obtain that Φ is a contraction mapping on B_R. From the contraction mapping principle, Φ has a unique fixed point in B_R, which gives a solution $u \in X(I)$ to (1.2). Furthermore, as we have mentioned, $u = \Phi(u) \in C^1(I; H^{s-2}(\mathbb{R}^N))$. On the other hand, since $X(I) \subset \bigcap_{j=0}^1 W^{j,\tilde{\gamma}}(I; B^{s-2j}_{\tilde{\rho}}(\mathbb{R}^N))$, it follows from Lemma 2.7 (i) that $f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$. Hence, we see $-\Delta u = i\partial_t u - f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$, so that $u \in C(I; H^s(\mathbb{R}^N))$. The uniqueness of solutions in $C(I; H^s(\mathbb{R}^N))$ has been proved in [3, Proposition 4.2.13]. Finally, we can easily check that $u \in L^q(I; B^s_r(\mathbb{R}^N))$ for any admissible pair (q, r), since $L^\infty(I; H^s(\mathbb{R}^N)) \cap L^2(I; B^s_{\rho^*}(\mathbb{R}^N)) \subset L^q(I; B^s_r(\mathbb{R}^N))$. □

Remark. A direct application of Lemma 2.2 gives $\|f(u)\|_{L^\gamma(B^{s-2}_\rho)} \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha-1} R$, which would not suffice to show that Φ should be a contraction mapping. This is why we estimate the difference $\|f(u) - f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-2}_\rho)}$ instead. On the other hand, we cannot estimate $\|f(u) - f(U(\cdot)\varphi)\|_{L^2(B^{s-2}_{\rho^*})}$ by Lemma 2.4, since the assumption $\rho^* < N/s$ for the lemma might not be satisfied. This is why we estimate $\|f(u)\|_{L^2(B^{s-2}_{\rho^*})}$ and $\|f(u)\|_{L^\gamma(B^{s-2}_\rho)}$ differently.

Proposition 3.2. *Let $N \geq 8$, $2 < s < N/2$ and let $\alpha = \alpha^*(s)$ satisfy $\max\{1; s-2\} < \alpha \leq s-1$. Let $\varphi \in H^s(\mathbb{R}^N)$ and let $\{\varphi_k\}_{k=1}^\infty \subset H^s(\mathbb{R}^N)$ satisfy $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^N)$. Then there exists $T > 0$ such that (1.2) has a unique solution $u \in C(I; H^s(\mathbb{R}^N)) \cap X(I)$ with $I = [0, T]$, and that (1.2) with φ replaced by φ_k has a unique solution $u_k \in C(I; H^s(\mathbb{R}^N)) \cap X(I)$ for sufficiently large k. Furthermore, $u_k \to u$ in $C(I; H^s(\mathbb{R}^N)) \cap X(I)$.*

Proof. Step 1. Let R and T satisfy (3.14) in the proof of Proposition 3.1. We note that we can take R arbitrarily small, if we choose T smaller so as to satisfy the first inequality of (3.14). By Proposition 3.1, the equation (1.2) has a unique solution $u \in C(I; H^s(\mathbb{R}^N)) \cap X(I)$. From Lemmas 2.1 and 2.4, we see $R_0(\varphi_k; T) \to R_0(\varphi; T)$ as $k \to \infty$, where R_0 is defined by (3.4). Especially, we use (2.12) to show that $\|f(U(\cdot)\varphi_k) - f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-2}_\rho)} \to 0$. Therefore, for sufficiently large k, the mapping Φ with φ replaced by φ_k is still a contraction on B_R, so that (1.2) with φ replaced by φ_k has a unique solution $u_k \in C(I; H^s(\mathbb{R}^N)) \cap X(I)$. We have $\|u_k\|_X \leq R$ and $\|u_k\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim R \vee \|\varphi\|_{H^s}$.

Step 2. From the equations for u and u_k, we have

$$\|u_k - u\|_X \lesssim \|u_k - u\|_{X_1} + \|\dot{u}_k - \dot{u}\|_{X_1} + \|f(u_k) - f(u)\|_{X_1}, \tag{3.15}$$

$$\|u_k - u\|_{L^\infty(H^s)} \lesssim \|u_k - u\|_{L^\infty(H^{s-2})} + \|\dot{u}_k - \dot{u}\|_{L^\infty(H^{s-2})} + \|f(u_k) - f(u)\|_{L^\infty(H^{s-2})}. \tag{3.16}$$

From Lemma 2.4 (ii), and (2.20) together with the mean value theorem, we obtain

$$\begin{aligned} &\|f(u_k) - f(u)\|_{L^\gamma(B^{s-2}_\rho)} \\ &\quad \lesssim (\|u_k\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|u\|_{L^\infty(B^{s-2}_{\kappa_1})})^{\alpha-1} (\|u_k\|_X \vee \|u\|_X) \|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\|f(u_k) - f(u)\|_{L^2(B^{s-2}_{\rho^*})} \\ &\quad \lesssim (\|u_k\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|u\|_{L^\infty(B^{s-2}_{\kappa_1})})^{\alpha/2} (\|u_k\|_X \vee \|u\|_X)^{\alpha/2} \|u_k - u\|_{L^\gamma(B^{s-2}_{\rho_1})}. \end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\|f(u_k) - f(u)\|_{X_1} \lesssim R \|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} + R^{\alpha/2} \|u_k - u\|_{L^\gamma(B^{s-2}_{\rho_1})}. \tag{3.19}$$

As in the proof of Proposition 3.1, we set $v(t) = u(t) - U(t)\varphi$, and analogously $v_k(t) = u_k(t) - U(t)\varphi_k$, so that $v(0) = v_k(0) = 0$. Like (3.5), we have $\|v_k - v\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|v_k - v\|_X$, so that

$$\|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|u_k - u\|_X + \|\varphi_k - \varphi\|_{H^s}. \tag{3.20}$$

It follows from (3.15), (3.19) and (3.20) that

$$\|u_k - u\|_X \lesssim \|u_k - u\|_{X_1} + \|\dot{u}_k - \dot{u}\|_{X_1} + (R \vee R^{\alpha/2}) \|u_k - u\|_X + \|\varphi_k - \varphi\|_{H^s}.$$

If $R > 0$ is small enough, the third term in the right-hand side is absorbed into the left-hand side, so that we obtain

$$\|u_k - u\|_X \lesssim \|u_k - u\|_{X_1} + \|\dot{u}_k - \dot{u}\|_{X_1} + \|\varphi_k - \varphi\|_{H^s}. \tag{3.21}$$

Step 3. We shall show

$$\lim_{k\to\infty} \|u_k - u\|_{L^\infty(H^{s-2})\cap X_1} = 0. \tag{3.22}$$

From Lemma 2.1 and the estimate (2.19) together with the mean value theorem, we have

$$\begin{aligned}\|u_k - u\|_{L^\infty(H^{s-2})\cap X_1} &\lesssim \|\varphi_k - \varphi\|_{H^{s-2}} + \|f(u_k) - f(u)\|_{L^{\gamma'}(B^{s-2}_{\rho'})}\\ &\lesssim \|\varphi_k - \varphi\|_{H^{s-2}} + (\|u_k\|_X \vee \|u\|_X)^\alpha \|u_k - u\|_{X_1}\\ &\lesssim \|\varphi_k - \varphi\|_{H^{s-2}} + R^\alpha \|u_k - u\|_{X_1}.\end{aligned}$$

If R is small enough, the second term in the right-hand side is absorbed into the left-hand side. Since $\|\varphi_k - \varphi\|_{H^{s-2}} \to 0$, we obtain (3.22).

Step 4. We shall next show

$$\lim_{k\to\infty} \|\dot{u}_k - \dot{u}\|_{L^\infty(H^{s-2})\cap X_1} = 0. \tag{3.23}$$

Again by Lemma 2.1,

$$\begin{aligned}\|\dot{u}_k - \dot{u}\|_{L^\infty(H^{s-2})\cap X_1} \lesssim{}& \|\dot{\varphi}_k - \dot{\varphi}\|_{H^{s-2}} + \|f'(u_k)(\dot{u}_k - \dot{u})\|_{L^{\gamma'}(B^{s-2}_{\rho'})}\\ &+ \|(f'(u_k) - f'(u))\dot{u}\|_{L^{\gamma'}(B^{s-2}_{\rho'})}.\end{aligned} \tag{3.24}$$

From (2.19), the second term in the right-hand side is bounded by $R^\alpha \|\dot{u}_k - \dot{u}\|_{X_1}$, and hence it is absorbed into the left-hand side provided that R is small enough. Therefore, to prove (3.23), it suffices to show that the third term goes to zero. To this end, let $\chi_l(\xi) = \chi(\xi/2^l)$, where χ is defined at the beginning of §2. We decompose u as

$$u = \chi_l(D)u + (1 - \chi_l(D))u \equiv u^L + u^H. \tag{3.25}$$

The supports of $\mathscr{F}u^L$ and $\mathscr{F}u^H$ are respectively contained in the region $|\xi| \lesssim 2^l$ and $|\xi| \gtrsim 2^l$. Therefore, $(1-\Delta)u^L \in X(I)$ for arbitrary l. For a while, we arbitrarily fix l. For the estimate of the low frequency part, we take $\varepsilon > 0$ satisfying $s - 2 + \varepsilon < \min\{s; \alpha\}$. Then, from a slight modification of the estimate (2.19) together with the mean value theorem, we see

$$\|(f'(u_k) - f'(u))\dot{u}^L\|_{L^{\gamma'}(B^{s-2+\varepsilon}_{\rho'})} \lesssim (\|u_k\|_X \vee \|u\|_X)^\alpha \|\dot{u}^L\|_{L^\gamma(B^{s-2+\varepsilon}_{\rho})} \lesssim R^\alpha \|(1-\Delta)\dot{u}^L\|_{X_1}.$$

Furthermore, by the Hölder inequality, we obtain

$$\|(f'(u_k) - f'(u))\dot{u}^L\|_{L^{\gamma'}(L^{\rho'})} \lesssim (\|u_k\|_X \vee \|u\|_X)^{\alpha-1} \|u_k - u\|_{L^\gamma(L^\rho)} \|\dot{u}^L\|_{X_1} \to 0$$

as $k \to \infty$. Hence, by interpolation, we have $\|(f'(u_k) - f'(u))\dot{u}^L\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \to 0$ for any l. On the other hand, it follows from (2.19) that

$$\|(f'(u_k) - f'(u))\dot{u}^H\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \lesssim (\|u_k\|_X \vee \|u\|_X)^\alpha \|\dot{u}^H\|_{L^\gamma(B^{s-2}_{\rho})} \lesssim R^\alpha \|\dot{u}^H\|_{X_1},$$

and the right-hand side is independent of k. Thus, we obtain

$$\overline{\lim_{k\to\infty}} \|(f'(u_k) - f'(u))\dot{u}\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \lesssim R^{\alpha}\|\dot{u}^H\|_{X_1}. \tag{3.26}$$

Furthermore, by the Lebesgue convergence theorem, we see that $\|u^H\|_X \to 0$ as $l \to \infty$. Therefore, the left-hand side of (3.26) must be zero.

Step 5. From (3.21)–(3.23), we obtain $\lim_{k\to\infty}\|u_k - u\|_X = 0$. Once this is proved, from (3.20) we obtain $\|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} \to 0$, which implies $\|f(u_k) - f(u)\|_{L^\infty(H^{s-2})} \to 0$ by Lemma 2.6 (i). Therefore, from (3.16), we obtain $\|u_k - u\|_{L^\infty(H^s)} \to 0$. □

We shall proceed to the case $\max\{2; s-3\} < \alpha^*(s) \le s-2$, which occurs only for $N \ge 11$. As in the previous case, we shall separately prove the unique existence (Proposition 3.3), and the continuous dependence on data (Proposition 3.4).

Proposition 3.3. *Let $N \ge 11$, $4 < s < N/2$ and let $\alpha = \alpha^*(s)$ satisfy* $\max\{2; s-3\} <$ $\alpha \le s-2$*. For any $\varphi \in H^s(\mathbb{R}^N)$, there exists $T > 0$ such that* (1.2) *has a unique solution $u \in C(I; H^s(\mathbb{R}^N))$, where $I = [0,T]$. Furthermore, $u \in L^q(I; B^s_r(\mathbb{R}^N))$ for any admissible pair (q,r).*

Proof. For $R > 0$, we define the metric space

$$\tilde{B}_R = \{u \in Y(I) : u(0) = \varphi, \dot{u}(0) = \dot{\varphi}, \|u\|_Y \le R\}$$

with metric $d(u_1,u_2) = \|u_1 - u_2\|_Z$. We note that $(\tilde{B}_R, d)$ is again a complete metric space. For suitable T and R to be specified later, we shall prove that the mapping Φ is a contraction mapping on $\tilde{B}_R$. We set

$$\begin{aligned} R_1 = R_1(\varphi;T) \equiv\, & \|U(\cdot)\varphi\|_X \vee \|U(\cdot)\dot{\varphi}\|_{X_1} \vee \|U(\cdot)\ddot{\varphi}\|_{X_2} \\ & \vee \|f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-4}_{\rho_1})} \vee \|f'(U(\cdot)\varphi)U(\cdot)\dot{\varphi}\|_{L^\gamma(B^{s-4}_{\rho})}. \end{aligned} \tag{3.27}$$

Then, we have $\lim_{T\to 0} R_1(\varphi;T) = 0$. For $u \in \tilde{B}_R$, we set $v(t) = u(t) - U(t)\varphi$, and $w(t) = u(t) - U(t)\psi$ with $\psi = \varphi + (-\Delta)^{-1}f(\varphi)$. Then $\psi \in H^s$, $\dot{w}(t) = \dot{u}(t) - U(t)\dot{\varphi}$ and $v(0) = \dot{w}(0) = 0$. Like (3.5), we see $\|v\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|v\|_Y$ and $\|\dot{w}\|_{L^\infty(B^{s-4}_{\kappa_1})} \lesssim (\|\dot{w}\|_{X_1}\|\ddot{w}\|_{X_2})^{1/2}$. It follows from these estimates together with Lemma 2.1 that

$$\|v\|_Y \lesssim \|u\|_Y \vee R_1, \qquad \max_{j=1,2}\|\partial_t^j w\|_{X_j} \lesssim \max_{j=1,2}\|\partial_t^j u\|_{X_j} \vee R_1, \tag{3.28}$$

$$\|u\|_{L^\infty(L^\mu)} \lesssim \|u\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|u\|_Y \vee \|\varphi\|_{H^s}. \tag{3.29}$$

Like (3.7) and (3.8), it follows from Lemmas 2.1 and 2.5 that

$$\|\Phi(u)\|_{X_1} \lesssim \|U(\cdot)\varphi\|_{X_1} + \|f(u)\|_{L^{\gamma'}(B^{s-2}_{\rho'})} \lesssim R_1 + \|u\|^{\alpha}_{Y_0}\|u\|_{X_1} \lesssim R_1 + R^{\alpha+1}, \tag{3.30}$$

$$\begin{aligned}\|\partial_t\Phi(u)\|_{X_2} &\lesssim \|U(\cdot)\dot\varphi\|_{X_2} + \|f'(u)\dot u\|_{L^{\gamma'}(B^{s-4}_{\rho'})} \\ &\lesssim R_1 + \|u\|^{\alpha}_{Y_0}\|\dot u\|_{X_2} \lesssim R_1 + R^{\alpha+1},\end{aligned} \tag{3.31}$$

$$\begin{aligned}\|\partial_t^2\Phi(u)\|_{X_2} &\lesssim \|U(\cdot)\ddot\varphi\|_{X_2} + \|f'(u)\ddot u + f''(u)\dot u\dot u\|_{L^{\gamma'}(B^{s-4}_{\rho'})} \\ &\lesssim R_1 + \|u\|^{\alpha}_{Y_0}\|\ddot u\|_{X_2} + \|u\|^{\alpha-1}_{Y_0}\|\dot u\|^2_{X_1} \lesssim R_1 + R^{\alpha+1}.\end{aligned} \tag{3.32}$$

Especially, we have used (2.17) for (3.30), (2.21) for (3.31), and (2.21) together with (2.24) for (3.32). Lemma 2.1 also shows that $\Phi(u) \in C(I; H^{s-2}) \cap C^2(I; H^{s-4})$, with the estimate

$$\|\Phi(u)\|_{L^\infty(H^{s-2})} \vee \max_{j=1,2}\|\partial_t^j\Phi(u)\|_{L^\infty(H^{s-4})} \lesssim \|\varphi\|_{H^{s-2}} \vee \|\dot\varphi\|_{H^{s-4}} \vee \|\ddot\varphi\|_{H^{s-4}} + R^{\alpha+1}. \tag{3.33}$$

On the other hand, for the estimates of $\Phi(u)$ in Y_0 and $\partial_t\Phi(u)$ in X_1, we use the equation (3.3) with $j = 0, 1$. We see

$$\|\Phi(u)\|_{Y_0} \sim \|(1-\Delta)\Phi(u)\|_{Y_1} \lesssim \|\Phi(u)\|_{X_1} + \|\partial_t\Phi(u)\|_{X_1} + \|f(u)\|_{Y_1}, \tag{3.34}$$

$$\begin{aligned}\|\partial_t\Phi(u)\|_{X_1} &\sim \|(1-\Delta)\partial_t\Phi(u)\|_{X_2} \\ &\lesssim \|\partial_t\Phi(u)\|_{X_2} + \|\partial_t^2\Phi(u)\|_{X_2} + \|f'(u)\dot u\|_{X_2}.\end{aligned} \tag{3.35}$$

Here, we have used $X_1 \subset Y_1$. Hence, we need to estimate $\|f(u)\|_{Y_1}$ and $\|f'(u)\dot u\|_{X_2}$. As in the proof of Proposition 3.1, we estimate $\|f(u)\|_{L^2(B^{s-4}_{\rho_1^*})}$ and $\|f(u)\|_{L^\gamma(B^{s-4}_{\rho_1})}$ separately. In $L^2(I; B^{s-4}_{\rho_1^*}(\mathbb{R}^N))$, we directly estimate $f(u)$ as before. Then, like (3.11), we have

$$\|f(u)\|_{L^2(B^{s-4}_{\rho_1^*})} \lesssim (\|u\|_Y \vee \|\varphi\|_{H^s})^{\alpha/2}\|u\|^{\alpha/2+1}_{Y_0} \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha/2}R^{\alpha/2+1}.$$

On the other hand, taking account of the inequality $\|f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-4}_{\rho_1})} \le R_1$, we estimate the difference $f(u) - f(U(\cdot)\varphi)$ in $L^\gamma(I; B^{s-4}_{\rho_1}(\mathbb{R}^N))$. It follows from Lemma 2.4 (ii) that

$$\begin{aligned}&\|f(u) - f(U(\cdot)\varphi)\|_{L^\gamma(B^{s-4}_{\rho_1})} \\ &\lesssim (\|u\|_{L^\infty(L^\mu)} \vee \|U(\cdot)\varphi\|_{L^\infty(L^\mu)})^{\alpha-1}(\|u\|_{L^\gamma(B^{s-4}_{\rho_2})} \vee \|U(\cdot)\varphi\|_{L^\gamma(B^{s-4}_{\rho_2})})\|v\|_{L^\infty(B^{s-2}_{\kappa_1})} \\ &\lesssim (\|u\|_Y \vee \|\varphi\|_{H^s})^{\alpha-1}(\|u\|_{Y_0} \vee \|U(\cdot)\varphi\|_{X_0})\|v\|_Y \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha-1}(R \vee R_1)^2.\end{aligned}$$

Combining these estimates, we obtain

$$\|f(u)\|_{Y_1} \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha-1}(R \vee R_1)^2 + R_1. \tag{3.36}$$

We next estimate $\|f'(u)\dot u\|_{X_2}$. We use (2.22) to obtain

$$\|f'(u)\dot u\|_{L^2(B^{s-4}_{\rho^*})} \lesssim (\|u\|_Y \vee \|\varphi\|_{H^s})^{\alpha/2}\|u\|^{\alpha/2}_{Y_0}\|\dot u\|_{Y_1} \lesssim (R \vee \|\varphi\|_{H^s})^{\alpha/2}R^{\alpha/2+1}.$$

On the other hand, for the estimate in $L^\gamma(I; B^{s-4}_\rho(\mathbb{R}^N))$, we write

$$f'(u)\dot u = f'(u)\dot w + \{f'(u) - f'(U(\cdot)\varphi)\}U(\cdot)\dot\varphi + f'(U(\cdot)\varphi)U(\cdot)\dot\varphi.$$

The last term is bounded by R_1 in $L^\gamma(I; B^{s-4}_\rho(\mathbb{R}^N))$, so it suffices to estimate the first two terms. From Lemma 2.4 (iii)–(iv), we see

$$\begin{aligned}
&\|f'(u)\dot{w}\|_{L^\gamma(B^{s-4}_\rho)} \lesssim \|u\|^{\alpha-1}_{L^\infty(L^\mu)}\|u\|_{L^\gamma(B^{s-2}_{\rho_1})}\|\dot{w}\|_{L^\infty(B^{s-4}_{\kappa_1})}\\
&\quad\lesssim (\|u\|_Y \vee \|\varphi\|_{H^s})^{\alpha-1}\|u\|_{Y_0}(\|\dot{w}\|_{X_1}\|\ddot{w}\|_{X_2})^{1/2} \lesssim (R\vee\|\varphi\|_{H^s})^{\alpha-1}R(R\vee R_1),\\
&\|\{f'(u) - f'(U(\cdot)\varphi)\}U(\cdot)\dot{\varphi}\|_{L^\gamma(B^{s-4}_\rho)}\\
&\quad\lesssim (\|u\|_{L^\infty(B^{s-4}_{\kappa_2})} \vee \|U(\cdot)\varphi\|_{L^\infty(B^{s-4}_{\kappa_2})})^{\alpha-1}\|v\|_{L^\infty(B^{s-4}_{\kappa_2})}\|U(\cdot)\dot{\varphi}\|_{L^\gamma(B^{s-4}_{\rho_1})}.\\
&\quad\lesssim (\|u\|_Y \vee \|\varphi\|_{H^s})^{\alpha-1}\|v\|_Y\|U(\cdot)\dot{\varphi}\|_{X_1} \lesssim (R\vee\|\varphi\|_{H^s})^{\alpha-1}(R\vee R_1)R_1.
\end{aligned}$$

Combining these estimates and taking $\alpha > 2$ into account, we obtain

$$\|f'(u)\dot{u}\|_{X_2} \lesssim (R\vee\|\varphi\|_{H^s})^{\alpha-1}(R\vee R_1)^2 + R_1. \tag{3.37}$$

Collecting the estimates (3.30)–(3.32), (3.34), (3.35) and (3.37), we obtain

$$\|\Phi(u)\|_Y \le CR_1 + C(R\vee\|\varphi\|_{H^s})^{\alpha-1}(R\vee R_1)^2$$

for some constant $C \ge 1$. Now, we choose R and T such that

$$2CR_1(\varphi; T) < R < \min\left\{\frac{1}{2C\|\varphi\|^{\alpha-1}_{H^s}}; (2C)^{-1/\alpha}\right\}. \tag{3.38}$$

Then, Φ is a contraction mapping from $\tilde{B}_R$ into itself, and hence there is a unique fixed point u of Φ in $\tilde{B}_R$, which gives a solution to (1.2). We shall show that $u \in X_0(I)\cap C(I; H^s(\mathbb{R}^N))$ by a sort of bootstrap argument. It follows from (3.29) and Lemma 2.4 (i) that $f(u) \in X_1(I)\cap L^\infty(I; H^{s-2}(\mathbb{R}^N))$. Then, from (3.34) with Y_0 and Y_1 respectively replaced with X_0 and X_1, we find that $u = \Phi(u) \in X_0(I)$. As mentioned above, we have $u \in C(I; H^{s-2}(\mathbb{R}^N)) \cap C^2(I; H^{s-4}(\mathbb{R}^N))$. Since $X_0(I) \cap Y(I) \subset \bigcap_{j=0}^2 W^{j,\tilde{\gamma}}(I; B^{s-2j}_{\tilde{\rho}}(\mathbb{R}^N))$, Lemma 2.7 (ii) shows that $f(u) \in C^1(I; H^{s-4}(\mathbb{R}^N))$. Hence, using the equation (3.3) with $j = 1$, we obtain $\partial_t u \in C(I; H^{s-2}(\mathbb{R}^N))$. Next, using the equation (3.3) with $j = 0$, we obtain $u \in L^\infty(I; H^s(\mathbb{R}^N))$. Once this is obtained, applying Lemma 2.7 (ii) again, we obtain $f(u) \in C(I; H^{s-2}(\mathbb{R}^N))$. Then, we go back to the equation (3.3) with $j = 0$ and obtain $u \in C(I; H^s(\mathbb{R}^N))$. The rest of the proof is the same as that of Proposition 3.1. □

Proposition 3.4. *Let $N \ge 11$, $4 < s < N/2$ and let $\alpha = \alpha^*(s)$ satisfy* $\max\{2; s-3\} < \alpha \le s-2$. *Let $\varphi \in H^s(\mathbb{R}^N)$ and let $\{\varphi_k\}_{k=1}^\infty \subset H^s(\mathbb{R}^N)$ satisfy $\varphi_k \to \varphi$ in $H^s(\mathbb{R}^N)$. Then there exists $T > 0$ such that* (1.2) *has a unique solution $u \in C(I; H^s(\mathbb{R}^N))\cap X(I)$ with $I = [0, T]$, and that* (1.2) *with φ replaced by φ_k has a unique solution $u_k \in C(I; H^s(\mathbb{R}^N)) \cap X(I)$ for sufficiently large k. Furthermore, $u_k \to u$ in $C(I; H^s(\mathbb{R}^N)) \cap X(I)$.*

Proof. The proof of Proposition 3.4 is similar to that of Proposition 3.2.

Step 1. As in the previous case, there exist positive numbers R and T such that (i) the equation (1.2) has a unique solution $u \in C(I; H^s(\mathbb{R}^N)) \cap X_0(I) \cap Y(I)$; (ii) for sufficiently large k, (1.2) with φ replaced by φ_k has a unique solution $u_k \in C(I; H^s(\mathbb{R}^N))\cap X_0(I)\cap Y(I)$;

(iii) $\|u_k\|_Y \le R$ and $\|u_k\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|\dot u_k\|_{L^\infty(H^{s-2})} \lesssim R \vee \|\varphi\|_{H^s}$. Choosing T sufficiently small, we may assume R to be arbitrarily small, so that $R^{\alpha-1} \ll R^{\alpha/2} \ll R \ll 1$, for we have $\alpha > 2$ by assumption. From Lemmas 2.1 and 2.4, we see $R_1(\varphi_k; T) \to R_1(\varphi; T)$, where R_1 is defined by (3.27).

Step 2. Like (3.34), (3.35), we have

$$\|u_k - u\|_{Y_0} \lesssim \|u_k - u\|_{X_1} + \|\dot u_k - \dot u\|_{X_1} + \|f(u_k) - f(u)\|_{Y_1}, \tag{3.39}$$

$$\|u_k - u\|_{X_1} \lesssim \|u_k - u\|_{X_2} + \|\dot u_k - \dot u\|_{X_2} + \|f(u_k) - f(u)\|_{X_2}, \tag{3.40}$$

$$\|\dot u_k - \dot u\|_{X_1} \lesssim \|\dot u_k - \dot u\|_{X_2} + \|\ddot u_k - \ddot u\|_{X_2} + \|f'(u_k)\dot u - f'(u_k)\dot u\|_{X_2}. \tag{3.41}$$

Like (3.19), we can obtain

$$\|f(u_k) - f(u)\|_{Y_1} \lesssim R(\|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} + \|u_k - u\|_{L^\gamma(B^{s-2}_{\rho_1})}), \tag{3.42}$$

$$\|f(u_k) - f(u)\|_{X_2} \lesssim R(\|u_k - u\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k - u\|_{L^\gamma(B^{s-4}_{\rho_1})}). \tag{3.43}$$

Here, we have used Lemma 2.4 (ii)–(iii) together with (2.22) and (2.23). On the other hand, it follows from Lemma 2.4 (iii)–(iv) and (2.22), (2.25) that

$$\begin{aligned}
&\|f'(u_k)\dot u_k - f'(u)\dot u\|_{L^\gamma(B^{s-4}_\rho)} \lesssim \|f'(u_k)(\dot u_k - \dot u)\|_{L^\gamma(B^{s-4}_\rho)} + \|(f'(u_k) - f'(u))\dot u\|_{L^\gamma(B^{s-4}_\rho)}\\
&\quad\lesssim \|u_k\|^{\alpha-1}_{L^\infty(B^{s-2}_{\kappa_1})}(\|u_k\|_{L^\gamma(B^{s-2}_{\rho_1})}\|\dot u_k - \dot u\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})}\|\dot u\|_{L^\gamma(B^{s-2}_\rho)})\\
&\quad\lesssim R(\|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} + \|\dot u_k - \dot u\|_{L^\infty(B^{s-4}_{\kappa_1})}),\\
&\|f'(u_k)\dot u_k - f'(u)\dot u\|_{L^2(B^{s-4}_{\rho^*})} \lesssim \|f'(u_k)(\dot u_k - \dot u)\|_{L^2(B^{s-4}_{\rho^*})} + \|(f'(u_k) - f'(u))\dot u\|_{L^2(B^{s-4}_{\rho^*})}\\
&\quad\lesssim \|u_k\|^{\alpha/2}_{L^\infty(B^{s-2}_{\kappa_1})}\|u_k\|^{\alpha/2}_{L^\gamma(B^{s-2}_{\rho_1})}\|\dot u_k - \dot u\|_{L^\gamma(B^{s-2}_\rho)}\\
&\qquad + (\|u_k\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|u\|_{L^\infty(B^{s-2}_{\kappa_1})})^{(\alpha-1)/2}(\|u_k\|_{L^\gamma(B^{s-2}_{\rho_1})} \vee \|u\|_{L^\gamma(B^{s-2}_{\rho_1})})^{(\alpha-1)/2}\\
&\qquad\qquad \times \|u_k - u\|^{1/2}_{L^\infty(B^{s-2}_{\kappa_1})}\|u_k - u\|^{1/2}_{L^\gamma(B^{s-2}_{\rho_1})}\|\dot u\|_{L^\gamma(B^{s-2}_\rho)}\\
&\quad\lesssim R(\|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} + \|\dot u_k - \dot u\|_{L^\infty(B^{s-4}_{\kappa_2})} + \|u_k - u\|_{L^\gamma(B^{s-2}_{\rho_1})} + \|\dot u_k - \dot u\|_{L^\gamma(B^{s-2}_\rho)}).
\end{aligned}$$

Therefore, we obtain

$$\|f'(u_k)\dot u_k - f'(u)\dot u\|_{X_2} \lesssim R(\|\dot u_k - \dot u\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} + \|u_k - u\|_Y). \tag{3.44}$$

Collecting the estimates (3.39)–(3.44), we obtain

$$\begin{aligned}
\|u_k - u\|_Y \lesssim{}& \|u_k - u\|_{X_2} + \|\dot u_k - \dot u\|_{X_2} + \|\ddot u_k - \ddot u\|_{X_2}\\
&+ R(\|\dot u_k - \dot u\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} + \|u_k - u\|_Y).
\end{aligned} \tag{3.45}$$

If $R > 0$ is small enough, then the last term in the right-hand side is absorbed into the left-hand side. Like (3.20), we can obtain

$$\|\dot u_k - \dot u\|_{L^\infty(B^{s-4}_{\kappa_1})} + \|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} \lesssim \|u_k - u\|_Y + \|\varphi_k - \varphi\|_{H^s} + \|\dot\varphi_k - \dot\varphi\|_{H^{s-2}}. \tag{3.46}$$

Substituting (3.46) into (3.45), we obtain

$$\|u_k - u\|_Y \lesssim \|u_k - u\|_{X_2} + \|\dot u_k - \dot u\|_{X_2} + \|\ddot u_k - \ddot u\|_{X_2} + \|\varphi_k - \varphi\|_{H^s} + \|\dot\varphi_k - \dot\varphi\|_{H^{s-2}}. \tag{3.47}$$

Step 3. We shall show $\lim_{k\to\infty} \|u_k - u\|_{L^\infty(H^{s-4})\cap X_2} = 0$, analogous to the previous case $\alpha > s-2$. Indeed, we have only to replace the index $(s-2)$ with $(s-4)$ and the space X_1 with X_2 in the proof of (3.22). From Lemma 2.1 and the estimate (2.21), we have

$$\begin{aligned}\|u_k - u\|_{L^\infty(H^{s-4})\cap X_2} &\lesssim \|\varphi_k - \varphi\|_{H^{s-4}} + \|f(u_k) - f(u)\|_{L^{\gamma'}(B^{s-4}_{\rho'})}\\ &\lesssim \|\varphi_k - \varphi\|_{H^{s-4}} + (\|u_k\|_Y \vee \|u\|_Y)^\alpha \|u_k - u\|_{X_2}\\ &\lesssim \|\varphi_k - \varphi\|_{H^{s-4}} + R^\alpha \|u_k - u\|_{X_2}.\end{aligned}$$

If R is small enough, the second term in the right-hand side is absorbed into the left-hand side. Hence $\|u_k - u\|_{L^\infty(H^{s-4})\cap X_2} \lesssim \|\varphi_k - \varphi\|_{H^{s-4}} \to 0$.

Step 4. We next estimate $\|\dot u_k - \dot u\|_{L^\infty(H^{s-4})\cap X_2}$ and $\|\ddot u_k - \ddot u\|_{L^\infty(H^{s-4})\cap X_2}$. From Lemma 2.1 and the estimates (2.21), (2.24), we obtain

$$\begin{aligned}&\|\dot u_k - \dot u\|_{L^\infty(H^{s-4})\cap X_2}\\ &\lesssim \|\dot\varphi_k - \dot\varphi\|_{H^{s-4}} + \|f'(u_k)(\dot u_k - \dot u)\|_{L^{\gamma'}(B^{s-4}_{\rho'})} + \|\{f'(u_k) - f'(u)\}\dot u\|_{L^{\gamma'}(B^{s-4}_{\rho'})}\\ &\lesssim \|\dot\varphi_k - \dot\varphi\|_{H^{s-4}} + \|u_k\|^\alpha_{L^\gamma(B^{s-2}_{\rho_1})} \|\dot u_k - \dot u\|_{L^\gamma(B^{s-4}_\rho)}\\ &\quad + (\|u_k\|_{L^\gamma(B^{s-2}_{\rho_1})} \vee \|u\|_{L^\gamma(B^{s-2}_{\rho_1})})^{\alpha-1} \|u_k - u\|_{L^\gamma(B^{s-2}_\rho)} \|\dot u\|_{L^\gamma(B^{s-2}_\rho)}\\ &\lesssim \|\dot\varphi_k - \dot\varphi\|_{H^{s-4}} + R^\alpha \|u_k - u\|_{X_1} + R^\alpha \|\dot u_k - \dot u\|_{X_2}.\end{aligned}$$

Using (3.47) and choosing R sufficiently small, we obtain

$$\|\dot u_k - \dot u\|_{L^\infty(H^{s-4})\cap X_2} \lesssim \|\varphi_k - \varphi\|_{H^s} + \|\dot\varphi_k - \dot\varphi\|_{H^{s-2}} + R\|u_k - u\|_{X_2} + R\|\ddot u_k - \ddot u\|_{X_2}. \tag{3.48}$$

Similarly,

$$\begin{aligned}&\|\ddot u_k - \ddot u\|_{L^\infty(H^{s-4})\cap X_2}\\ &\lesssim \|\ddot\varphi_k - \ddot\varphi\|_{H^{s-4}} + \|f'(u_k)(\ddot u_k - \ddot u)\|_{L^{\gamma'}(B^{s-4}_{\rho'})} + \|\{f'(u_k) - f'(u)\}\ddot u\|_{L^{\gamma'}(B^{s-4}_{\rho'})}\\ &\quad + \|f''(u_k)(\dot u_k \dot u_k - \dot u \dot u)\|_{L^{\gamma'}(B^{s-4}_{\rho'})} + \|\{f''(u_k) - f''(u)\}\dot u \dot u\|_{L^{\gamma'}(B^{s-4}_{\rho'})}\\ &\equiv \|\ddot\varphi_k - \ddot\varphi\|_{H^{s-4}} + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

As before, the terms I, II and III are estimated by the Leibniz rule. Namely, from (2.21) and (2.24),

$$\begin{aligned}\mathrm{I} &\lesssim \|u_k\|^\alpha_{L^\gamma(B^{s-2}_{\rho_1})} \|\ddot u_k - \ddot u\|_{L^\gamma(B^{s-4}_\rho)},\\ \mathrm{II} &\lesssim (\|u_k\|_{L^\gamma(B^{s-2}_{\rho_1})} \vee \|u\|_{L^\gamma(B^{s-2}_{\rho_1})})^{\alpha-1} \|u_k - u\|_{L^\gamma(B^{s-2}_{\rho_1})}, \|\ddot u\|_{L^\gamma(B^{s-4}_\rho)},\\ \mathrm{III} &\lesssim (\|u_k\|_{L^\gamma(B^{s-2}_{\rho_1})} \vee \|u\|_{L^\gamma(B^{s-2}_{\rho_1})})^{\alpha-1} (\|\dot u_k\|_{L^\gamma(B^{s-2}_\rho)} \vee \|\dot u\|_{L^\gamma(B^{s-2}_\rho)}) \|\dot u_k - \dot u\|_{L^\gamma(B^{s-2}_\rho)},\end{aligned}$$

so that $\mathrm{I}+\mathrm{II}+\mathrm{III} \lesssim R^\alpha \|u_k - u\|_Y$. Therefore

$$\|\ddot{u}_k - \ddot{u}\|_{L^\infty(H^{s-4})\cap X_2} \lesssim \|\ddot{\varphi}_k - \ddot{\varphi}\|_{H^{s-4}} + R^\alpha \|u_k - u\|_Y + \mathrm{IV}. \tag{3.49}$$

The term $\mathrm{IV} \equiv \|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}\|_{L^{\gamma'}(B^{s-4}_{\rho'})}$ is estimated as in Step 4 of the proof of Proposition 3.2. Namely, we further decompose

$$\begin{aligned}\mathrm{IV} &\le \|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}^L\|_{L^{\gamma'}(B^{s-4}_{\rho'})} + \|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}^H\|_{L^{\gamma'}(B^{s-4}_{\rho'})}\\ &\equiv \mathrm{IV}_1 + \mathrm{IV}_2.\end{aligned}$$

Recall that the decomposition $u = u^L + u^H$ is defined by (3.25), so that the supports of $\mathscr{F}u^L$ and $\mathscr{F}u^H$ are respectively contained in the region $|\xi| \lesssim 2^l$ and $|\xi| \gtrsim 2^l$. For arbitrarily fixed l, we can show $\mathrm{IV}_1 \to 0$ as $k \to \infty$. Indeed, it follows from the relation $1/\rho_2' = (\alpha-2)/\nu + 1/\rho + 2(1/\rho - (s-2)/N)$ together with the Hölder and the Sobolev inequalities that

$$\begin{aligned}&\|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}^L\|_{L^{\gamma'}(B^{-4}_{\rho'})} \lesssim \|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}^L\|_{L^{\gamma'}(L^{\rho_2'})}\\ &\lesssim (\|u_k\|_{L^\gamma(L^\nu)} \vee \|u\|_{L^\gamma(L^\nu)})^{\alpha-2}\|u_k - u\|_{L^\gamma(L^\rho)}\|\dot{u}\|^2_{L^\gamma(B^{s-2}_\rho)} \lesssim R^\alpha \|u_k - u\|_{X_2} \to 0.\end{aligned}$$

We take $\varepsilon > 0$ satisfying $s - 4 + \varepsilon < \min\{s-2; \alpha - 1\}$. Let $1/\rho_{j\varepsilon} = 1/\rho_j + \varepsilon/N$, $j = 1, 2$. From the Sobolev inequality, we have $B^{s-4+2j}_\rho \subset B^{s-4+\varepsilon}_{\rho_{j\varepsilon}}$. It follows from a slight modification of (2.24) that

$$\begin{aligned}\|f''(u_k)\dot{u}\dot{u}^L\|_{L^{\gamma'}(B^{s-4+\varepsilon}_{\rho'})} &\lesssim \|u_k\|^{\alpha-2}_{L^\gamma(L^\nu)}\|u_k\|_{L^\gamma(B^{s-4+\varepsilon}_{\rho_{2\varepsilon}})}\|\dot{u}\|_{L^\gamma(B^{s-4+\varepsilon}_{\rho_{1\varepsilon}})}\|\dot{u}^L\|_{L^\gamma(B^{s-4+\varepsilon}_{\rho_1})}\\ &\lesssim \|u_k\|^{\alpha-1}_{L^\gamma(B^s_\rho)}\|\dot{u}\|_{L^\gamma(B^{s-2}_\rho)}\|\dot{u}^L\|_{L^\gamma(B^{s-2+\varepsilon}_\rho)} \lesssim R^\alpha \|\dot{u}^L\|_{L^\gamma(B^{s-2+\varepsilon}_\rho)},\end{aligned}$$

and $\|f''(u)\dot{u}\dot{u}^L\|_{L^{\gamma'}(B^{s-4+\varepsilon}_{\rho'})} \lesssim R^\alpha \|\dot{u}^L\|_{L^\gamma(B^{s-2+\varepsilon}_\rho)}$. Hence, we see that $\|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}^L\|_{L^{\gamma'}(B^{s-4+\varepsilon}_{\rho'})}$ is bounded for arbitrarily fixed l. Therefore, we obtain $\mathrm{IV}_1 \to 0$ as $k \to \infty$ by interpolation. On the other hand, we have

$$\mathrm{IV}_2 \lesssim (\|u_k\|_{L^\gamma(B^{s-2}_{\rho_1})} \vee \|u\|_{L^\gamma(B^{s-2}_{\rho_1})})^{\alpha-1}\|\dot{u}\|_{L^\gamma(B^{s-2}_\rho)}\|\dot{u}^H\|_{L^\gamma(B^{s-2}_\rho)} \lesssim R^\alpha \|\dot{u}^H\|_{L^\gamma(B^{s-2}_\rho)},$$

which goes to zero as $l \to \infty$, uniformly with respect to k. Therefore, we see $\mathrm{IV} \to 0$. Now, collecting the estimates (3.47), (3.48) and (3.49), we have

$$\begin{aligned}\|u_k - u\|_Y &\lesssim \|\varphi_k - \varphi\|_{H^s} + \|\dot{\varphi}_k - \dot{\varphi}\|_{H^{s-2}} + \|\ddot{\varphi}_k - \ddot{\varphi}\|_{H^{s-4}}\\ &\quad + \|u_k - u\|_{X_2} + \|\{f''(u_k) - f''(u)\}\dot{u}\dot{u}\|_{L^{\gamma'}(B^{s-4}_{\rho'})}.\end{aligned}$$

This estimate shows that $\lim_{k\to\infty}\|u_k - u\|_Y = 0$. Going back to (3.46), (3.48) and (3.49), we also obtain

$$\lim_{k\to\infty}\left\{\|u_k - u\|_{L^\infty(B^{s-2}_{\kappa_1})} \vee \|\dot{u}_k - \dot{u}\|_{L^\infty(B^{s-4}_{\kappa_1}\cap H^{s-4})} \vee \|\ddot{u}_k - \ddot{u}\|_{L^\infty(H^{s-4})}\right\} = 0.$$

Step 5. We shall finally show that $\lim_{k\to\infty} \|u_k - u\|_{L^\infty(H^s)} = 0$. It follows from the equations for u_k and u that

$$\|u_k - u\|_{L^\infty(H^s)} \lesssim \|u_k - u\|_{L^\infty(L^2)} + \|\dot{u}_k - \dot{u}\|_{L^\infty(H^{s-2})} + \|f(u_k) - f(u)\|_{L^\infty(H^{s-2})},$$
$$\|\dot{u}_k - \dot{u}\|_{L^\infty(H^{s-2})} \lesssim \|\dot{u}_k - \dot{u}\|_{L^\infty(L^2)} + \|\ddot{u}_k - \ddot{u}\|_{L^\infty(H^{s-4})} + \|f'(u_k)\dot{u}_k - f'(u)\dot{u}\|_{L^\infty(H^{s-4})},$$

so that

$$\begin{aligned}
&\|u_k - u\|_{L^\infty(H^s)} + \|\dot{u}_k - \dot{u}\|_{L^\infty(H^{s-2})} \\
&\quad\lesssim \|u_k - u\|_{L^\infty(L^2)} + \|\dot{u}_k - \dot{u}\|_{L^\infty(L^2)} + \|\ddot{u}_k - \ddot{u}\|_{L^\infty(H^{s-4})} \\
&\qquad + \|f(u_k) - f(u)\|_{L^\infty(H^{s-2})} + \|f'(u_k)\dot{u}_k - f'(u)\dot{u}\|_{L^\infty(H^{s-4})}.
\end{aligned}$$

By Steps 3–4, the first three terms in the right-hand side converge to zero. Furthermore, it follows from Lemma 2.6 (ii)–(iii) that the last two terms converge to zero. This completes the proof. □

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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The authors have no conflicts of interest to declare that are relevant to the content of this article.

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