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Teichmüller space and the mapping class group of the twice punctured torus

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Abstract. We introduce coordinate systems to the Teichmüller space of the twice-punctured torus and give matrix representations for the points of Teichmüller space. The coordinate systems allow representation of the mapping class group of the twice punctured torus as a group of rational transformations and provide several applications to the mapping class group and also to Kleinian groups.

1. Introduction.

The Teichmüller space $\mathcal{T}_{g,n}$ of the closed oriented surface $M_{g,n}$ of genus g with n punctures with $2g-2+n>0$ is homeomorphic to $\mathbb{R}^D$, $D=6g-6+2n$, (see, for example, [**16**, 34.3]) and there are several global coordinate-systems for $\mathcal{T}$ which realize it as a D-cell in $\mathbb{R}^m$ for some m. The Fenchel–Nielsen coordinate-system and coordinate-systems by using a set of geodesic length functions or equivalently trace functions are the most popular among them (see, for example, [**10**], [**12**], [**13**] and [**14**].) In particular in [**10**] and [**11**] a coordinate-system by a set of $D+1$ trace functions is introduced to $\mathcal{T}_{g,n}$ by which the action of mapping class group $\mathcal{MC}_{g,n}$ of $M_{g,n}$ can be described as a group of rational transformations.

Let $\mathcal{T}=\mathcal{T}_{1,2}$ denote the Teichmüller space of the twice puncture torus M, the space of all marked complete hyperbolic structures on M of finite area. In this paper we regard $\mathcal{T}$ as a space quasiconformal deformations of twice punctured torus groups, or Fuchsian groups with signature $(1;\infty,\infty)$. For this particular case, Button gave in [**2**] a coordinate-system by which $\mathcal{T}$ can be identified with an open subset of $\mathbb{R}^4$ defined by a single simple inequality and also gave matrix representations of the twice punctured torus group in $PSL(2,\mathbb{R})$. The objective of this paper is also to give coordinate-systems to $\mathcal{T}$. But we put emphasis on their applications to the mapping class group $\mathcal{MC}=\mathcal{MC}_{1,2}$ of the twice punctured torus. We apply our coordinate systems to give a rational representation of the mapping class group $\mathcal{MC}$ as a group of rational transformations.

Any twice punctured torus group in $PSL(2,\mathbb{R})$ is a subgroup of Fuchsian groups with signature $(0;2,2,2,2,\infty)$, [**4**, Theorem 3A], [**15**, Thereom 1]. We consider in Section 2 Fuchsian groups of signature $(0;2,2,2,2,\infty)$ and give their matrix representations in $PSL(2,\mathbb{R})$. The matrix representations naturally lead to those of twice punctured torus groups. In Section 3, we give two global coordinate systems of the Teichmüller space $\mathcal{T}$.

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One coordinate-system is defined by trace functions on the Teichmüller space $\mathcal{T}$ of twice puncture torus groups as in papers cited above. The other coordinate-system uses trace functions on the Teichmüller space of Fuchsian groups with signature $(0; 2, 2, 2, 2, \infty)$. With the second coordinate-system the Teichmüller space $\mathcal{T}$ is embedded into the locus of a very simple polynomial equation in $\mathbb{R}^5$. In this section we also give matrix representations of twice punctured torus groups by matrices whose entries are described as rational functions in the coordinates introduced here. In Section 4, we consider the mapping class group $\mathcal{MC}$ of the twice punctured torus. We will show that the action of $\mathcal{MC}$ on the Teichmüller space $\mathcal{T}$ is represented by a group of rational transformations in the coordinates introduced in the previous section. In Section 5, we treat periodic elements of the mapping class group $\mathcal{MC}$ and describe them as a product of Gervais generators [**3**] up to conjugation. A periodic element has fixed points in $\mathcal{T}$. We also give Fuchsian groups related to the fixed points. The matrix representations of the twice punctured torus group obtained in Section 3 extend naturally to those in $SL(2, \mathbb{C})$. We conclude this paper by examples of twice punctured torus groups in $PSL(2, \mathbb{C})$. Two of them have extensions to Kleinian groups of finite covolume and the other one is a Kleinian group of the second kind which is a combination of two conjugations of $PSL(2, \mathbb{Z})$.

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2. Twice punctured torus groups.

This section is devoted to a preparatory argument for a matrix representation of Fuchsian group Γ with signature $(1; \infty, \infty)$ or a twice punctured torus group. This matrix representation is a lift of Γ to $SL(2, \mathbb{R})$. We also consider a matrix representation of a group which projects to a Fuchsian group G with signature $(0; 2, 2, 2, 2, \infty)$. Contrary to the case of Γ, there are no lifts of G to $SL(2, \mathbb{R})$, since G contains elliptic elements of order 2.

2.1. Matrix representations of twice punctured torus groups.

Let $\mathbb{H} = \{z = x + iy : y > 0\}$ be the upper half plane equipped with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Each matrix A of $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ as a conformal isometry:

$$A(z) = \frac{az + b}{cz + d} \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } z \in \mathbb{H}.$$

We denote by I the identity matrix in $SL(2, \mathbb{R})$. Two matrices A and B of $SL(2, \mathbb{R})$ define an identical isometry if and only if $B = \pm A$. The group of all conformal isometries on $\mathbb{H}$ is $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$ ([**1**, Theorem 7.4.1]). The *trace* $\mathrm{tr}A$ of $A \in SL(2, \mathbb{R})$ is invariant under conjugation. If $|\mathrm{tr}A| > 2$, then A is called hyperbolic and A has two fixed points p_A and q_A with $|A'(p_A)| > 1$ and $|A'(q_A)| < 1$, and they are called *repulsive* and *attractive* fixed point of A, respectively. We assume that the *axis* $ax(A)$ of A, the hyperbolic line connecting p_A and q_A, is oriented from p_A to q_A. We call a matrix E of $SL(2,\mathbb{R})$ *elliptic of order* 2, if $\mathrm{tr}E = 0$. If E is elliptic of order 2, then $E^2 = -I$. So the order 2 means the order of E when it is regarded as an element of $PSL(2,\mathbb{R})$.

Let Γ be a Fuchsian group in $PSL(2,\mathbb{R})$ with signature $(1;\infty,\infty)$:

$$\Gamma = \langle a, b, c, d : aba^{-1}b^{-1}cd = 1 \rangle, \tag{2.1}$$

where c and d are parabolic elements. The factor surface for the action of Γ on $\mathbb{H}$ is a twice punctured torus. The group Γ always has an extension G of index 2 which has signature $(0;2,2,2,2,\infty)$ (see [**4**, Theorem 3A], [**15**, Thereom 1]):

$$G = \langle e_1, e_2, e_3, e_4, d : e_1^2 = e_2^2 = e_3^2 = e_4^2 = e_1e_2e_3e_4d = 1 \rangle, \tag{2.2}$$

where

$$a = e_1e_3, \quad b = e_3e_2, \quad c = e_4^{-1}de_4 = e_3e_2e_1e_4. \tag{2.3}$$

We normalize G by a conjugation with an element of $PSL(2,\mathbb{R})$ in order to have canonical generating systems (a,b,c,d) of Γ and (e_1,e_2,e_3,e_4) of G so that the position of the fixed points and axes of some of their elements is as illustrated in Figure 2.1, in which we identify the upper half plane $\mathbb{H}$ with the unit disk by conformal maps.

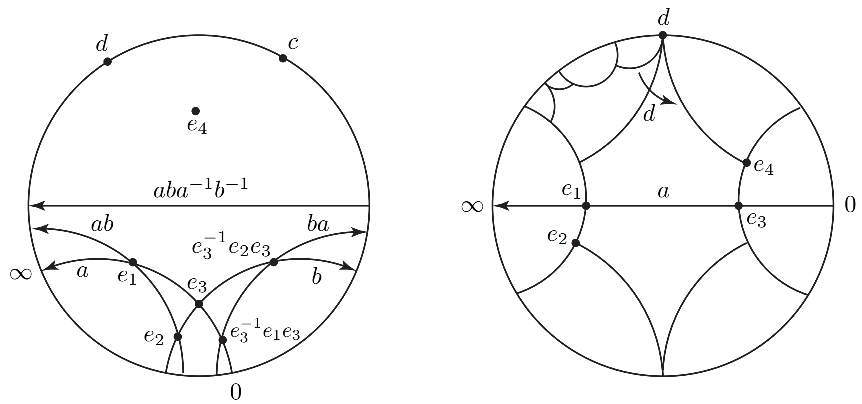


Figure 2.1. The two unit disks in the figure are images of $\mathbb{H}$ under different conformal maps.

The following lemmas for elliptic elements of order 2 may be well known. Let $c(E)$ denote the $(2,1)$-entry of an elliptic element E in $SL(2,\mathbb{R})$. We remark that in $SL(2,\mathbb{R})$ there are no elliptic elements E of order 2 with $c(E) = 0$.

Lemma 2.1. *The sign of $c(E)$ is invariant under conjugation.*

Proof. Let

$$E = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \tag{2.4}$$

be elliptic of order 2. Suppose that $r = c(E) > 0$. From $-qr = 1 + p^2 > 0$, $q < 0$. If

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a matrix of $SL(2, \mathbb{R})$, then

$$\begin{aligned} c(P^{-1}EP) &= a^2 r + c^2(-q) - 2acp \geq 2(|ac|\sqrt{-qr} - acp) \\ &= 2(|ac|\sqrt{1+p^2} - acp) > 0. \end{aligned}$$

If $c(E) < 0$, it suffices to apply the same argument to $-E$. □

Lemma 2.2. *Let E be elliptic of order 2 and D be a conjugate of*

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

in $SL(2, \mathbb{R})$. Then $\mathrm{tr}ED > 0$ if and only if $c(E) < 0$.

Proof. By our assumption, D can be written with real numbers c and d as

$$D = \begin{pmatrix} -1 - cd & -d^2 \\ c^2 & -1 + cd \end{pmatrix}.$$

If E is as in (2.4), then $-c(E) \cdot \mathrm{tr}ED = -r\mathrm{tr}ED = c^2 + (cp + rd)^2 > 0$. □

Lemma 2.3. *Let E_1, E_2 and E_3 be elliptic of order 2 with distinct fixed points. Let ϵ_j be the sign of $c(E_j)$ for $j = 1, 2, 3$. If $\mathrm{tr}E_1E_3 > 0$, $\mathrm{tr}E_3E_2 > 0$ and $\mathrm{tr}E_1E_2 < 0$, then $(\epsilon_1, \epsilon_2, \epsilon_3)$ equals $(+1, +1, -1)$ or $(-1, -1, +1)$.*

Proof. By Lemma 2.1 we may assume that

$$E_1 = \epsilon_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \epsilon_2 \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \quad E_3 = \epsilon_3 \begin{pmatrix} 0 & -1/\lambda \\ \lambda & 0 \end{pmatrix}, \tag{2.6}$$

where $r > 0$, $\lambda > 1$ and $q = -(p^2 + 1)/r < 0$. Then $\epsilon_j = \mathrm{sign}(c(E_j))$ for $i = 1, 2, 3$. Since

$$\mathrm{tr}E_1E_3 = -\epsilon_1\epsilon_3(\lambda + 1/\lambda), \ \mathrm{tr}E_3E_2 = \epsilon_2\epsilon_3(-r/\lambda + q\lambda), \ \mathrm{tr}E_1E_2 = \epsilon_1\epsilon_2(-r + q),$$

(see (2.10) below), we have $\epsilon_1\epsilon_3 < 0$, $\epsilon_2\epsilon_3 < 0$ and $\epsilon_1\epsilon_2 > 0$ under our assumption. □

Theorem 2.1 (matrix representations). (1) *Let*

$$G = \langle E_1, E_2, E_3, E_4, D : E_1^2 = E_2^2 = E_3^2 = E_4^2 = -I, E_1E_2E_3E_4D = I \rangle$$

be a Fuchsian group with signature $(0; 2, 2, 2, 2, \infty)$ such that the fixed point of $e_j = E_j$ for $j = 1, 2, 3, 4$ is as depicted in Figure 2.1. Then the matrices E_1, E_2, E_3 and E_4 can be chosen so that

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 1/\lambda \\ -\lambda & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} s & t \\ u & -s \end{pmatrix}, \tag{2.7}$$

where

$$p^2 + qr = -1, \ s^2 + tu = -1, \ p < 0, \ r > 0, \ s < 0, \ u < 0, \ \lambda > 1,$$
$$ps + ru + (ps + qt)\lambda^2 - 2\lambda = 0. \tag{2.8}$$

(2) *Let* $\Gamma = \langle A, B, C, D : ABA^{-1}B^{-1}CD = I \rangle$ *be a Fuchsian group with signature* $(1; \infty, \infty)$ *such that the axes of* $a = A$ *and* $b = B$ *are as depicted in Figure* 2.1 *and* $\mathrm{tr}D = -2$*. Then the matrices can be chosen so that* $A = E_1E_3$*,* $B = E_3E_2$*,* $C = E_4^{-1}DE_4 = E_3E_2E_1E_4$ *with matrices in* (2.7)*. They satisfy*

$$\mathrm{tr}A > 0, \ \mathrm{tr}B > 0, \ \mathrm{tr}AB > 0, \ \mathrm{tr}C = -2. \tag{2.9}$$

Proof. We choose E_1, E_2 and E_3 in (2.6) with $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, +1, -1)$. Then $r > 0$, $q < 0$ and $\lambda > 1$. The traces of $A = E_1E_3$, $B = E_3E_2$ and $AB = -E_1E_2$ are positive, since

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \ B = \begin{pmatrix} r/\lambda & -p/\lambda \\ -p\lambda & -q\lambda \end{pmatrix}, \ AB = \begin{pmatrix} r & -p \\ -p & -q \end{pmatrix}. \tag{2.10}$$

By Figure 2.1, the real part of fixed point $(p+i)/r$ of E_2 is negative. Hence $p < 0$. Since

$$E_1E_2E_3 = \begin{pmatrix} -p\lambda & -r/\lambda \\ -q\lambda & p/\lambda \end{pmatrix},$$

we have $\mathrm{tr}E_4D = \mathrm{tr}(E_1E_2E_3)^{-1} = \mathrm{tr}E_1E_2E_3$ and

$$\mathrm{tr}E_1E_2E_3 = -p(\lambda - 1/\lambda) > 0. \tag{2.11}$$

If $\mathrm{tr}D = -2$, by Figure 2.1 D is a conjugate of the matrix in (2.5). By Lemma 2.2 and (2.11), $u < 0$. Since by Figure 2.1 the real part of fixed point $(s+i)/u$ of E_4 is positive, $s < 0$. Finally we obtain (2.8) from $\mathrm{tr}D = -2$. It follows from $E_3^2 = E_4^2 = -I$ that

$$ABA^{-1}B^{-1}CD = E_1E_2E_3(E_1E_2E_3E_4D)E_4D = I.$$

We remark that from $\mathrm{tr}E_1E_4 = -u + t$, $\mathrm{tr}E_2E_4 = 2ps + qu + rt$ and $\mathrm{tr}E_3E_4 = u\lambda^{-1} - \lambda t$ follow

$$\mathrm{tr}E_1E_4 > 0, \quad \mathrm{tr}E_2E_4 > 0, \quad \mathrm{tr}E_3E_4 < 0. \tag{2.12}$$

□

Lemma 2.4. *Let* A*,* B *and* C *be as in Theorem* 2.1*. Then* $\mathrm{tr}AC > 0$*,* $\mathrm{tr}BC < 0$ *and* $\mathrm{tr}ABC > 0$*.*

Proof. We use the matrices in the proof of Theorem 2.1. Since

$$C = E_3E_2E_1E_4 = \begin{pmatrix} -(ps + ru)/\lambda & (rs - pt)/\lambda \\ (pu - qs)\lambda & -(ps + qt)\lambda \end{pmatrix},$$

we have

$$\begin{aligned}\mathrm{tr}AC &= (-q)t + (-u)r - 2ps \geq 2(\sqrt{(qr)(tu)} - ps)\\ &= 2(\sqrt{(p^2+1)(s^2+1)} - ps) > 0,\end{aligned}$$

$$\begin{aligned}\mathrm{tr}ABC &= (q^2t + (-u)p^2 + 2pqs)\lambda + (p^2t + (-u)r^2 - 2prs)/\lambda\\ &\geq 2(|pq|\sqrt{t(-u)} + pqs)\lambda + 2(|pr|\sqrt{t(-u)} - prs)/\lambda\\ &= 2(|pq|\lambda + |pr|/\lambda)(\sqrt{s^2+1} - |s|) > 0\end{aligned}$$

and from (2.8) we have $\mathrm{tr}BC = -2(r/\lambda + (-q)\lambda) - (t-u) < 0$. □

The group G can be written as $G = \langle E_3\rangle \ltimes \Gamma$. E_3 acts on Γ by

$$\begin{aligned} &E_3A = A^{-1}E_3, && E_3B = B^{-1}E_3,\\ &E_3C = B^{-1}A^{-1}C^{-1}BAE_3, && E_3D = B^{-1}A^{-1}D^{-1}BAE_3. \end{aligned} \tag{2.13}$$

3. Teichmüller space of twice punctured torus groups.

3.1. Trace identities.

We will use the following trace identities in $SL(2,\mathbb{C})$ and hence in $SL(2,\mathbb{R})$.

(I1) $\mathrm{tr}A = \mathrm{tr}A^{-1}$,
(I2) $\mathrm{tr}A\mathrm{tr}B = \mathrm{tr}AB + \mathrm{tr}AB^{-1}$,
(I3) $\mathrm{tr}ABC = \mathrm{tr}A\mathrm{tr}BC + \mathrm{tr}B\mathrm{tr}CA + \mathrm{tr}C\mathrm{tr}AB - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}ACB$,
(I4) $\mathrm{tr}[A,B] = \mathrm{tr}ABA^{-1}B^{-1} = (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB - 2$,
(I5) $\mathrm{tr}ABA^{-1}C = \mathrm{tr}A\mathrm{tr}ABC + \mathrm{tr}B\mathrm{tr}C - \mathrm{tr}AB\mathrm{tr}AC - \mathrm{tr}BC$,
(I6) $\mathrm{tr}ABAC = \mathrm{tr}AB\mathrm{tr}AC - \mathrm{tr}B\mathrm{tr}C + \mathrm{tr}BC$.

The identities (I1)–(I4) are found in [**9**, 3.4]. (I5) and (I6) are derived from the rest.

3.2. Teichmüller space of twice punctured torus groups.

We fix a Fuchsian group Γ_0 and a canonical generating system (A_0, B_0, C_0, D_0) of Γ_0 as in Theorem 2.1(2). Let $\mathcal{M}$ denote the space of all tuples (A, B, C, D) of matrices in $SL(2,\mathbb{R})$ such that there exists a quasiconformal automorphism w of $\mathbb{H}$ satisfying $A = wA_0w^{-1}$, $B = wB_0w^{-1}$, $C = wC_0w^{-1}$ and $D = wD_0w^{-1}$. Two tuples (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) in $\mathcal{M}$ are said to be *equivalent* if there exists a Möbius transformation h preserving $\mathbb{H}$ such that $A_2 = hA_1h^{-1}$, $B_2 = hB_1h^{-1}$, $C_2 = hC_1h^{-1}$ and $D_2 = hD_1h^{-1}$, that is, as tuples of matrices, they are simultaneously conjugate to each other in $SL(2,\mathbb{R})$.

Definition 3.1. The *Teichmüller space* $\mathcal{T} = \mathcal{T}_{1,2}$ of the twice punctured torus is the space of all equivalent classes of tuples in $\mathcal{M}$.

Our objective here is to give a coordinate-system of $\mathcal{T}$. We will identify a point of $\mathcal{T}_{1,2}$ with one of its representatives (A, B, C, D). Since $D = C^{-1}BAB^{-1}A^{-1}$, we often omit D and denote the point by (A, B, C). As before we assume that $\mathrm{tr}A$ and $\mathrm{tr}B$ are positive, $\mathrm{tr}C = \mathrm{tr}D = -2$. Let

$$a = \mathrm{tr}A, \quad b = \mathrm{tr}B, \quad z = \mathrm{tr}AB, \quad u = \mathrm{tr}AC, \quad w = \mathrm{tr}ABC, \tag{3.1}$$

and we define auxiliary positive parameters

$$\begin{aligned} t &= \sqrt{-\mathrm{tr}ABA^{-1}B^{-1} + 2} = \sqrt{4 + abz - a^2 - b^2 - z^2}, \\ s &= \sqrt{2 - \mathrm{tr}CABC^{-1}A^{-1}B^{-1}} = \sqrt{2 - \mathrm{tr}[B^{-1}C, AB]}. \end{aligned} \tag{3.2}$$

If we choose E_1, E_2, E_3 as in Theorem 2.1, then

$$\begin{aligned} s^2 &= -\mathrm{tr}CABC^{-1}A^{-1}B^{-1} + 2 = \mathrm{tr}(E_4E_1E_2)^2 + 2 = (\mathrm{tr}E_4E_1E_2)^2, \\ t^2 &= -\mathrm{tr}ABA^{-1}B^{-1} + 2 = \mathrm{tr}(E_1E_2E_3)^2 + 2 = (\mathrm{tr}E_1E_2E_3)^2. \end{aligned}$$

By (2.11) and Lemma 2.2 we have $t = \mathrm{tr}E_1E_2E_3$ and $s = \mathrm{tr}E_1E_2E_4 = \mathrm{tr}E_3(E_4DE_4^{-1})$.

THEOREM 3.1. *Each of the tuples (a, b, z, u, w) and (a, b, z, w, s) gives a global coordinate system to the Teichmüller space $\mathcal{T}$ of the twice punctured torus. In other words, $\Phi_u(A, B, C) = (a, b, z, u, w)$ and $\Phi_s(A, B, C) = (a, b, z, w, s)$ are embeddings of $\mathcal{T}_{1,2}$ into $\mathbb{R}^5$. The parameters satisfy*

$$s^2 + t^2 + (z + w)^2 - stw = 0. \tag{3.3}$$

PROOF. It is known that a finite set of traces parametrizes $\mathcal{T}$ globally (see for example [**5**], [**7**], [**10**], [**12**], [**13**], [**14**]). On the other hand, if G is a group generated by (A, B, C), the trace $\mathrm{tr}g$ of each $g \in G$ is a polynomial in

$$(a, b, z, u, v, w) = (\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB, \mathrm{tr}AC, -\mathrm{tr}BC, \mathrm{tr}ABC)$$

with integer coefficients (see [**9**, Lemma 3.5.1]). Here we employ $-\mathrm{tr}BC$ instead of $\mathrm{tr}BC$, because $\mathrm{tr}BC < 0$. By trace identities we have

$$\begin{aligned} -2 &= \mathrm{tr}ABA^{-1}(B^{-1}C) \\ &= \mathrm{tr}A\mathrm{tr}AB(B^{-1}C) + \mathrm{tr}B\mathrm{tr}BC^{-1} - \mathrm{tr}AB\mathrm{tr}AB^{-1}C - \mathrm{tr}B(B^{-1}C) \\ &= au + b(-2b + v) - z(bu - w) + 2 \\ &= (a - bz)u + bv - 2b^2 + zw + 2. \end{aligned} \tag{3.4}$$

By (3.4)

$$v = \frac{(bz - a)u + 2b^2 - zw - 4}{b}. \tag{3.5}$$

Hence we can omit v and conclude that (a, b, z, u, w) parametrizes $\mathcal{T}$ globally. The equation (3.3) is a consequence of trace identities

$$\begin{aligned} s^2 - 2 &= -\mathrm{tr}C(AB)C^{-1}(BA)^{-1} \\ &= -\mathrm{tr}C\mathrm{tr}D^{-1} - \mathrm{tr}AB\mathrm{tr}(BA)^{-1} + \mathrm{tr}CAB\mathrm{tr}CA^{-1}B^{-1} + \mathrm{tr}[A, B] \\ &= -4 - z^2 + w \cdot \mathrm{tr}CA^{-1}B^{-1} + (-t^2 + 2) \end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}CA^{-1}B^{-1} &= \mathrm{tr}(E_3E_1)(E_2E_3)(E_3E_2E_1E_4) = \mathrm{tr}E_3E_4^{-1} \\ &= \mathrm{tr}E_1E_2E_3\mathrm{tr}E_1E_2E_4 - \mathrm{tr}E_1E_2E_3E_1E_2E_4 \\ &= st + \mathrm{tr}E_1E_2E_3E_1E_2(C^{-1}E_3E_2E_1) \\ &= st - \mathrm{tr}E_1E_2C^{-1} \\ &= st + \mathrm{tr}ABC^{-1} = st - 2z - w.\end{aligned}$$

From this follows

$$\mathrm{tr}ACB = w - st, \tag{3.6}$$

and then by (I3)

$$w - st = \mathrm{tr}ACB = -av - 2z + ub + 2ab - w. \tag{3.7}$$

Solve the linear equations (3.4) and (3.7) in u and v to obtain

$$u = \frac{a(4+wz) + b(st - 2w - 2z)}{t^2 + z^2 - 4}, \quad v = \frac{a(2z + 2w - st) + b(2t^2 + stz - wz - 4)}{t^2 + z^2 - 4}. \tag{3.8}$$

So we can replace u by s and conclude that (a, b, z, w, s) also parametrizes $\mathcal{T}$. □

We remark that the identity (3.3) in coordinates a, b, z, u, w is

$$\begin{aligned}(aw - uz)^2 &+ 4(abw + buz + 2b^2 - 2au - 2wz - 4) \\ &= (buw - b^2 - u^2 - w^2)(abz - a^2 - b^2 - z^2).\end{aligned} \tag{3.9}$$

We let $\mathbb{R}_{>2} = \{x \in \mathbb{R} : x > 2\}$ and define (T, V, π) by

$$\begin{aligned}T &= \{(a, b, z, w, s, t) \in \mathbb{R}^6_{>2} : t^2 = 4 + abz - a^2 - b^2 - z^2, s^2 + t^2 + (z+w)^2 = stw\}, \\ V &= \{(a, b, z, t) \in \mathbb{R}^4_{>2} : abz - a^2 - b^2 - z^2 = t^2 - 4\}\end{aligned}$$

and $\pi : T \to V$ is a natural projection. Each slice

$$V_t = \{(a, b, z) \in \mathbb{R}^3_{>2} : abz - a^2 - b^2 - z^2 = t^2 - 4\}$$

of V is homeomorphic to $\mathbb{R}^2$ and identified with the Teichmüller space of the hyperbolic torus with one totally geodesic boundary curve of length $\ell(t)$ satisfying $t^2 - 2 = 2\cosh(\ell(t)/2)$ (see [**16**, Section 33.D].) We have $V = \bigcup_{t>2} V_t \cong \mathbb{R}^3$. For each $(a, b, z, t) \in V$, $\pi^{-1}(a, b, z, t)$ is a branch of the hyperbola $(t+2)X^2 - (t-2)Y^2 + A = 0$, where

$$X = \frac{s - w}{\sqrt{2}} - \frac{2z}{\sqrt{2}(t+2)}, \ Y = \frac{s + w}{\sqrt{2}} - \frac{2z}{\sqrt{2}(t-2)}, \ A = \frac{2t^2(z^2 + t^2 - 4)}{t^2 - 4}.$$

Therefore T is homeomorphic to $\mathbb{R}^4$. Let $\varpi : \mathcal{T}_{1,2} \to V$ send (A, B, C) to (a, b, z, t), $a = \mathrm{tr}A$, $b = \mathrm{tr}B$, $z = \mathrm{tr}AB$ and $t = \sqrt{abz - a^2 - b^2 - z^2 + 4}$. For each $(a, b, z, t) \in V$, we fix a point (A, B, C_0) in $\varpi^{-1}(a, b, z, t)$. We recall the Fenchel–Nielsen coordinates for $\mathcal{T}_{1,2}$. Then all other points (A, B, C) in $\varpi^{-1}(a, b, z, t)$ are obtained from (A, B, C_0) by a Nielsen twist along the curve determined by the axis of $ABA^{-1}B^{-1}$. In other words, if $H_\tau \in SL(2, \mathbb{R})$ has the same axis as $ABA^{-1}B^{-1}$ and its trace is $2\cosh(\tau/2)$ $(-\infty < \tau < \infty)$, then there exists a unique τ such that $C = H_\tau C_0 H_\tau^{-1}$. The parameter $w = \mathrm{tr}ABC$ is a real analytic convex function of τ and tends to $+\infty$ as $\tau \to \pm\infty$. This implies that the map sending τ to the branch of hyperbola is homeomorphic. Now we establish the well known fact that $\mathcal{T}_{1,2}$ is homeomorphic to $T \cong \mathbb{R}^4$.

We conclude this section by a matrix representation of a canonical generating system (A, B, C, D) of a twice punctured torus group described in terms of (a, b, z, w, s). It is normalized so that $C(z) = z + 1$ and D fixes 0:

$$\begin{aligned}
A &= \frac{1}{P_6}\begin{pmatrix} t^{-1}(aP_1 + bP_2) & t^{-2}[aP_3 + b(-tP_2 + P_4)] \\ a(tP_1 + P_3) + bP_4 & aP_6 - t^{-1}(aP_1 + bP_2) \end{pmatrix}, \\
B &= \frac{1}{P_6}\begin{pmatrix} t^{-1}[aP_2 + b(P_1 + tP_3)] & t^{-2}[a(-tP_2 + P_4) + bP_5] \\ aP_4 + bP_3 & bP_6 - t^{-1}[aP_2 + b(P_1 + tP_3)] \end{pmatrix}, \\
C &= \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ t^2 & -1 \end{pmatrix},
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
&P_1 = -2t - sz, \quad && P_2 = 2s - st^2 + tw + 2tz, \quad && P_3 = t^2 + stz - wz - z^2, \\
&P_4 = -st + 2w + 2z, \quad && P_5 = -P_3 + z(tP_2 - P_4), \quad && P_6 = t^2 + z^2 - 4.
\end{aligned}$$

To calculate the following matrices, we use (3.3) to reduce entries of matrices so that no terms are multiples of stw.

$$\begin{aligned}
AB &= \begin{pmatrix} -s/t & (st - z - w)/t^2 \\ -z - w & z + s/t \end{pmatrix}, \\
A^{-1}B^{-1} &= \begin{pmatrix} s/t - st + w + 2z & (-2st + st^3 + w - t^2w + z - 2t^2z)/t^2 \\ -st + z + w & -s/t + st - z - w \end{pmatrix}, \\
ABC &= \begin{pmatrix} s/t & (z + w)/t^2 \\ z + w & -s/t + w \end{pmatrix}, \\
CABC^{-1}A^{-1}B^{-1} &= \begin{pmatrix} 1 - 2s^2 - stz + s(w + z)/t & 2s^2 + stz - 2s(z + w)/t - s^2/t^2 \\ -s(s + tz) & 1 + s^2 + stz - s(w + z)/t \end{pmatrix}.
\end{aligned}$$

4. Mapping class group.

4.1. Mapping class group and its generators.

Let $\mathcal{MC} = \mathcal{MC}_{1,2}$ denote the mapping class group of the twice punctured torus. Each element of $\mathcal{MC}$ acts on the twice punctured torus group as an outer automorphism. We employ a generating system $(\omega_1, \omega_2, \omega_3, \omega_4)$ of $\mathcal{MC}$ from [**3**]. We describe ω by the change of canonical generators (up to simultaneous conjugation) caused by ω:

$$\begin{aligned}
\omega_1(a,b,c,d) &= (ab^{-1}, b, c, d),\\
\omega_2(a,b,c,d) &= (a, ba, c, d),\\
\omega_3(a,b,c,d) &= (b^{-1}ca, b, c, b^{-1}cdc^{-1}b),\\
\omega_4(a,b,c,d) &= (a, b, d, d^{-1}cd).
\end{aligned}$$

From these expressions we can verify that

$$\omega_1\omega_3 = \omega_3\omega_1,\ \omega_1\omega_4 = \omega_4\omega_1, \omega_2\omega_4 = \omega_4\omega_2, \tag{4.1}$$

$$\omega_1\omega_2\omega_1 = \omega_2\omega_1\omega_2,\ \omega_3\omega_2\omega_3 = \omega_2\omega_3\omega_2. \tag{4.2}$$

Relations in (4.1) arise from that Dehn twists along disjoint simple loops are commutative. Those in (4.2) are the *braid relations*. Both $(\omega_3\omega_4)^2$ and $(\omega_4\omega_3)^2$ send (a,b,c,d) to $(b^{-1}(ab^{-2})b, b, b^{-1}cb, b^{-1}db)$. Since we ignore a difference by an inner automorphism,

$$\omega_1^2 = (\omega_3\omega_4)^2 = (\omega_4\omega_3)^2. \tag{4.3}$$

4.2. Permutations of "Weierstrass points".

Let Γ and G be the groups in (2.1) and (2.2). Let $\pi : \mathbb{H} \to M = \mathbb{H}/\Gamma$ be the canonical projection, where M is a twice punctured torus. Let $w_j \in M$ denote the projection under π of the fixed point of e_j for $j = 1,2,3,4$. The mapping class

$$\zeta_0 = \omega_1\omega_2\omega_3\omega_4\omega_3\omega_2\omega_1$$

induces an involution of M fixing the set $W = \{w_1, w_2, w_3, w_4\}$ pointwise. Moreover ζ_0 commutes with all elements of $\mathcal{MC}$ (see Section 5.1). Therefore, each $\omega \in \mathcal{MC}$ preserves W and defines an element $\sigma = \tau(\omega)$ of the permutation group S_4 such that $\omega(w) = w_{\sigma(i)}$. Let $\gamma(g)$ denote the projection of the axis of a hyperbolic element g of Γ. Since $a = e_1e_3$, the geodesic loop $\gamma(a)$ passes w_1 and w_3. Likewise $\gamma(b)$ for $b = e_3e_2$ passes w_2 and w_3, $\gamma(ab)$ for $ab = e_1e_2$ passes w_1 and w_2. Since

$$ab^{-1} = e_1 \cdot e_3^{-1}e_2e_3, \quad b = (e_3^{-1}e_2e_3)\cdot e_3,$$

$\gamma(ab^{-1})$ passes w_1 and w_2, and $\gamma(b)$ passes w_2 and w_3. Hence ω_1 fixes each of w_1 and w_4 and interchanges w_2 and w_3. So ω_1 induces the permutation $(1)(2,3)(4)$. Likewise

$$\begin{aligned}
a &= e_3 \cdot e_3^{-1}e_1e_3, & ba &= e_3^{-1}e_1e_3 \cdot e_3^{-1}e_1^{-1}e_2e_1e_3,\\
b^{-1}ca &= e_1^{-1}e_4e_1 \cdot e_3, & b &= e_3 \cdot e_2,\\
a &= e_1 \cdot e_3, & b &= e_3 \cdot e_2.
\end{aligned}$$

give rise to

$$\tau(\omega_1) = (1)(2,3)(4),\ \tau(\omega_2) = (1,3)(2)(4),\ \tau(\omega_3) = (1,4)(2)(3),\ \tau(\omega_4) = 1. \tag{4.4}$$

4.3. Mapping classes as rational transformations.

The trace identities yield rational representations of Gervais generators:

$$\omega_{1*}(a,b,z,u,w) = (ab-z, b, a, bu-w, u), \tag{4.5}$$
$$\omega_{1*}^{-1}(a,b,z,u,w) = (z, b, bz-a, w, bw-u), \tag{4.6}$$
$$\omega_{2*}(a,b,z,u,w) = (a, z, az-b, u, zu+2b-v), \tag{4.7}$$
$$\omega_{2*}^{-1}(a,b,z,u,w) = (a, ab-z, b, u, aw-2b-zu+v), \tag{4.8}$$
$$\omega_{3*}(a,b,z,u,w) = (bu-w, b, u, -2bu+uv-z, -2bw-bz+vw+a), \tag{4.9}$$
$$\omega_{3*}^{-1}(a,b,z,u,w) = (-2z+st-w, b, -2bz+vz-u, z, bz-a), \tag{4.10}$$
$$\omega_{4*}(a,b,z,u,w) = (a, b, z, b(st-w)-zv+u, -2z-w+st), \tag{4.11}$$

and

$$\omega_{4*}^{-1}(a,b,z,u,w) = \big(a, b, z, -4a+b(w-st)+(t^2-3)u+vz, -st+(t^2-1)w-2z\big), \tag{4.12}$$

where t and v are given in (3.2) and (3.5). By the first equation in (3.8)

$$st = \frac{u(abz-a^2-b^2)-a(4+wz)}{b} + 2w + 2z.$$

So each entry of (4.5)–(4.12) is a rational function of (a,b,z,u,w). We obtain (4.5)–(4.12) as in the following way: Since $\omega_1(A,B,C) = (AB^{-1}, B, C)$,

$$\begin{aligned}\omega_{1*}(a,b,z,u,w) &= \big(\mathrm{tr}AB^{-1}, \mathrm{tr}B, \mathrm{tr}(AB^{-1})B, \mathrm{tr}(AB^{-1})C, \mathrm{tr}(AB^{-1})BC\big)\\ &= (ab-z, b, a, bu-w, u).\end{aligned}$$

From this we obtain (4.6). Since $\omega_2(A,B,C) = (A, BA, C)$,

$$\omega_{2*}(a,b,z,u,w) = \big(a, z, \mathrm{tr}A(BA), u, \mathrm{tr}A(BA)C\big).$$

From $\mathrm{tr}C = -2$, $\mathrm{tr}BC = -v$ and the trace identity (I6),

$$\begin{aligned}\mathrm{tr}A(BA) &= \mathrm{tr}A\mathrm{tr}AB - \mathrm{tr}A^{-1}BA = az-b,\\ \mathrm{tr}A(BA)C &= \mathrm{tr}AB\mathrm{tr}AC - \mathrm{tr}B\mathrm{tr}C + \mathrm{tr}BC = zu+2b-v.\end{aligned}$$

Since $\omega_2^{-1}(A,B,C) = (A, BA^{-1}, C)$,

$$\omega_{2*}^{-1}(a,b,z,u,w) = \big(a, \mathrm{tr}BA^{-1}, \mathrm{tr}A(BA^{-1}), u, \mathrm{tr}A(BA^{-1})C\big).$$

We have $\mathrm{tr}BA^{-1} = \mathrm{tr}B\mathrm{tr}A^{-1} - \mathrm{tr}AB = ab-z$ and by (I5)

$$\mathrm{tr}A(BA^{-1})C = \mathrm{tr}A\mathrm{tr}ABC + \mathrm{tr}B\mathrm{tr}C - \mathrm{tr}AB\mathrm{tr}AC - \mathrm{tr}BC = aw-2b-zu+v.$$

Since $\omega_3(A,B,C)=(B^{-1}CA,B,C)$,

$$\omega_{3*}(a,b,z,u,w)=(\mathrm{tr}B^{-1}CA,b,u,\mathrm{tr}B^{-1}CAC,\mathrm{tr}B^{-1}CABC).$$

By using the trace identities, we have

$$\begin{aligned}\mathrm{tr}B^{-1}CA&=\mathrm{tr}B\mathrm{tr}CA-\mathrm{tr}BCA=bu-w,\\ \mathrm{tr}B^{-1}CAC&=\mathrm{tr}B^{-1}C\mathrm{tr}AC-\mathrm{tr}BA=(-2b+v)u-z,\\ \mathrm{tr}B^{-1}CABC&=\mathrm{tr}BCB^{-1}CA=\mathrm{tr}B\mathrm{tr}BC^2A+\mathrm{tr}C\mathrm{tr}CA-\mathrm{tr}BC\mathrm{tr}BCA-\mathrm{tr}C^2A\\ &=b(\mathrm{tr}C\mathrm{tr}BCA-\mathrm{tr}AB)-2u+vw-(\mathrm{tr}C\mathrm{tr}AC-\mathrm{tr}A)\\ &=-2bw-bz+vw+a.\end{aligned}$$

Since $\omega_3^{-1}(A,B,C)=(C^{-1}BA,B,C)$,

$$\omega_{3*}^{-1}(a,b,z,u,w)=(\mathrm{tr}C^{-1}BA,b,\mathrm{tr}C^{-1}BAB,z,\mathrm{tr}C^{-1}BABC).$$

By using (3.6) and the trace identities, we have

$$\begin{aligned}\mathrm{tr}C^{-1}BA&=\mathrm{tr}C\mathrm{tr}AB-\mathrm{tr}CBA=-2z-(w-st),\\ \mathrm{tr}C^{-1}BAB&=\mathrm{tr}C^{-1}B\mathrm{tr}AB-\mathrm{tr}CA=(-2b+v)z-u,\\ \mathrm{tr}C^{-1}BABC&=\mathrm{tr}AB^2=bz-a.\end{aligned}$$

Since $\omega_4(A,B,C)=(A,B,D)$, where $D=C^{-1}BAB^{-1}A^{-1}$,

$$\omega_{4*}(a,b,z,u,w)=(a,b,z,\mathrm{tr}AD,\mathrm{tr}ABD).$$

By using the trace identities, we have

$$\begin{aligned}\mathrm{tr}AD&=\mathrm{tr}BA^{-1}B^{-1}C=\mathrm{tr}A\mathrm{tr}BB^{-1}C-\mathrm{tr}BAB^{-1}C\\ &=-2a-(\mathrm{tr}B\mathrm{tr}BAC+\mathrm{tr}A\mathrm{tr}C-\mathrm{tr}BA\mathrm{tr}BC-\mathrm{tr}AC)\\ &=-b(w-st)-zv+u,\\ \mathrm{tr}ABD&=\mathrm{tr}A^{-1}B^{-1}C=\mathrm{tr}C^{-1}BA=-2z-(w-st).\end{aligned}$$

Since $\omega_4^{-1}(A,B,C)=(A,B,CDC^{-1})$,

$$\omega_{4*}^{-1}(a,b,z,u,w)=(a,b,z,\mathrm{tr}ACDC^{-1},\mathrm{tr}ABCDC^{-1}).$$

By using the trace identities, we have

$$\begin{aligned}\mathrm{tr}ACDC^{-1}&=\mathrm{tr}CDC^{-1}A=\mathrm{tr}C\mathrm{tr}CDA+\mathrm{tr}D\mathrm{tr}A-\mathrm{tr}CD\mathrm{tr}CA-\mathrm{tr}AD\\ &=-2\mathrm{tr}BA^{-1}B^{-1}-2\mathrm{tr}A-\mathrm{tr}[A,B]\mathrm{tr}AC-\mathrm{tr}BA^{-1}B^{-1}C\\ &=-4a+(t^2-2)u-(\mathrm{tr}A\mathrm{tr}BB^{-1}C-\mathrm{tr}BAB^{-1}C)\\ &=-2a+(t^2-2)u+(\mathrm{tr}B\mathrm{tr}BAC+\mathrm{tr}A\mathrm{tr}C-\mathrm{tr}BA\mathrm{tr}BC-\mathrm{tr}AC)\\ &=-4a+(t^2-2)u+b(w-st)+zv-u,\end{aligned}$$

$$\begin{aligned}\mathrm{tr}ABCDC^{-1} &= \mathrm{tr}CDC^{-1}AB = \mathrm{tr}C\mathrm{tr}CDAB + \mathrm{tr}D\mathrm{tr}AB - \mathrm{tr}CD\mathrm{tr}CAB - \mathrm{tr}ABD \\ &= -2\mathrm{tr}A^{-1}B^{-1} - 2\mathrm{tr}AB - \mathrm{tr}[A,B]\mathrm{tr}ABC - \mathrm{tr}A^{-1}B^{-1}C \\ &= -4z + (t^2-2)w - (\mathrm{tr}C\mathrm{tr}AB - \mathrm{tr}ACB) \\ &= -4z + (t^2-2)w - (-2z - (w - st)).\end{aligned}$$

5. Mapping classes of finite order.

5.1. Mapping classes of finite order.

A mapping class ζ of finite order acts as a holomorphic automorphism on a Riemann surface R homeomorphic to a twice punctured torus [**6**]. We define

$$\begin{aligned}&\zeta_0 = \omega_1\omega_2\omega_3\omega_4\omega_3\omega_2\omega_1, \quad \zeta_1 = \omega_1\omega_4^{-1}\omega_3^{-1}, \quad \zeta_2 = \omega_1^2\omega_2\omega_3, \\ &\zeta_3 = \omega_1\omega_2\omega_3\omega_4, \qquad\qquad \zeta_4 = \omega_1\omega_2\omega_3, \qquad \zeta_5 = \omega_2\omega_3\omega_4.\end{aligned} \tag{5.1}$$

In this section we complete the following table.

ζ	order			
ζ_0	2	$(0;2,2,2,2,\infty)$	1	$+$
ζ_1	2	$(1;\infty)$	$(1,4)(2,3)$	$+$
ζ_2	3	$(0;3,\infty,\infty)$	$(1,4,3)(2)$	$-$
ζ_3	4	$(0;4,4,\infty)$	$(1,4,2,3)$	$+$
ζ_4	4	$(0;2,\infty,\infty)$	$(1,4,2,3)$	$-$
ζ_5	6	$(0;2,6,\infty)$	$(1,4,3)(2)$	$+$

The third column of the table shows the orbifold type of $R/\langle\zeta\rangle$, or the signature of a Fuchsian group K such that $\mathbb{H}/K = R/\langle\zeta\rangle$. We will verify this in Section 5.2. The fourth column shows the images under $\tau : \mathcal{MC} \to S_4$. This follows from (4.4). The sign $(+)$ in the fifth column means that ζ interchanges the two punctures and $(-)$ means otherwise. Note that among ω_1, ω_2, ω_3 and ω_4, only ω_4 interchanges the punctures. The element ζ_0 satisfies $\zeta_0^2 = 1$ and belongs to the center of $\mathcal{MC}$ (see Section 5.2).

LEMMA 5.1.

$$\zeta_0 = \zeta_5^3, \tag{5.2}$$

$$\zeta_1 = \omega_1\omega_2^{-1}\zeta_3^{-2}\omega_2\omega_1^{-1}\cdot\zeta_0 = \omega_4^{-1}\omega_2\zeta_4^{-2}\omega_2^{-1}\omega_4\cdot\zeta_0, \tag{5.3}$$

$$\zeta_2 = \omega_3^{-1}\omega_1^2\zeta_5^2\omega_1^{-2}\omega_3. \tag{5.4}$$

PROOF. We obtain (5.2) by

$$\begin{aligned}\zeta_5^3 &= \omega_2\omega_3\underline{\omega_4\omega_2}\omega_3\omega_4\omega_2\omega_3\omega_4 = \underline{\omega_2\omega_3\omega_2}\omega_4\omega_3\omega_4\omega_2\omega_3\omega_4 \\ &= \omega_3\omega_2\underline{(\omega_3\omega_4)^2}\omega_2\omega_3\omega_4 = \omega_3\omega_2\omega_1^2\omega_2\omega_3\omega_4 = \omega_3\omega_2\omega_1\zeta_0\omega_1^{-1}\omega_2^{-1}\omega_3^{-1} \\ &= \zeta_0.\end{aligned}$$

In this calculation, to the underlined parts (4.1)–(4.3) are applied. The last equation is due to the fact that ζ_0 is in the center of $\mathcal{MC}$. The equations of (5.3) follow from

$$\begin{aligned}\zeta_3^2 &= (\omega_1\omega_2\omega_3\omega_4)^2 = \zeta_0\omega_1^{-1}\omega_2^{-1}\underline{\omega_3^{-1}\omega_1}\omega_2\omega_3\omega_4\\ &= \underline{\omega_1^{-1}\omega_2^{-1}\omega_1}\cdot\underline{\omega_3^{-1}\omega_2\omega_3}\omega_4\zeta_0 = \omega_2\omega_1^{-1}\omega_2^{-1}\cdot\omega_2\omega_3\underline{\omega_2^{-1}\omega_4}\zeta_0\\ &= \omega_2\omega_1^{-1}\omega_3\omega_4\omega_2^{-1}\zeta_0\\ &= \omega_2\omega_1^{-1}\zeta_1^{-1}\omega_1\omega_2^{-1}\zeta_0\end{aligned}$$

and

$$\begin{aligned}\zeta_4^2 &= \omega_1\omega_2\underline{\omega_3\omega_1}\omega_2\omega_3 = \omega_1\omega_2\omega_1\underline{\omega_3\omega_2\omega_3} = \omega_1\underline{\omega_2\omega_1\omega_2}\omega_3\omega_2\\ &= \underline{\omega_1^2}\omega_2\omega_1\omega_3\omega_2 = \omega_4\omega_3\omega_4\omega_3\omega_2\omega_1\omega_3\omega_2\\ &= \underline{\omega_4\omega_2^{-1}}\omega_1^{-1}\zeta_0\omega_3\omega_2 = \omega_2^{-1}\underline{\omega_4\omega_1^{-1}\omega_3}\omega_2\zeta_0\\ &= \omega_2^{-1}\omega_4(\omega_3\omega_4\omega_1^{-1})\omega_4^{-1}\omega_2\zeta_0\\ &= \omega_2^{-1}\omega_4\zeta_1^{-1}\omega_4^{-1}\omega_2\zeta_0.\end{aligned}$$

We obtain (5.4) by

$$\begin{aligned}\zeta_5^2 &= \omega_2\omega_3\underline{\omega_4\omega_2}\omega_3\omega_4 = \underline{\omega_2\omega_3\omega_2}\omega_4\omega_3\omega_4\\ &= \omega_3\omega_2\underline{(\omega_3\omega_4)^2} = \omega_1^{-2}\underline{\omega_1^2\omega_3}\omega_2\omega_3\omega_3^{-1}\omega_1^2\\ &= \omega_1^{-2}\omega_3\zeta_2\omega_3^{-1}\omega_1^2.\end{aligned}$$ □

5.2. Finite extensions of twice punctured torus groups.

We denote by $T(g;\nu_1,\nu_2,\ldots,\nu_k)$ the Teichmüller space of the Fuchsian groups with signature $(g;\nu_1,\nu_2,\ldots,\nu_k)$ $(2\le\nu_1\le\nu_2\le\cdots\le\nu_k\le\infty)$. As before we write $\mathcal{T}=\mathcal{T}_{1,2}=T(1;\infty,\infty)$ and also $\mathcal{T}_{1,1}=T(1;\infty)$ the Teichmüller space of the once punctured torus. If $g=0$ we write $T(\nu_1,\nu_2,\ldots,\nu_k)$ instead of $T(0;\nu_1,\nu_2,\ldots,\nu_k)$. The Teichmüller space $T(p,q,r)$ of a triangle group consists of a single point of the conjugacy class of a Fuchsian group with presentation $\langle L,M,N : L^p=M^q=N^r=LMN=1\rangle$. If, for example, $r=\infty$, N in this notation is parabolic and we omit $N^r=1$. We have an identification

$$T(2,p,\infty)=T(p,p,\infty). \tag{5.5}$$

We find the *fixed point sets* $\{x\in\mathcal{T}:\zeta_j(x)=x\}$. We take $\zeta_2=\omega_1^2\omega_2\omega_3$ as an example. The mapping class ζ_2 sends (A,B,C) to $(B^{-1}A^{-1}C^{-1},CA,C)$ (up to simultaneous conjugation) and yields a rational mapping $\zeta_{2*}(a,b,z,u,w)=(w,u,b,z,b^{-1}(buz-au-zw-4))$. Solving the equations

$$\left(w,u,b,z,\frac{buz-au-zw-4}{b}\right)=(a,b,z,u,w)$$

and (3.9), we obtain a unique fixed point $(a,b,z,u,w)=(5,4,4,4,5)$ of ζ_{2*} and from (3.8)

$$t=\sqrt{abz-a^2-b^2-z^2+4}=3\sqrt{3},\quad s=3\sqrt{3}.$$

Substituting $(a,b,z,w,s)=(5,4,4,5,3\sqrt{3})$ to the matrix representation (3.10) we obtain the fixed point of ζ_2 represented by a canonical generating system (A,B,C) of a Fuchsian group Γ_1. (A, B and C in Section 5.2.3 below are replaced by their conjugates so that their entries are integers.) Since $(B^{-1}A^{-1}C^{-1}, CA, C)$ differs from (A,B,C) by an inner automorphism, there exists a matrix $M\in SL(2,\mathbb{R})$ such that

$$B^{-1}A^{-1}C^{-1}=M^{-1}AM,\ CA=M^{-1}BM,\ C=M^{-1}CM.$$

These equations determine M uniquely up to sign. We obtain an extension Γ_2 of Γ_1 by adding M to it. In Section 5.2.3 we give also the signature and a canonical generating system of Γ_2. Similar arguments apply to other cases in Sections 5.2.1–5.2.6.

5.2.1. $\boldsymbol{\zeta_0=\omega_1\omega_2\omega_3\omega_4\omega_3\omega_2\omega_1}$**.**

Since ζ_0 sends (A,B,C,D) to $(A^{-1},B^{-1},B^{-1}A^{-1}DAB,A^{-1}B^{-1}CBA)$ induced from the inner automorphism $(E_1,E_2,E_3,E_4)\to(E_3E_1E_3^{-1},E_3E_2E_3^{-1},E_3E_3E_3^{-1},$ $E_3E_4E_3^{-1})$ of the group with signature $(2,2,2,2,\infty)$, we see that $\zeta_0^2=1$, ζ_0 belongs to the center of $\mathcal{MC}$ and fixes all points of $\mathcal{T}$ ([**4**], [**15**]). The map

$$\iota:(E_1,E_2,E_3,E_4,D)\to(A,B,C)=(E_1E_3,E_3E_2,E_3E_2E_1E_4) \tag{5.6}$$

in Theorem 2.1 gives a homeomorphism $\iota: T(2,2,2,2,\infty)\xrightarrow{\cong}\mathcal{T}_{1,2}$.

5.2.2. $\boldsymbol{\zeta_1=\omega_1\omega_4^{-1}\omega_3^{-1}}$**.**

We show the following inclusions

$$\begin{array}{ccc} T(2,2,2,\infty) & \underset{\cong}{\xrightarrow{\iota_1}} & \mathcal{T}_{1,1}\\ \iota_2\ \cap & & \iota_3\ \cap\\ T(2,2,2,2,\infty) & \underset{\cong}{\xrightarrow{\iota}} & \mathcal{T}_{1,2}\end{array} \tag{5.7}$$

with the embeddings

$$\begin{array}{l}(S,T,U,V)\ \xrightarrow{\iota_1}\ (P,Q,R)=(SU,-UT,-V^2)\\ \quad\iota_2\ \downarrow\\ (E_1,E_2,E_3,E_4,D)=(S,TUT,U,VTU,-V^2),\end{array} \tag{5.8}$$

$$(P,Q,R)\xrightarrow{\iota_3}(A,B,C,D)=(P,Q^2,QRQ^{-1},R) \tag{5.9}$$

and ι as in (5.6). As is well known, the Teichmüller space $\mathcal{T}_{1,1}$ is identified with

$$\mathbb{M}=\left\{(x,y,z):x,y,z>2,x^2+y^2+z^2=xyz\right\}$$

by the map

$$(P,Q,R)\mapsto(x,y,z)=(\mathrm{tr}P,\mathrm{tr}Q,\mathrm{tr}PQ)$$

and also identified with $T(2,2,2,\infty)$ by ι_1 ([**17**], [**4**], [**15**]). The point of $T(2,2,2,\infty)$ associated to $(x,y,z)\in\mathbb{M}$ is the conjugacy class of (S,T,U,V) such that

$$\begin{aligned}
S &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
T &= \frac{1}{\sqrt{x^2-4}}\begin{pmatrix} 2 & y-z(x-\sqrt{x^2-4})/2 \\ y-z(x+\sqrt{x^2-4})/2 & -2 \end{pmatrix}, \\
U &= \begin{pmatrix} 0 & (x-\sqrt{x^2-4})/2 \\ -(x+\sqrt{x^2-4})/2 & 0 \end{pmatrix}, \\
V &= \frac{1}{\sqrt{x^2-4}}\begin{pmatrix} x-\sqrt{x^2-4} & (xy-2z-y\sqrt{x^2-4})/2 \\ (xy-2z+y\sqrt{x^2-4})/2 & -x-\sqrt{x^2-4} \end{pmatrix}.
\end{aligned} \tag{5.10}$$

The conjugacy class of $(E_1,E_2,E_3,E_4)=(S,TUT,U,VTU)$ is a point of $T(2,2,2,2,\infty)$. Compare this and (2.7). The point of $\mathcal{T}_{1,1}$ associated to $(x,y,z)\in\mathbb{M}$ is the conjugacy class of $(P,Q,R)=(SU,-UT,-V^2)$ with

$$\begin{aligned}
P &= \begin{pmatrix} (x+\sqrt{x^2-4})/2 & 0 \\ 0 & (x-\sqrt{x^2-4})/2 \end{pmatrix}, \\
Q &= \frac{1}{\sqrt{x^2-4}}\begin{pmatrix} z-y(x-\sqrt{x^2-4})/2 & x-\sqrt{x^2-4} \\ x+\sqrt{x^2-4} & -z+y(x+\sqrt{x^2-4})/2 \end{pmatrix}, \\
R &= \frac{1}{\sqrt{x^2-4}}\begin{pmatrix} 2x-\sqrt{x^2-4} & xy-2z-y\sqrt{x^2-4} \\ xy-2z+y\sqrt{x^2-4} & -2x-\sqrt{x^2-4} \end{pmatrix}.
\end{aligned} \tag{5.11}$$

Since the action of $\zeta_1(A,B,C,D)=(C^{-1}BAB^{-1},B,BDB^{-1},C)$ restricted to $\iota_3(\mathcal{T}_{1,1})$ is

$$\zeta_1(\iota_3(P,Q,R)) = Q(P,Q^2,QRQ^{-1},R)Q^{-1} = Q\iota_3(P,Q,R)Q^{-1},$$

ζ_1 fixes each point of $\iota_3(\mathcal{T}_{1,1})$. If we identify $\mathcal{T}_{1,1}$ with $\mathbb{M}$ and $\mathcal{T}_{1,2}$ with $\Phi_u(\mathcal{T}_{1,2})$, $\iota_3(x,y,z)$ is expressed by

$$\begin{aligned}
(a,b,z,u,w) = \iota_3(x,y,z) &= (\mathrm{tr}P, \mathrm{tr}Q^2, \mathrm{tr}PQ^2, \mathrm{tr}PQRQ^{-1}, \mathrm{tr}PQ^3RQ^{-1}) \\
&= (x, y^2-2, yz-x, 4yz-3x, 3x-4xy^2-7yz+4y^3z).
\end{aligned}$$

But if we replace ι_3 by $\omega_{4*}\iota_3$, then ι_3 has a simpler form

$$\iota_3(x,y,z) = (x, y^2-2, yz-x, x, yz-x)$$

and, if we identify $\mathcal{T}_{1,2}$ with $\Phi_s(\mathcal{T}_{1,2})$, from (3.8) the s-coordinate is $2z$.

Remark. The triangle group $G(2,4,\infty)=\langle L,M,N : L^2=M^4=LMN=1\rangle$ has a matrix representation such that

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1/\sqrt{2} & -1+1/\sqrt{2} \\ 1+1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$
$$N = \begin{pmatrix} -1+1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1-1/\sqrt{2} \end{pmatrix}. \tag{5.12}$$

Let $S = L$, $T = M^3LM$ and $U = -M^2$. Then S, T and U are all elliptic of order 2 and $STU = -S^{-1}N^2S$ is parabolic with trace -2. The map $\iota_4(L, M, N) = (S, T, U)$ gives an embedding $\iota_4 : T(2, 4, \infty) = T(4, 4, \infty) \to T(2, 2, 2, \infty)$.

The triangle group $G(2, 3, \infty) = \langle L, M, N : L^2 = M^3 = LMN = 1\rangle$ has a matrix representation such that

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} -(5+\sqrt{5})/10 & (-5+\sqrt{5})/5 \\ (5+\sqrt{5})/5 & (-5+\sqrt{5})/10 \end{pmatrix},$$
$$N = \begin{pmatrix} -1+\sqrt{5}/5 & -(5-\sqrt{5})/10 \\ (5+\sqrt{5})/10 & -1-\sqrt{5}/5 \end{pmatrix}.$$

Let $S = L$, $T = -MLM^{-1}$ and $U = -M^2LM^{-2}$. Then S, T and U are all elliptic of order 2 and $STU = N^{-3}$ is parabolic with trace -2. The map $\iota_5(L, M, N) = (S, T, U)$ gives an embedding $\iota_5 : T(2, 3, \infty) = T(3, 3, \infty) \to T(2, 2, 2, \infty)$.

5.2.3. $\zeta_2 = \omega_1^2\omega_2\omega_3$.

The triangle group $G(3, \infty, \infty) = \langle L, M, N : L^3 = LMN = 1\rangle$ has a matrix representation such that

$$L = \begin{pmatrix} -8 & 19 \\ -3 & 7 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -10 & 27 \\ -3 & 8 \end{pmatrix}.$$

Let $A = M^2L$, $B = L^{-2}M^{-1}$ and $C = M^{-3}$. Then

$$A = \begin{pmatrix} -2 & 5 \\ -3 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & -11 \\ 3 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -3 \\ 0 & -1 \end{pmatrix}, \tag{5.13}$$

and $\Gamma = \langle A, B, C\rangle$ is a twice punctured torus group with

$$D = C^{-1}BAB^{-1}A^{-1} = \begin{pmatrix} -1 & 0 \\ 9 & -1 \end{pmatrix}.$$

The mapping class ζ_2 sends (A, B, C) to $(B^{-1}A^{-1}C^{-1}, CA, C)$. The inner automorphism under conjugation by M yields $B^{-1}A^{-1}C^{-1} = M^{-1}AM$, $CA = M^{-1}BM$ and $C = M^{-1}CM$. Hence (A, B, C) is a fixed point of ζ_2. The map $\iota_6(L, M, N) = (A, B, C)$ gives an embedding $\iota_6 : T(3, \infty, \infty) \to \mathcal{T}_{1,2}$.

Since $\Gamma \subset SL(2, \mathbb{Z})$, $\boldsymbol{x}_1 = (\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB, \mathrm{tr}AC, \mathrm{tr}ABC) = (5, 4, 4, 4, 5)$ and hence all points of the $\mathcal{MC}$-orbit of $\boldsymbol{x}_1$ are positive integer solutions of the equation (3.9).

5.2.4. $\zeta_3 = \omega_1\omega_2\omega_3\omega_4$.

If L, M, N are the matrices in (5.12), then $(L^{-1}ML, M, -N^2)$ is a canonical generating system of a triangle group with signature $(0; 4, 4, \infty)$. However, instead of it, we consider the triangle group $G(4,4,\infty) = \langle L, M, N : L^4 = M^4 = LMN = 1\rangle$, where

$$L = \begin{pmatrix} 4\sqrt{2} & -5\sqrt{2} \\ 5/\sqrt{2} & -3\sqrt{2} \end{pmatrix}, \quad M = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -5 & 8 \\ -2 & 3 \end{pmatrix}.$$

The group Γ generated by $A = ML^{-1}$, $B = M^{-1}L$ and $C = M^3LMLM^2$, where

$$A = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -6 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -8 \\ 0 & -1 \end{pmatrix},$$

is a twice punctured torus group with

$$D = C^{-1}BAB^{-1}A^{-1} = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}.$$

The mapping class ζ_3 sends (A, B, C) to (B^{-1}, DA, D). The inner automorphism under conjugation by M yields $B^{-1} = M^{-1}AM$, $DA = M^{-1}BM$ and $D = M^{-1}CM$. Hence (A, B, C) is a fixed point of ζ_3. The map $\iota_7(L, M, N) = (A, B, C)$ gives an embedding $\iota_7 : T(4, 4, \infty) \to \mathcal{T}_{1,2}$.

Since $\Gamma \subset SL(2, \mathbb{Z})$, $\boldsymbol{x}_2 = (\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB, \mathrm{tr}AC, \mathrm{tr}ABC) = (4, 4, 10, 20, 70)$ and hence all points of the $\mathcal{MC}$-orbit of $\boldsymbol{x}_2$ are positive integer solutions of the equation (3.9).

5.2.5. $\zeta_4 = \omega_1\omega_2\omega_3$.

The triangle group $G(2, \infty, \infty) = \langle L, M, N : L^2 = LMN = 1\rangle$ has a matrix representation such that

$$L = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}.$$

Let $A = L^{-1}M^{-2}$, $B = M^3LM^{-1}$ and $C = -M^4$. Then

$$A = \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & -11 \\ 2 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix},$$

and $\Gamma = \langle A, B, C\rangle$ is a twice punctured torus group with

$$D = C^{-1}BAB^{-1}A^{-1} = \begin{pmatrix} -1 & 0 \\ 8 & -1 \end{pmatrix}.$$

The mapping class ζ_4 sends (A, B, C) to (B^{-1}, CA, C). The inner automorphism under conjugation by M^{-1} yields $B^{-1} = MAM^{-1}$, $CA = MBM^{-1}$ and $C = MCM^{-1}$. Hence (A, B, C) is a fixed point of ζ_4. The map $\iota_8(L, M, N) = (A, B, C)$ gives an embedding $\iota_8 : T(2, \infty, \infty) = T(\infty, \infty, \infty) \to \mathcal{T}_{1,2}$.

Since $\Gamma \subset SL(2, \mathbb{Z})$, $\boldsymbol{x}_3 = (\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB, \mathrm{tr}AC, \mathrm{tr}ABC) = (4, 4, 6, 4, 10)$ and hence all points of the $\mathcal{MC}$-orbit of $\boldsymbol{x}_3$ are positive integer solutions of the equation (3.9).

5.2.6. $\zeta_5 = \omega_2\omega_3\omega_4$.

The triangle group $G(2,6,\infty) = \langle L, M, N : L^2 = M^6 = LMN = 1\rangle$ has a matrix representation such that

$$L = \begin{pmatrix} 0 & -1/\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \sqrt{3} & -1/\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ -3 & -1 \end{pmatrix}.$$

Let $A = M^{-1}L^{-1}M^{-1}LM^2$, $B = ML^{-1}M^2$ and $C = (ML)^3$. Then

$$A = \begin{pmatrix} -2 & 1 \\ -15 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -3 \\ 0 & -1 \end{pmatrix},$$

and $\Gamma = \langle A, B, C\rangle$ is a twice punctured torus group with

$$D = C^{-1}BAB^{-1}A^{-1} = (LM)^3 = \begin{pmatrix} -1 & 0 \\ 9 & -1 \end{pmatrix}.$$

The mapping class ζ_5 sends (A, B, C) to $(B^{-1}DA, DA, D)$. The inner automorphism under conjugation by M yields $B^{-1}DA = M^{-1}AM$, $DA = M^{-1}BM$ and $D = M^{-1}CM$. Hence (A, B, C) is a fixed point of ζ_5.

Let $P = M^{-2}$, $Q = M^2L^{-1}M^{-1}$ and $R = (PQ)^{-1} = ML$. Then (P, Q, R) is a canonical generating system of a triangle group with signature $(3, \infty, \infty)$. So we can identify $T(2, 6, \infty)$ with $T(3, \infty, \infty)$.

Remark. As a point of $\mathcal{T}$, (A, B, C, D) is sent by $\omega_4\omega_2^{-1}$ to the point given in (5.13). More precisely, if

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (A, BA^{-1}, D, D^{-1}CD) = \omega_4\omega_2^{-1}(A, B, C, D),$$

and

$$U = \begin{pmatrix} 0 & 1/\sqrt{3} \\ -\sqrt{3} & 3\sqrt{3} \end{pmatrix},$$

then $(U^{-1}\tilde{A}U, U^{-1}\tilde{B}U, U^{-1}\tilde{C}U, U^{-1}\tilde{D}U)$ is identical with (A, B, C, D) in (5.13).

6. Examples.

6.1. Isometric spheres.

The matrix representations (3.10) can be extended to $SL(2,\mathbb{C})$. Using (3.10) or their conjugates in $SL(2,\mathbb{C})$ we show some examples of Kleinian groups. We identify a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2,\mathbb{C})$ with the Möbius transformation

$$A(z) = \frac{az+b}{cz+d}.$$

The matrix A acts also on the 3-dimensional hyperbolic space $\mathbb{H}^3 = \{\zeta = z + tj : t > 0\}$ (in the space of quaternions) by the Poincaré extension (see Beardon [**1**, Chapter 4]). Its action can be described by

$$A(z+jt) = \frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2 + tj}{|cz+d|^2 + |c|^2t^2}$$

[**1**, (4.1.4)]. If $c \neq 0$, we denote by $I(A)$ the *isometric sphere* of A, the Euclidean sphere with center $-d/c$ and radius $1/|c|$. Isometric spheres have following properties:

(1) $A(I(A)) = I(A^{-1})$.

(2) It holds that

$$I(M_1AM_2) = I(AM_2) = M_2^{-1}(I(A)) \tag{6.1}$$

for Euclidean motions

$$M_1 = \begin{pmatrix} \alpha_1 & \mu_1 \\ 0 & \beta_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \alpha_2 & \mu_2 \\ 0 & \beta_2 \end{pmatrix}, \qquad \left|\frac{\alpha_1}{\beta_1}\right| = \left|\frac{\alpha_2}{\beta_2}\right| = 1.$$

6.2. The Poincaré polyhedron theorem.

The Poincaré polyhedron theorem provides conditions for a polyhedron D in $\mathbb{H}^3$ to be a fundamental domain of a Kleinian group. Assume that for each side s of D there is a side s' of D and $g_s \in PSL(2,\mathbb{C})$ satisfying

(i) $g_s(s) = s'$.

(ii) $g_{s'} = g_s^{-1}$.

(iii) $g_s(D) \cap D = \emptyset$.

This g_s is called a *side pairing transformation.* Let $\bar{D}$ denote the closure of D in $\mathbb{H}^3$. For two points z and w of $\bar{D}$, define an equivalence relation $z \sim w$ by that $z = w$ or there is a side pairing transformation g such that $g(z) = w$. Let $p : \bar{D} \to D^* = \bar{D}/\sim$ be the projection to the quotient space. The fourth condition is:

(iv) For each $z \in D^*$, $p^{-1}(z)$ is a finite set.

Let $e_1, e_2, \ldots, e_k$ be distinct edges of D. Assume that e_j is on the boundary of two sides s_j and s'_j of D and that a side-pairing transformation g_j sends s'_j to s_{j+1} for $i = 1, 2, \ldots, k$ with $s_{k+1} = s_1$. Then $\sigma = \{e_1, e_2, \ldots, e_k\}$ is called a *cycle of edges* and $h = g_k \circ \cdots \circ g_1$ the *cycle transformation.* The next condition is

(v) For each cycle transformation h, there is a positive integer t such that $h^t = 1$.

For an edge e, let $\alpha(e)$ be the dihedral angle at e measured from inside D. For a cycle of edges $\sigma = \{e_1, e_2, \ldots, e_k\}$, let $\alpha(\sigma) = \alpha(e_1) + \alpha(e_2) + \cdots + \alpha(e_k)$. Then we need the following condition

(vi) For each cycle of edges σ, $\alpha(\sigma) = \sum_{m=1}^{k} \alpha(e_m) = 2\pi/t$ for some positive integer t.

The last condition is

(vii) D^* is complete.

Let $x = x_1, x_2, \ldots, x_k$ be points in $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that (a) two sides s_j and $\tilde{s}_j$ of D are tangent at x_j, (b) $\tilde{s}_{j+1} = g_{s_j}(s_j)$ and (c) $x_{j+1} = g_{s_j}(x_j)$, where $x_{k+1} = x_1$. Then x is called an *infinite edge* and $h = g_k \circ \cdots \circ g_1$ the *infinite cycle transformation* at x. If D is finite sided and every infinite cycle transformation at every infinite edge is parabolic, then we can omit condition (vii) because it is a consequence of conditions (i)–(vi) ([**8**, Proposition IV.I.6]).

Theorem 6.1 (the Poincaré polyhedron theorem [**8**, Theorem IV.H.11]). *If conditions* (i)–(vii) *are satisfied, then the group G generated by side pairing transformations is a discrete group, that is, a Kleinian group and D is a fundamental domain for G.*

In the following, we let $S(a, r)$ denote the sphere $\{\zeta : |\zeta - a| = r\}$. If $E \in SL(2, \mathbb{C})$ is elliptic, its fixed point set $Fix(E)$ is a hyperbolic line in $\mathbb{H}^3$.

6.3. Examples.
Example 1. We consider the mapping class $\varphi_1 = \omega_2\omega_3^{-1}\omega_1\omega_2$:

$$(A, B, C, D) \to \bigl(C^{-1}BAB^{-1}, BA(C^{-1}BAB^{-1}), C, C^{-1}BADA^{-1}B^{-1}C\bigr).$$

Let G_1 be the group generated by

$$A = \begin{pmatrix} i & -i \\ \sqrt{3} - i & -\sqrt{3} \end{pmatrix}, \quad B = \begin{pmatrix} 2\sqrt{3} + i & -(1/2)(\sqrt{3} + 3i) \\ \sqrt{3} + i & -i \end{pmatrix},$$
$$C = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ -2(1 + \sqrt{3}i) & -1 \end{pmatrix}.$$

These matrices satisfy $ABA^{-1}B^{-1}CD = I$, where I is the unit matrix. Let

$$M = \begin{pmatrix} -1 & -\omega \\ 0 & -1 \end{pmatrix}, \quad \omega = \frac{1}{2}(-1 + \sqrt{3}i). \tag{6.2}$$

Then

$$\begin{aligned} M^{-1}AM &= C^{-1}BAB^{-1}, & M^{-1}BM &= BAC^{-1}BAB^{-1}, \\ M^{-1}CM &= C, & M^{-1}DM &= C^{-1}BADA^{-1}B^{-1}C. \end{aligned} \tag{6.3}$$

Let $\Gamma_1 = \langle M \rangle \ltimes G_1$ be the group defined by (6.3). We list isometric spheres of some elements of Γ_1 and some planes orthogonal to the complex plane:

$$
\begin{aligned}
&I(A^{-1}) = S\left(\frac{\omega}{2}, \frac{1}{2}\right), && I(A) = S\left(\frac{\omega}{2}+1, \frac{1}{2}\right),\\
&I(MB) = S\left(\frac{\omega}{2}+\frac{1}{2}, \frac{1}{2}\right), && I(B^{-1}M^{-1}) = S\left(\frac{\omega}{2}+\frac{3}{2}, \frac{1}{2}\right),\\
&I(CA) = S\left(\frac{\omega}{2}+1, \frac{1}{2}\right), && I(A^{-1}C^{-1}) = S\left(\frac{\omega}{2}+2, \frac{1}{2}\right),\\
&I(MBA^{-1}) = S\left(\omega+\frac{1}{2}, \frac{1}{2}\right), && I(AB^{-1}M^{-1}) = S\left(\omega+\frac{3}{2}, \frac{1}{2}\right)\\
&I(BA^{-1}M) = S\left(\frac{1}{2}, \frac{1}{2}\right), && I(M^{-1}AB^{-1}) = S\left(\frac{3}{2}, \frac{1}{2}\right),\\
&I(C) = \{\zeta = z+jt : \mathrm{Im}\,(z\bar{\omega}) = 0\}, && I(C^{-1}) = \big\{\zeta = z+jt : \mathrm{Im}\,((z-1)\bar{\omega}) = 0\big\},\\
&I(M) = \{\zeta = z+jt : \mathrm{Im}\, z = 0\}, && I(M^{-1}) = \{\zeta = z+jt : \mathrm{Im}\, z = \mathrm{Im}\,\omega\}.
\end{aligned} \tag{6.4}
$$

Note that $I(C)$ is the vertical plane passing through 0 and ω, $I(M)$ is the vertical plane passing through 0 and 2, $I(C^{-1}) = C(I(C))$ and $I(M^{-1}) = M(I(M))$. For convenience, we refer to them as the isometric spheres of C, M, C^{-1} and M^{-1}, respectively. Let $\mathscr{R}$ denote the region bounded by these isometric spheres. Figure 6.1 depicts the faces of $\mathscr{R}$. We denote by s_i the face labeled as in Figure 6.1:

$$
\begin{array}{llll}
s_1 \subset I(A^{-1}), & s_2 \subset I(MB), & s_3 \subset I(A), & s_4 \subset I(A) = I(CA),\\
s_5 \subset I(MB^{-1}M^{-1}), & s_6 \subset I(A^{-1}C^{-1}), & s_7 \subset I(MBA^{-1}), & s_8 \subset I(AB^{-1}M^{-1}),\\
s_9 \subset I(BA^{-1}M), & s_{10} \subset I(M^{-1}AB^{-1}), & s_{11} \subset I(C), & s_{12} \subset I(C^{-1}),\\
s_{13} \subset I(M), & s_{14} \subset I(M), & s_{15} \subset I(M^{-1}), & s_{16} \subset I(M^{-1}).
\end{array}
$$

Side-pairings of $\mathscr{R}$ are as follows:

$$
\begin{array}{llll}
\{s_1, s_3; A^{-1}\}, & \{s_2, s_5; MB\}, & \{s_4, s_6; CA\}, & \{s_7, s_8; MBA^{-1}\},\\
\{s_9, s_{10}; BA^{-1}M\}, & \{s_{11}, s_{12}; C\}, & \{s_{13}, s_{15}; M\}, & \{s_{14}, s_{16}; M\}.
\end{array}
$$

Here $\{s_i, s_j, P\}$ means $P(s_i) = s_j$. The subgroup generated by the side pairing transformations equals Γ_1, since $B = M^{-1}(MB)$.

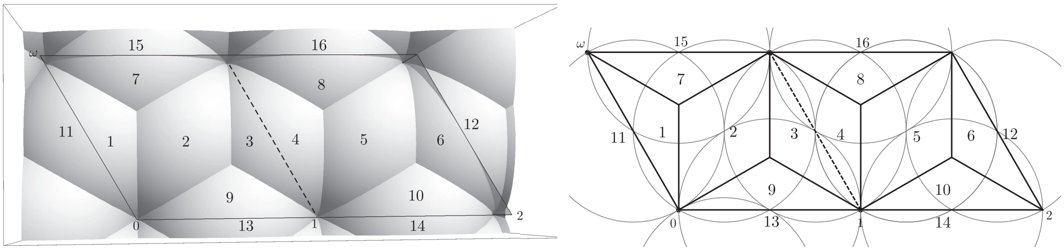


Figure 6.1. A bird's-eye view of isometric spheres (left). Faces s_{11}, s_{12}, ..., s_{16} are on vertical planes.

We define Euclidean motions

$$S=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad T=\begin{pmatrix}-i&(2+\omega)i\\0&i\end{pmatrix},\quad U=\begin{pmatrix}-i&(1+\omega)i\\0&i\end{pmatrix}.$$

Then we have

$$\begin{aligned}
&SAS^{-1}=A^{-1}C^{-1}, && SBS^{-1}=M^{-1}CB^{-1}M^{-1}, && SCS^{-1}=C, && SMS^{-1}=M,\\
&TAT^{-1}=CA, && TBT^{-1}=AB^{-1}CA, && TCT^{-1}=C^{-1}, && TMT^{-1}=M^{-1},\\
&UAU^{-1}=A^{-1}, && UBU^{-1}=MC^{-1}MB, && UCU^{-1}=C^{-1}, && UMU^{-1}=M^{-1}.
\end{aligned}$$

Thus $S\Gamma_1 S^{-1}=T\Gamma_1 T^{-1}=U\Gamma_1 U^{-1}=\Gamma_1$ and hence Γ_1 is invariant under the translation $S(z)=z+1$, the rotation $T(z)=-z+(2+\omega)$ about $1+\omega/2$ through the angle π and the rotation $U(z)=-z+(1+\omega)$ about $(1+\omega)/2$ through the angle π. This accounts for symmetries shown in Figure 6.1.

We will find cycles of edges. Let $s_{i,j}=s_{j,i}$ be the common edge of s_i and s_j. For example, $\sigma_1=\{s_{2,1},s_{3,9},s_{10,5}\}$ is a cycle of $\mathscr{R}$. Since $A^{-1}(s_1)=s_3$, $BA^{-1}M(s_9)=s_{10}$ and $B^{-1}M^{-1}(s_5)=s_2$, the cycle transformation is $B^{-1}M^{-1}\circ BA^{-1}M\circ A^{-1}$, which equals 1 by (6.3). We identify two cycles σ and σ' if σ' is a cyclic permutation of σ or of σ with reversed order. Then all cycles of edges and cycle transformations of $\mathscr{R}$ are

$$\begin{aligned}
&\sigma_1=\{s_{2,1},s_{3,9},s_{10,5}\}, && B^{-1}M^{-1}\circ BA^{-1}M\circ A^{-1}=1,\\
&\sigma_2=\{s_{2,9},s_{10,6},s_{4,5}\}, && B^{-1}M^{-1}\circ A^{-1}C^{-1}\circ BA^{-1}M=1,\\
&\sigma_3=\{s_{5,6},s_{4,8},s_{7,2}\}, && MB\circ AB^{-1}M^{-1}\circ A^{-1}C^{-1}=1,\\
&\sigma_4=\{s_{5,8},s_{7,1},s_{3,2}\}, && MB\circ A^{-1}\circ AB^{-1}M^{-1}=1,\\
&\sigma_5=\{s_{1,11},s_{12,6},s_{4,3}\}, && A\circ A^{-1}C^{-1}\circ C=1,\\
&\sigma_6=\{s_{9,13},s_{15,7},s_{8,16},s_{14,10}\}, && M^{-1}AB^{-1}\circ M^{-1}\circ MBA^{-1}\circ M=1.
\end{aligned}$$

Note that σ_1 and σ_3 are equivalent under T and so are σ_2 and σ_4.

The condition (vi) in Section 6.2 is satisfied by each cycle since

$$\begin{aligned}
\alpha(\sigma_1)&=\alpha(s_{1,2})+\alpha(s_{3,9})+\alpha(s_{10,5})=2\pi/3+2\pi/3+2\pi/3=2\pi,\\
\alpha(\sigma_5)&=\alpha(s_{1,11})+\alpha(s_{12,6})+\alpha(s_{4,3})=\pi/2+\pi/2+\pi=2\pi,\\
\alpha(\sigma_6)&=\alpha(s_{9,13})+\alpha(s_{15,7})+\alpha(s_{8,16})+\alpha(s_{14,10})=\pi/2+\pi/2+\pi/2+\pi/2=2\pi,
\end{aligned}$$

and that $\alpha(\sigma_2)=\alpha(\sigma_3)=\alpha(\sigma_4)=2\pi$ follows from $\alpha(\sigma_1)=2\pi$ and the symmetries S, T and U. Since $\mathscr{R}$ has finitely many sides, the other conditions (i) through (vi) in Section 6.2 are satisfied. The condition (vii) is satisfied by Proposition IV.I.6 in [**8**]. The Poincaré polyhedron theorem concludes that Γ_1 is a Kleinian group and has $\mathscr{R}$ as a fundamental polyhedron.

Example 2. We consider $\varphi_2=\omega_2^2\omega_3^{-1}\omega_1\omega_2^2$:

$$(A,B,C,D)\to(C^{-1}BAB^{-1},BA^2(C^{-1}BAB^{-1})^2,C,C^{-1}BA^2DA^{-2}B^{-1}C).$$

Let a be a root of $a^6-4a^4+16=0$ such that $a=(-1.721433\cdots)+(0.352201\cdots)i$. We have $a^4-2a^2-4=ia^3$. Let G_2 be the group generated by

$$A = \begin{pmatrix} (1/8)a^3(a^2-2) & -(1/16)a^3(a^2-2) \\ -2a & -(1/8)a(a-2)(a+2)(a^2+2) \end{pmatrix},$$

$$B = \begin{pmatrix} -(1/8)ia(a^4+6a^2-8) & -(1/16)ia(a^4-4a^2-8) \\ -ia^3 & -(1/8)ia(a-2)(a+2)(a^2+2) \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ a^4-8 & -1 \end{pmatrix}.$$

If

$$M = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix},$$

where

$$\omega = \frac{1}{8}a^2(2-a^2), \tag{6.5}$$

then

$$\begin{aligned} &M^{-1}AM = C^{-1}BAB^{-1}, && M^{-1}BM = BA^2(C^{-1}BAB^{-1})^2, \\ &M^{-1}CM = C, && M^{-1}DM = C^{-1}BA^2DA^{-2}B^{-1}C. \end{aligned} \tag{6.6}$$

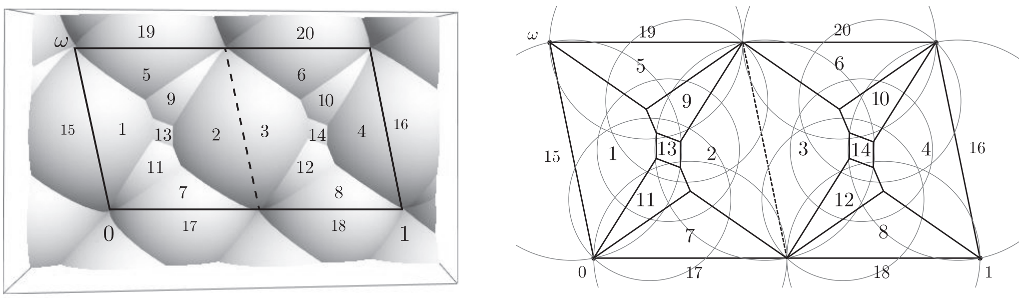


Figure 6.2. A bird's-eye view of isometric spheres (left).

Let $\Gamma_2 = \langle M \rangle \ltimes G_2$ be the group defined by (6.6). We define Euclidean motions

$$S = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -i & (1+\omega)i \\ 0 & i \end{pmatrix}, \quad U = \begin{pmatrix} -i & (1/2+\omega)i \\ 0 & i \end{pmatrix}.$$

Then we have

$$\begin{aligned} &SAS^{-1} = A^{-1}C^{-1}, && SBS^{-1} = M^{-1}CB^{-1}M^{-1}, && SCS^{-1} = C, && SMS^{-1} = M, \\ &TAT^{-1} = CA, && TBT^{-1} = A^2B^{-1}(CA)^2, && TCT^{-1} = C^{-1}, && TMT^{-1} = M^{-1}, \\ &UAU^{-1} = A^{-1}, && UBU^{-1} = M^2C^{-1}B, && UCU^{-1} = C^{-1}, && UMU^{-1} = M^{-1} \end{aligned}$$

and thus $S\Gamma_2S^{-1} = T\Gamma_2T^{-1} = U\Gamma_2U^{-1} = \Gamma_2$. Hence Γ_2 is invariant under the translation $S(z) = z + 1/2$, the rotation $T(z) = -z + (1+\omega)$ about $(1+\omega)/2$ through the angle

π and the rotation $U(z) = -z + (1/2 + \omega)$ about $(1/2+\omega)/2$ through the angle π. This accounts for symmetries shown in Figure 6.2.

Let $\mathscr{R}$ denote the region bounded by the isometric spheres

$$I(A^{-1}) = S\left(\frac{\omega}{2}, \frac{1}{2|a|}\right), \qquad I(A) = S\left(\frac{\omega+1}{2}, \frac{1}{2|a|}\right),$$
$$I(A^{-1}C^{-1}) = S\left(\frac{\omega}{2}+1, \frac{1}{2|a|}\right), \qquad I(BA^{-2}) = S\left(\omega + \frac{1}{4}, \frac{1}{4}\right),$$
$$I(MA^2B^{-1}M^{-1}) = S\left(\omega + \frac{3}{4}, \frac{1}{4}\right), \qquad I(M^{-1}BA^{-2}M) = S\left(\frac{1}{4}, \frac{1}{4}\right),$$
$$I(A^2B^{-1}) = S\left(\frac{3}{4}, \frac{1}{4}\right), \qquad I(BA^{-1}) = S\left(\frac{3a^2-a^4}{8}, \frac{2}{|a|^4}\right),$$
$$I(AB^{-1}M^{-1}) = S\left(\frac{4+3a^2-a^4}{8}, \frac{2}{|a|^4}\right), \quad I(MBA) = S\left(\frac{4-a^2}{8}, \frac{2}{|a|^4}\right),$$
$$I(A^{-1}B^{-1}M^{-1}) = S\left(\frac{8-a^2}{8}, \frac{2}{|a|^4}\right), \qquad I(B) = S\left(\frac{\omega}{2}+\frac{1}{4}, \frac{1}{|a|^3}\right),$$
$$I(MB^{-1}M^{-1}) = S\left(\frac{\omega}{2}+\frac{3}{4}, \frac{1}{|a|^3}\right)$$

and the planes orthogonal to the complex plane

$$I(C) = \{z+tj : \mathrm{Im}\,(z\bar{\omega}) = 0\}, \; I(C^{-1}) = \{z+tj : \mathrm{Im}\,((z-1)\bar{\omega}) = 0\},$$
$$I(M) = \{z+tj : \mathrm{Im}\,(z) = 0\}, \quad I(M^{-1}) = \{z+tj : \mathrm{Im}\,(z-\omega) = 0\}.$$

For convenience we call $I(C)$ the isometric sphere of C, and so on. Since

$$SMBAS^{-1} = CB^{-1}M^{-1}A^{-1}C^{-1}, \; TMBAT^{-1} = AB^{-1}M^{-1},$$
$$UMBAU^{-1} = MC^{-1}BA^{-1},$$

we have

$$S(I(MBA)) = I(B^{-1}M^{-1}A^{-1}), \; T(I(MBA)) = I(AB^{-1}M^{-1}),$$
$$U(I(MBA)) = I(BA^{-1}).$$

The faces s_i of $\mathscr{R}$ labeled as in Figure 6.2 are:

$s_1 \subset I(A^{-1})$, $s_2 \subset I(A)$, $s_3 \subset I(A) = I(CA)$, $s_4 \subset I(A^{-1}C^{-1})$,
$s_5 \subset I(BA^{-2})$, $s_6 \subset I(MA^2B^{-1}M^{-1})$, $s_7 \subset I(M^{-1}BA^{-2}M)$, $s_8 \subset I(A^2B^{-1})$,
$s_9 \subset I(BA^{-1})$, $s_{10} \subset I(AB^{-1}M^{-1})$, $s_{11} \subset I(MBA)$, $s_{12} \subset I(A^{-1}B^{-1}M^{-1})$,
$s_{13} \subset I(B)$, $s_{14} \subset I(MB^{-1}M^{-1})$, $s_{15} \subset I(C)$, $s_{16} \subset I(C^{-1})$,
$s_{17} \subset I(M)$, $s_{18} \subset I(M)$, $s_{19} \subset I(M^{-1})$, $s_{20} \subset I(M^{-1})$.

Side-pairings of $\mathscr{R}$ are:

$$\{s_1, s_2; A^{-1}\}, \quad \{s_3, s_4; CA\}, \quad \{s_5, s_6; MBA^{-2}\}, \{s_7, s_8; BA^{-2}M\},$$
$$\{s_9, s_{10}; MBA^{-1}\}, \{s_{11}, s_{12}; MBA\}, \{s_{13}, s_{14}; MB\}, \quad \{s_{15}, s_{16}; C\},$$
$$\{s_{17}, s_{19}; M\}, \quad \{s_{18}, s_{20}; M\}.$$

Here again $\{s_i, s_j, P\}$ means $P(s_i) = s_j$. Let $s_{i,j} = s_{j,i}$ be the common edge of s_i and s_j. The cycles of edges and cycle transformations of $\mathscr{R}$ are

$$\begin{aligned}
&\sigma_1 = \{s_{7,11}, s_{12,3}, s_{4,8}\}, && M^{-1}A^2B^{-1} \circ CA \circ MBA = 1,\\
&\sigma_2 = \{s_{1,11}, s_{12,8}, s_{7,2}\}, && A \circ M^{-1}A^2B^{-1} \circ MBA = 1,\\
&\sigma_3 = \{s_{10,6}, s_{5,1}, s_{2,9}\}, && MBA^{-1} \circ A^{-1} \circ A^2B^{-1}M^{-1} = 1,\\
&\sigma_4 = \{s_{4,10}, s_{9,5}, s_{6,3}\}, && CA \circ MBA^{-2} \circ AB^{-1}M^{-1} = 1,\\
&\sigma_5 = \{s_{11,13}, s_{14,3}, s_{4,12}\}, && A^{-1}B^{-1}M^{-1} \circ CA \circ MB = 1,\\
&\sigma_6 = \{s_{2,13}, s_{14,10}, s_{9,1}\}, && A^{-1} \circ AB^{-1}M^{-1} \circ MB = 1,\\
&\sigma_7 = \{s_{9,13}, s_{14,4}, s_{3,10}\}, && AB^{-1}M^{-1} \circ A^{-1}C^{-1} \circ MB = 1,\\
&\sigma_8 = \{s_{1,13}, s_{14,12}, s_{11,2}\}, && A \circ A^{-1}B^{-1}M^{-1} \circ MB = 1,\\
&\sigma_9 = \{s_{1,15}, s_{16,4}, s_{3,2}\}, && A \circ A^{-1}C^{-1} \circ C = 1,\\
&\sigma_{10} = \{s_{7,17}, s_{19,5}, s_{6,20}, s_{18,8}\}, && M^{-1}A^2B^{-1} \circ M^{-1} \circ MBA^{-2} \circ M = 1.
\end{aligned}$$

We will verify that each cycle satisfies the condition (vi) in Section 6.2. Let $S_1 = \{\zeta : |\zeta - c_1| = r_1\}$ and $S_2 = \{\zeta : |\zeta - c_2| = r_2\}$ be two spheres with $c_1, c_2 \in \mathbb{C}$ and D be the region $\{\zeta : |\zeta - c_1| > r_1, |\zeta - c_2| > r_2\}$. If S_1 and S_2 meets, then the dihedral angle between them measured from inside D is $\pi - \arg \Phi(S_1, S_2)$, where

$$\Phi(S_1, S_2) = r_1^2 + r_2^2 - d^2 + \sqrt{\Delta(d, r_1, r_2)}, \tag{6.7}$$

with $d = |c_1 - c_2|$ and

$$\Delta(d, r_1, r_2) = (d + r_1 + r_2)(d - r_1 + r_2)(d + r_1 - r_2)(d - r_1 - r_2).$$

For the cycle σ_1,

$$\begin{aligned}
\alpha(s_{7,11}) &= \pi - \arg \Phi\big(I(M^{-1}BA^{-2}M), I(MBA)\big),\\
\alpha(s_{12,3}) &= \pi - \arg \Phi\big(I(A^{-1}B^{-1}M^{-1}), I(CA)\big),\\
\alpha(s_{4,8}) &= \pi - \arg \Phi\big(I(A^{-1}C^{-1}), I(A^2B^{-1})\big).
\end{aligned}$$

If $S_1 = I(M^{-1}BA^{-2}M)$ and $S_2 = I(MBA)$, then $d = |a^2 - 2|/8 = 1/|a|^3$, since $a^2 - 2 = 8i/a^3$, $r_1 = 1/4$ and $r_2 = 2/|a|^4$, and hence

$$\Phi_1 = \Phi\big(I(M^{-1}BA^{-2}M), I(MBA)\big) = \frac{1}{16x^8}\left(x^8 + 4^3 - 4^2x^2 + \sqrt{\Delta}\right),$$

where $x = |a|$ and

$$\Delta = (x-2)(x+2)(x^3 - 2x^2 + 4x - 4)(x^3 + 2x^2 + 4x + 4)(x^4 - 4x + 8)(x^4 + 4x + 8).$$

If $S_1 = I(A^{-1}B^{-1}M^{-1})$ and $S_2 = I(A)$, then $d = |a^4 - 4a^2 + 8|/16 = 4/|a|^5$, since $a^4 - 4a^2 + 8 = 64i/a^5$, $r_1 = 2/|a|^4$ and $r_2 = 1/(2|a|)$, and hence

$$\Phi_2 = \Phi\big(I(A^{-1}B^{-1}M^{-1}), I(A)\big) = \frac{1}{4x^{10}}\left(x^8 - 4^3 + 4^2x^2 + \sqrt{\Delta}\right).$$

If $S_1 = I(A^{-1}C^{-1})$ and $S_2 = I(A^2B^{-1})$, then $d = |a^4 - 2a^2 - 4|/16 = |a|^3/16$, since $a^4 - 2a^2 - 4 = ia^3$, $r_1 = 1/2|a|$ and $r_2 = 1/4$, and hence

$$\Phi_3 = \Phi\big(I(A^{-1}C^{-1}), I(A^2B^{-1})\big) = \frac{1}{16x^2}\left(-x^8 + 4^3 + 4^2x^2 + \sqrt{\Delta}\right).$$

Therefore $\arg(\Phi_1\Phi_2\Phi_3) = \arg(-2^{13}x^{10}) = \pi$ and we have

$$\alpha(\sigma_1) = 3\pi - \arg(\Phi_1\Phi_2\Phi_3) = 2\pi.$$

By using the symmetries S, T and U we have also $\alpha(\sigma_2) = \alpha(\sigma_3) = \alpha(\sigma_4) = 2\pi$.

For the cycle σ_5,

$$\begin{aligned}\alpha(s_{11,13}) &= \pi - \arg\Phi\big(I(MBA), I(B)\big),\\ \alpha(s_{14,3}) &= \pi - \arg\Phi\big(I(MB^{-1}M^{-1}), I(CA)\big),\\ \alpha(s_{4,12}) &= \pi - \arg\Phi\big(I(A^{-1}C^{-1}), I(A^{-1}B^{-1}M^{-1})\big).\end{aligned}$$

If $S_1 = I(MBA)$ and $S_2 = I(B)$, then $d = |a^2 - 2|^2/16 = 4/|a|^6$, since $a^2 - 2 = 8i/a^3$, $r_1 = 2/|a|^4$ and $r_2 = 1/|a|^3$, and hence

$$\Phi_1 = \Phi\big(I(MBA), I(B)\big) = \frac{1}{x^{12}}\left(-4^2 + 4x^4 + x^6 + \sqrt{\Delta}\right),$$

where $x = |a|$ and

$$\Delta = (-4 - 2x^2 + x^3)(4 - 2x^2 + x^3)(-4 + 2x^2 + x^3)(4 + 2x^2 + x^3).$$

If $S_1 = I(MB^{-1}M^{-1})$ and $S_2 = I(CA)$, then $d = 1/4$, $r_1 = 1/|a|^3$ and $r_2 = 1/(2|a|)$, and hence

$$\Phi_2 = \Phi\big(I(MB^{-1}M^{-1}), I(CA)\big) = \frac{1}{4^2x^6}\left(4^2 + 4x^4 - x^6 + \sqrt{\Delta}\right).$$

If $S_1 = I(A^{-1}C^{-1})$ and $S_2 = I(A^{-1}B^{-1}M^{-1})$, then $d = |4a^2 - a^4|/16 = 1/|a|^2$, since $a^2 - 4 = -16/a^4$, $r_1 = 1/2|a|$ and $r_2 = 2/|a|^4$, and hence

$$\Phi_3 = \Phi\big(I(A^{-1}C^{-1}), I(A^{-1}B^{-1}M^{-1})\big) = \frac{1}{4x^8}\left(4^2 + x^6 - 4x^4 + \sqrt{\Delta}\right).$$

Therefore $\arg(\Phi_1\Phi_2\Phi_3) = \arg(-8x^{-16}) = \pi$ and we have

$$\alpha(\sigma_5) = 3\pi - \arg(\Phi_1\Phi_2\Phi_3) = 2\pi.$$

By using the symmetries we have also $\alpha(\sigma_6) = \alpha(\sigma_7) = \alpha(\sigma_8) = 2\pi$. For the cycles σ_9

and σ_{10},

$$\alpha(\sigma_9) = \alpha(s_{1,15}) + \alpha(s_{16,4}) + \alpha(s_{3,2}) = \pi/2 + \pi/2 + \pi = 2\pi,$$

and

$$\alpha(\sigma_{10}) = \alpha(s_{7,17}) + \alpha(s_{19,5}) + \alpha(s_{6,20}) + \alpha(s_{18,8}) = \pi/2 + \pi/2 + \pi/2 + \pi/2 = 2\pi.$$

(Note that $s_{15}, s_{16}, \ldots, s_{20}$ are planes vertical to $\mathbb{C}$.) By the Poincaré polyhedron theorem we conclude that the subgroup generated by the side-pairing transformations is a Kleinian group. Since C, CA, M and MB are side-pairing transformations and $A = C^{-1}(CA)$, $B = M^{-1}(MB)$, this group coincides with Γ_2.

Example 3. Let

$$A = \begin{pmatrix} -1 & 2i \\ 2i & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3i & 2-i \\ 2+i & -2i \end{pmatrix},$$
$$C = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -1+2i & 1 \\ 4 & -1-2i \end{pmatrix}.$$

The group Γ_3 generated by A, B and C is a subgroup of $PSL(2, \mathbb{Z}[i])$ and hence a Kleinian group. Besides C and D, the matrices A,

$$BA^{-1}B^{-1} = \begin{pmatrix} 3-2i & -4 \\ -2i & -1+2i \end{pmatrix} \text{ and } CD = [B, A] = \begin{pmatrix} -3-2i & 2i \\ -4 & 1+2i \end{pmatrix}$$

are also parabolic. Although $ABA^{-1}B^{-1}CD = I$ holds, Γ_3 is not a faithful representation of the twice punctured torus group, for B^2AC and $BABC$ are elliptic of order 2. Let

$$M = \begin{pmatrix} \sqrt{2} & -i\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

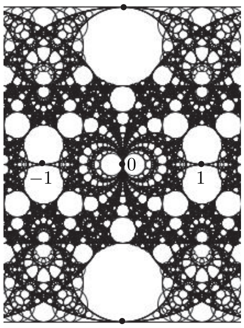
and replace $\gamma \in \Gamma_3$ by $M\gamma M^{-1}$. So

$$A = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1+i & -2 \\ 1+i/2 & -1 \end{pmatrix},$$
$$C = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -1-2i & 2 \\ 2 & -1+2i \end{pmatrix}.$$

Let ℓ_1 be the imaginary axis, ℓ_2 the vertical line through 1 and ℓ_3 the real axis. The limit set of Γ_3 is symmetric in the lines ℓ_1, ℓ_2 and ℓ_3. To see this, let $R_1(z) = -\bar{z}$, $R_2(z) = -\bar{z} + 2$ and $R_3(z) = \bar{z}$ be the reflections in ℓ_1, ℓ_2 and ℓ_3, respectively. If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the Möbius transformations $J_k(M) = R_j \circ M \circ R_j^{-1}$ for $j = 1, 2, 3$ is represented by

$$J_1(M) = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}, \quad J_2(M) = \begin{pmatrix} \bar{a} - 2\bar{c} & -2\bar{a} - \bar{b} + 4\bar{c} + 2\bar{d} \\ -\bar{c} & 2\bar{c} + \bar{d} \end{pmatrix},$$

2i



−2i

Figure 6.3. A part of the limit set of Γ_3.

$$J_3(M) = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

We have

$$J_1(A) = A, \;\; J_1(B) = C^{-1}A^{-1}B^{-1}C, \;\; J_1(C) = C^{-1},$$
$$J_2(A) = BAB^{-1}, \;\; J_2(B) = A^{-1}B^{-1}, \;\; J_2(C) = C^{-1},$$

and

$$J_3(A) = A^{-1}, \;\; J_3(B) = CAC^{-1}B, \;\; J_3(C) = C.$$

Hence Γ_3 is invariant under J_1, J_2 and J_3. Let

$$N = \begin{pmatrix} 1/\sqrt{2} & -i\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}.$$

Then

$$NB^2ACN^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad NCN^{-1} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Thus the subgroup $K_1 = \langle B^2A, C\rangle$ is a conjugate of the modular group $\Gamma_0 = PSL(2,\mathbb{Z})$ in $PSL(2,\mathbb{C})$, and so is the subgroup $K_2 = \langle BAB, C\rangle = J_1J_3(K_1)$. The region of discontinuity of K_1 is the union of $\Omega^+(K_1) = \{z : \mathrm{Im}\,(z) > 2\}$ and its reflection in the line $\mathrm{Im}\,z = 2$, and that of K_2 is the union of $\Omega^-(K_2) = \{z : \mathrm{Im}\,(z) < -2\}$ and its reflection in the line $\mathrm{Im}\,z = -2$. Figure 6.4 shows orthogonal projections of isometric spheres:

$$I(B^2AC) = S(2i,2), \;\; I(BABC) = S(-2i,2),$$
$$I(B^2AB) = S(1,1), \;\; I(C^{-1}B^2ABC) = S(-1,1),$$
$$I(C^{-1}) = \{\zeta = z + tj : \mathrm{Re}\,z = -1\}, \;\; I(C) = \{\zeta = z + tj : \mathrm{Re}\,z = 1\}.$$

We call $I(C)$ and $I(C^{-1})$ the isometric spheres of C and C^{-1} for convenience.

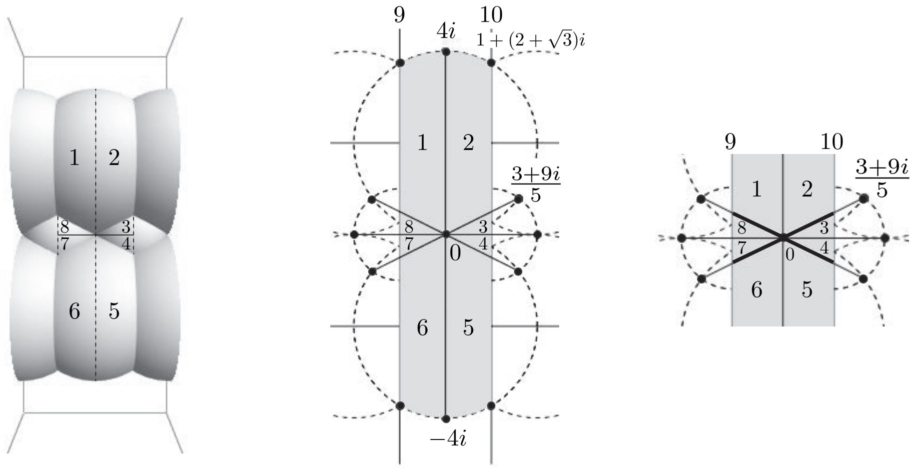


Figure 6.4. A bird's-eye view of isometric spheres (left).

Let $\mathscr{R}$ denote the region bounded by these planes and s_i $(i = 1, 2, \ldots, 10)$ the face of $\mathscr{R}$ labeled as in Figure 6.4: $s_1, s_2 \subset I(B^2AC)$, $s_3, s_4 \subset I(B^2AB)$, $s_5, s_6 \subset I(BABC)$, $s_7, s_8 \subset I(C^{-1}B^2ABC)$, $s_9 \subset I(C^{-1})$, $s_{10} \subset I(C)$. If $s_{i,j} = s_{j,i}$ denote the common edge of s_i and s_j, then

$$s_{1,2} \subset Fix(B^2AC),\ s_{3,4} \subset Fix(B^2AB),\ s_{2,10} \subset Fix(B^2A).$$

Here note that B^2AC and B^2AB are elliptic of order 2 and B^2A is of order 3. It can be shown by using symmetry that other edges are also in the fixed point sets of elliptic elements. Side-pairings of $\mathscr{R}$ are as follows:

$$\{s_1, s_2; B^2AC\},\ \{s_3, s_4; B^2AB\},\ \{s_5, s_6; BABC\},$$
$$\{s_7, s_8; C^{-1}B^2ABC\},\ \{s_9, s_{10}; C\}.$$

Here, again $\{s_i, s_j, P\}$ means $P(s_i) = s_j$. The subgroup of Γ_3 generated by the side-pairing transformations is identical with Γ_3 because

$$B = (B^2AB)C(BABC)^{-1}, \quad A = B^{-2}(B^2AC)C^{-1}.$$

The cycles of edges for $\mathscr{R}$ are:

$$\sigma_1 = \{s_{1,2}\},\ \sigma_2 = \{s_{3,4}\},\ \sigma_3 = \{s_{5,6}\},\ \sigma_4 = \{s_{7,8}\},$$
$$\sigma_5 = \{s_{1,9}, s_{10,2}\},\ \sigma_6 = \{s_{5,10}, s_{9,6}\},$$
$$\sigma_7 = \{s_{2,3}, s_{4,5}, s_{6,7}, s_{8,1}\}.$$

Dihedral angles at the edges are:

$$\alpha(s_{1,2}) = \alpha(s_{3,4}) = \alpha(s_{5,6}) = \alpha(s_{7,8}) = \pi,$$
$$\alpha(s_{1,9}) = \alpha(s_{2,10}) = \alpha(s_{5,10}) = \alpha(s_{6,9}) = \pi/3,$$
$$\alpha(s_{2,3}) = \alpha(s_{4,5}) = \alpha(s_{6,7}) = \alpha(s_{1,8}) = \pi/2.$$

Hence, by Poincaré polyhedron theorem, $\mathscr{R}$ is a fundamental polyhedron for Γ_3. The closure of $\mathscr{R}$ in $\{\zeta = z + tj : t \geq 0\}$ meets $\mathbb{C}$ in the sets $D_+ \cup D_-$, where

$$D_\pm = \left\{z : -1 \leq \operatorname{Re} z \leq 1, \pm(\operatorname{Im} z) > 2, |z - (\pm 2i)| \geq 2\right\},$$

and D_+ is a fundamental set for K_1 and D_- is a fundamental set for K_2. Therefore, if $\Omega(\Gamma_3)$ denotes the region of discontinuity for Γ_3, then

$$\Omega(\Gamma_3)/\Gamma_3 = \Omega^+(K_1)/K_1 \cup \Omega^-(K_2)/K_2.$$

Each of its two components is an orbifold of type $(0; 2, 3, \infty)$. Although Γ_3 and $PSL(2,\mathbb{Z})$ are not conjugate in $PSL(2,\mathbb{C})$, we have an isometry $\Omega(\Gamma_3)/\Gamma_3 \cong (\mathbb{C} - \mathbb{R})/PSL(2,\mathbb{Z})$.

Figures 6.1–6.4 are produced by Wolfram Mathematica.

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