



島根大学学術情報リポジトリ
S W A N
Shimane University Web Archives of kNnowledge

Title

Signed analogue of line graphs and their smallest eigenvalues

Author(s)

Alexander L. Gavriilyuk, Akihiro Munemasa, Yoshio Sano, Tetsuji Taniguchi

Journal

Journal of Graph Theory Volume 98, Issue 2 p. 309-325

Published

08 June 2021

URL

<https://doi.org/10.1002/jgt.22699>

これは出版社版ではありません。引用時は出版社版をご確認のうえご利用ください。

This is the pre-peer reviewed version of the following article:

<https://onlinelibrary.wiley.com/doi/full/10.1002/jgt.22699>, which has been published in final form at <https://doi.org/10.1002/jgt.22699>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Use of Self-Archived Versions."

SIGNED ANALOGUE OF LINE GRAPHS AND THEIR SMALLEST EIGENVALUES

ALEXANDER L. GAVRILYUK, AKIHIRO MUNEMASA, YOSHIO SANO,
AND TETSUJI TANIGUCHI

ABSTRACT. In this paper, we show that every connected signed graph with smallest eigenvalue strictly greater than -2 and large enough minimum degree is switching equivalent to a complete graph. This is a signed analogue of a theorem of Hoffman. The proof is based on what we call Hoffman's limit theorem which we formulate for Hermitian matrices, and also the extension of the concept of Hoffman graph and line graph for the setting of signed graphs.

1. INTRODUCTION

Let G be a simple graph with the vertices-versus-edges incidence $(0, 1)$ -matrix N . It is well known that the line graph $L(G)$ of G , whose vertices are the edges of G with two edges being adjacent whenever they are incident, has adjacency matrix

$$A(L(G)) = N^\top N - 2I_{|E(G)|},$$

and hence its smallest eigenvalue λ_{\min} is at least -2 . Although this property is not exclusive, a theorem of Cameron, Goethals, Shult, and Seidel [4], which is one of the most beautiful results in algebraic graph theory, classifies connected graphs having $\lambda_{\min} \geq -2$. Namely, such a graph L on at least 37 vertices must be a generalized line graph in the sense that its adjacency matrix satisfies

$$A(L) = N^\top N - 2I,$$

for some $(0, \pm 1)$ -matrix N . (A combinatorial definition of generalized line graphs can be found in [7].) The proof relies on the classification of root systems in Euclidean space [4].

Another approach, which sheds light on the structure of graphs with $\lambda_{\min} \geq -2$, was developed by Hoffman [7]. It is based on Ramsey's theorem and a special class of vertex-colored graphs, which are called Hoffman graphs by Woo and Neumaier [16]. Hoffman's results not only

Date: April 6, 2021.

2010 Mathematics Subject Classification. 05C50, 05C22, 15A18, 15B57.

show that graphs with $\lambda_{\min} \geq -2$ and on sufficiently large number of vertices are generalized line graphs, but also establish the existence of some limit points of the smallest eigenvalues of a sequence of graphs G with increasing minimum degree $\delta(G)$.

Theorem 1.1 ([7]). *There exists an integer valued function h defined on the half-open interval $(-1 - \sqrt{2}, -1]$, such that*

- (1) *for each $\lambda \in (-2, -1]$, if G is a connected graph with $\lambda_{\min}(G) \geq \lambda$ and $\delta(G) \geq h(\lambda)$, then G is a clique and $\lambda_{\min}(G) = -1$;*
- (2) *for each $\lambda \in (-1 - \sqrt{2}, -2]$, if G is a connected graph with $\lambda_{\min}(G) \geq \lambda$ and $\delta(G) \geq h(\lambda)$, then G is a generalized line graph and $\lambda_{\min}(G) = -2$.*

This means that the intervals $(-2, -1)$ and $(-1 - \sqrt{2}, -2)$ are ignorable if our concern is the smallest eigenvalues of graphs with sufficiently large minimum degree. Note that Hoffman [7] stated this theorem in a slightly weaker manner, but his proof shows that the above statement is valid (see [16, Theorem 5.5 and Remark 5.6]).

As a tool for the proof of Theorem 1.1, Hoffman showed what we would call Hoffman's limit theorem (sometimes attributed to Ostrowski and Hoffman due to their unpublished work; see [8]). A very terse proof of this theorem was given by Hoffman in [9, Lemma 2.2] (see also [10, Theorem 2.14]).

Signed graphs can be thought of as simple graphs whose edges get labels from the set $\{+1, -1\}$. A natural extension of $(0, 1)$ -matrices associated with graphs to $(0, \pm 1)$ -matrices allows to study the spectra of signed graphs. The spectral theory of signed graphs has received much attention recently [1], in particular, Problem 3.17 from [1] suggests to extend the so-called Hoffman theory, including the above-mentioned Theorem 1.1, to signed graphs.

In this paper, we first provide a detailed proof of a slightly generalized version of Hoffman's limit theorem (see Theorem 3.6), which is then used to show our main result: an analogue of Part (1) of Theorem 1.1 for signed graphs (see Section 2 for precise definitions).

Theorem 1.2. *There exists an integer valued function f defined on the half-open interval $(-2, -1]$ such that, for each $\lambda \in (-2, -1]$, if a connected signed graph S satisfies $\lambda_{\min}(S) \geq \lambda$, and $\delta(S) \geq f(\lambda)$, then S is switching equivalent to a complete graph (and hence $\lambda_{\min}(S) = -1$).*

Note that the proof of Theorem 1.1 in [7] uses Ramsey's theorem, which produces astronomical estimates for $h(\lambda)$. In proving Theorem 1.2 in Section 4, we introduce the notion of signed Hoffman graphs,

and then involve a structural classification of signed graphs with smallest eigenvalue greater than -2 from [6], which in turn relies on the root systems. This enables one to obtain close to tight estimates for $f(\lambda)$.

In Section 5, we consider a signed analogue of generalized line graphs. It would be interesting to see whether an analogue of Part (2) of Theorem 1.1 can be shown. Note that Woo and Neumaier [16] went on further to extend Hoffman's ideas to graphs with smallest eigenvalue at least α , where $\alpha \approx -2.4812$ is a zero of the cubic polynomial $x^3 + 2x^2 - 2x - 2$. Finally, Koolen, Yang and Yang [12] recently proved a version of Theorem 1.1 for simple graphs with smallest eigenvalue at least -3 .

2. PRELIMINARIES

A **signed graph** S is a triple (V, E^+, E^-) of a set V of vertices, a set E^+ of 2-subsets of V (called **(+)-edges**, or **positive** edges), and a set E^- of 2-subsets of V (called **(-)-edges**, or **negative** edges) such that $E^+ \cap E^- = \emptyset$. A signed graph in which $E^- = \emptyset$ is called an **unsigned graph** or simply, a graph.

Let S be a signed graph. We denote the set of vertices of S by $V(S)$, the set of (+)-edges of S by $E^+(S)$, and the set of (-)-edges of S by $E^-(S)$. By a **subgraph** $S' = (V(S'), E^+(S'), E^-(S'))$ of S we mean a vertex induced signed subgraph, i.e., $V(S') \subseteq V(S)$ and $E^\pm(S') = \{\{x, y\} \in E^\pm(S) \mid x, y \in V(S')\}$. If S' is a subgraph of S , then we say that S **contains** S' . The **underlying graph** $U(S)$ of S is the unsigned graph $(V(S), E^+(S) \cup E^-(S))$. The **minimum degree** $\delta(S)$ of S is defined to be the minimum degree of $U(S)$. The signed graph S is **connected** if $U(S)$ is connected.

Two signed graphs S and S' are said to be **isomorphic** if there exists a bijection $\phi : V(S) \rightarrow V(S')$ such that $\{u, v\} \in E^+(S)$ if and only if $\{\phi(u), \phi(v)\} \in E^+(S')$ and that $\{u, v\} \in E^-(S)$ if and only if $\{\phi(u), \phi(v)\} \in E^-(S')$. For a signed graph S , we define its **adjacency matrix** $A(S)$ by

$$(A(S))_{uv} = \begin{cases} 1 & \text{if } \{u, v\} \in E^+(S), \\ -1 & \text{if } \{u, v\} \in E^-(S), \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of S are defined to be those of $A(S)$.

A **switching** at a vertex v is the process of swapping the signs of each edge incident to v . Two signed graphs S and S' are said to be **switching equivalent** if there exists a subset $W \subseteq V(S)$ such that S' is isomorphic to the graph obtained by switching at each vertex in W .

Note that switching equivalent signed graphs have the same multiset of eigenvalues.

Let S be a signed graph with smallest eigenvalue at least -2 . A **representation** of S is a mapping ϕ from $V(S)$ to \mathbb{R}^n for some positive integer n such that $(\phi(u), \phi(v)) = \pm 1$ if $\{u, v\} \in E^\pm(S)$ respectively, and $(\phi(u), \phi(v)) = 2\delta_{u,v}$ otherwise, where $\delta_{u,v}$ is Kronecker's delta, i.e., $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ if $u \neq v$. Since $A(S) + 2I$ is positive semidefinite, it is the Gram matrix of a set of vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$. These vectors satisfy $(\mathbf{x}_i, \mathbf{x}_i) = 2$ and $(\mathbf{x}_i, \mathbf{x}_j) = 0, \pm 1$ for $i \neq j$. Sets of vectors satisfying these conditions determine line systems. We denote by $[\mathbf{x}]$ the line determined by a nonzero vector \mathbf{x} , in other words, $[\mathbf{x}]$ is the one-dimensional subspace spanned by \mathbf{x} . We say that S is **represented** by the line system \mathcal{S} if S has a representation ϕ such that $\mathcal{S} = \{[\phi(v)] : v \in V(S)\}$.

Below we give descriptions of three line systems, A_n , D_n and E_8 . Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be an orthonormal basis for \mathbb{R}^n .

$$\begin{aligned} A_n &= \{[\mathbf{e}_i - \mathbf{e}_j] : 1 \leq i < j \leq n+1\} \quad (n \geq 1), \\ D_n &= A_{n-1} \cup \{[\mathbf{e}_i + \mathbf{e}_j] : 1 \leq i < j \leq n\} \quad (n \geq 4), \\ E_8 &= D_8 \cup \left\{ \left[\frac{1}{2} \sum_{i=1}^8 \epsilon_i \mathbf{e}_i \right] : \epsilon_i \in \{\pm 1\}, \prod_{i=1}^8 \epsilon_i = 1 \right\}. \end{aligned}$$

These line systems are used in the following classical result of Cameron, Goethals, Shult, and Seidel.

Theorem 2.1 ([4]). *Let G be a connected graph with $\lambda_{\min}(G) \geq -2$. Then G is represented by a subset of either D_n or E_8 .*

Let S be a signed graph represented by a line system \mathcal{S} . If \mathcal{S} can be embedded into \mathbb{Z}^n for some n , then we say that S is **integrally represented** or that S has an **integral representation**. By Theorem 2.1, for a signed graph S with $\lambda_1(S) \geq -2$, G has an integral representation if and only if S is represented by a subset of D_n for some n . Let S be a connected signed graph with $\lambda_1(S) \geq -2$. We call S **exceptional** if it does not have an integral representation. Clearly there are only finitely many exceptional signed graphs.

Let S be a signed graph with smallest eigenvalue greater than -2 . Assume that S has an integral representation ϕ in \mathbb{R}^n . This means that, with $m = |V(S)|$, there exists an $n \times m$ matrix

$$M = (\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_m),$$

with entries in \mathbb{Z} , such that $(\mathbf{v}_i, \mathbf{v}_j) = \pm 1$ if $\{i, j\} \in E^\pm(S)$ respectively, and $(\mathbf{v}_i, \mathbf{v}_j) = 2\delta_{i,j}$ otherwise. We may assume that M has no rows

consisting only of zeros. Since $\mathbf{v}_i \in \mathbb{Z}^n$, \mathbf{v}_i has two entries equal to ± 1 , and all other entries 0. Let H be the graph with vertex set $\{1, \dots, n\}$, where vertices i and j are joined by the edge k whenever \mathbf{v}_k has ± 1 in its i th and j th positions. Note that the graph H may have multiple edges. A graph without multiple edges is called **simple**. We call H the **representation graph** of S associated with the representation ϕ . Note that H has no isolated vertex. If S is connected, then so is H .

Let S be an m -vertex connected signed graph having an integral representation ϕ and smallest eigenvalue greater than -2 . Let H be the n -vertex representation graph of S associated with the representation ϕ . Then by [6, Lemma 5], we have $n \in \{m, m+1\}$. Moreover, if $n = m$, then H is a unicyclic graph or a tree with a double edge and if $n = m+1$, then H is a tree.

For a simple graph H , we denote by $\mathfrak{L}(H)$ the line graph of H . If u and v are adjacent vertices in a graph, then we denote the edge $\{u, v\}$ by uv for brevity.

Let H be a unicyclic graph whose unique cycle C has at least 4 vertices and let $G = \mathfrak{L}(H)$. Then for each edge e of G there exists a unique maximal clique that contains e . For such a graph G , we denote by $\mathfrak{C}_G(e)$ the unique maximal clique of G containing the edge e . Let uu' be an edge of $\mathfrak{L}(C)$. Define $\mathfrak{L}^\dagger(H, uu')$ to be the signed graph (V, E^+, E^-) , where $V = V(\mathfrak{L}(H))$,

$$E^- = \{uv \in E(\mathfrak{L}(H)) \mid v \in \mathfrak{C}_{\mathfrak{L}(H)}(uu')\}$$

and $E^+ = E(\mathfrak{L}(H)) \setminus E^-$. Observe that, for all edges uu' and vv' of $\mathfrak{L}(C)$, the graph $\mathfrak{L}^\dagger(H, uu')$ is switching equivalent to $\mathfrak{L}^\dagger(H, vv')$.

Let H be a tree with a double edge u and u' , and let $H' = H - u'$ be the simple tree obtained from H by removing u' . We define $\mathfrak{L}(H)$ to be the signed graph obtained from the line graph $\mathfrak{L}(H')$ by attaching a new vertex u' , and join u' by (+)-edges to every vertex of a clique in the neighborhood of u , (−)-edges to every vertex of the other clique in the neighborhood of u . Note that there are two different ways to assign signs to edges from u' , but the resulting two signed graphs are switching equivalent.

Theorem 2.2 ([6, Theorem 6]). *Let S be a connected integrally represented signed graph having smallest eigenvalue greater than -2 . Let H be the representation graph of S for some integral representation. Then one of the following statements holds:*

- (i) H is a simple tree, and S is switching equivalent to the line graph $\mathfrak{L}(H)$,

- (ii) H is unicyclic with an odd cycle, and S is switching equivalent to the line graph $\mathfrak{L}(H)$,
- (iii) H is unicyclic with an even cycle C , and S is switching equivalent to $\mathfrak{L}^\dagger(H, uu')$ where uu' is an edge of $\mathfrak{L}(C)$.
- (iv) H is a tree with a double edge, and S is switching equivalent to $\mathfrak{L}(H)$.

Conversely, if S is a signed graph described by (i)–(iv) above, then S is integrally represented and has smallest eigenvalue greater than -2 .

Corollary 2.3. *Let S be a connected integrally represented signed graph having smallest eigenvalue greater than -2 . Then there exists a tree H such that $\mathfrak{L}(H)$ is switching equivalent to S with possibly one vertex removed.*

Proof. The assertion is clear if Theorem 2.2(i) holds. For the case (ii) of Theorem 2.2, let e be an edge of H contained in the unique cycle of H . Regarding e as a vertex of $\mathfrak{L}(H)$, we have $\mathfrak{L}(H) - e = \mathfrak{L}(H - e)$. Since S is switching equivalent to $\mathfrak{L}(H)$, $\mathfrak{L}(H) - e = \mathfrak{L}(H - e)$ is switching equivalent to S with one vertex removed. Since $H - e$ is a tree, the assertion holds.

For the case (iii) of Theorem 2.2, we proceed in a similar manner. Since $\mathfrak{L}^\dagger(H, uu') - u$ has no $(-)$ -edge, we have $\mathfrak{L}^\dagger(H, uu') - u = \mathfrak{L}(H - u)$, which is the line graph of a tree. Since S is switching equivalent to $\mathfrak{L}^\dagger(H, uu')$, $\mathfrak{L}(H - u)$ is switching equivalent to S with one vertex removed.

Finally, for the case (iv) of Theorem 2.2, let H be a tree with a double edge u and u' . Then $\mathfrak{L}(H) - u' = \mathfrak{L}(H - u')$ by construction. Since S is switching equivalent to $\mathfrak{L}(H)$, $\mathfrak{L}(H - u')$ is switching equivalent to S with one vertex removed. Since $H - u'$ is a simple tree, the assertion holds. \square

3. HOFFMAN'S LIMIT THEOREM

For $z \in \mathbb{C}$ and $\varepsilon > 0$, we define

$$\mathcal{D}(z; \varepsilon) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\}.$$

By a polynomial, we mean a polynomial with coefficients in \mathbb{C} .

Lemma 3.1. *Let $(f_n(z))_{n \in \mathbb{N}}$ be a sequence of polynomials of bounded degree. Suppose that this sequence converges to a nonzero polynomial $f(z)$ coefficient-wise. Then the following statements are equivalent for $\zeta \in \mathbb{C}$.*

- (i) $f(\zeta) = 0$,

(ii) for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$(1) \quad \forall n > n_0(\varepsilon), \{z \in \mathcal{D}(\zeta; \varepsilon) \mid f_n(z) = 0\} \neq \emptyset.$$

Proof. Since $\{z \in \mathbb{C} \mid f(z) = 0\}$ is finite, there exists $\varepsilon_0 > 0$ such that

$$\overline{\mathcal{D}(\zeta, \varepsilon_0)} \cap \{z \in \mathbb{C} \mid f(z) = 0\} \subseteq \{\zeta\}.$$

For $\varepsilon \in (0, \varepsilon_0]$, define

$$c_\varepsilon = \min\{|f(z)| \mid z \in \mathbb{C}, |z - \zeta| = \varepsilon\}.$$

Then $c_\varepsilon > 0$, and hence there exists $m_0(\varepsilon) \in \mathbb{N}$ such that

$$\forall n > m_0(\varepsilon), \max\{|f_n(z) - f(z)| \mid z \in \mathbb{C}, |z - \zeta| = \varepsilon\} < c_\varepsilon.$$

This implies that, for $n > m_0(\varepsilon)$, $|f_n(z) - f(z)| < |f(z)|$ for $z \in \mathbb{C}$, $|z - \zeta| = \varepsilon$. By Rouché's theorem [13, Theorem 1.3.7], we have, as multisets,

$$(2) \quad \forall n > m_0(\varepsilon), |\{z \in \mathcal{D}(\zeta, \varepsilon) \mid f(z) = 0\}| = |\{z \in \mathcal{D}(\zeta, \varepsilon) \mid f_n(z) = 0\}|.$$

(i) \Rightarrow (ii). Let $\varepsilon > 0$. Define $n_0(\varepsilon) = m_0(\min\{\varepsilon, \varepsilon_0\})$, and let $n > n_0(\varepsilon)$. Since $\zeta \in \{z \in \mathcal{D}(\zeta, \varepsilon) \mid f(z) = 0\}$, the left-hand side of (2) is nonzero. Thus $\{z \in \mathcal{D}(\zeta, \varepsilon) \mid f_n(z) = 0\} \neq \emptyset$. This proves (1).

(ii) \Rightarrow (i). Let $\varepsilon \in (0, \varepsilon_0]$. Then for $n > \max\{m_0(\varepsilon), n_0(\varepsilon)\}$, (1) and (2) imply

$$\mathcal{D}(\zeta; \varepsilon) \cap \{z \in \mathbb{C} \mid f(z) = 0\} \neq \emptyset.$$

Since $\varepsilon \in (0, \varepsilon_0]$ was arbitrary, we conclude $f(\zeta) = 0$. \square

Lemma 3.2. *Let $(f_n(z))_{n \in \mathbb{N}}$ be a sequence of real-rooted polynomials of bounded degree. Suppose that this sequence converges to a nonzero real-rooted polynomial $f(z)$ coefficient-wise, and the limit*

$$\delta = \lim_{n \rightarrow \infty} \min\{x \in \mathbb{R} \mid f_n(x) = 0\}$$

exists. Then $f(x)$ has a real root and

$$\delta = \min\{x \in \mathbb{R} \mid f(x) = 0\}.$$

Proof. By Lemma 3.1(ii) \Rightarrow (i), we obtain $f(\delta) = 0$. Suppose $\zeta < \delta$ and $f(\zeta) = 0$. Then by Lemma 3.1(i) \Rightarrow (ii), there exists $n_0 \in \mathbb{N}$ such that

$$\forall n > n_0, \{x \in \mathcal{D}(\zeta; (\delta - \zeta)/2) \mid f_n(x) = 0\} \neq \emptyset.$$

Since $f_n(z)$ are real-rooted, this implies

$$\forall n > n_0, \exists x < \frac{\zeta + \delta}{2}, f_n(x) = 0.$$

Thus, we obtain

$$\lim_{n \rightarrow \infty} \min\{x \in \mathbb{R} \mid f_n(x) = 0\} \leq \frac{\zeta + \delta}{2} < \delta,$$

which contradicts the assumption. \square

Lemma 3.3. *Let*

$$g(t, z) = h(z)t^N + \sum_{i=0}^{N-1} h_i(z)t^i \in \mathbb{C}[t, z]$$

be a polynomial, where $h(z)$ is a nonzero real-rooted polynomial. Suppose that the sequence $(g(t, z))_{t \in \mathbb{N}}$ of polynomials in z satisfies the following conditions:

- (i) $g(t, z)$ is real-rooted for all $t \in \mathbb{N}$,
- (ii) $\delta = \lim_{t \rightarrow \infty} \min\{x \in \mathbb{R} \mid g(t, x) = 0\}$ exists.

Then

$$\delta = \min\{x \in \mathbb{R} \mid h(x) = 0\}.$$

Proof. Let

$$f_t(z) = \frac{1}{t^N} g(t, z) = h(z) + \sum_{i=0}^{N-1} h_i(z)t^{i-N} \quad (t \in \mathbb{N}).$$

Then $(f_t(z))_{t \in \mathbb{N}}$ is a sequence of real-rooted polynomials of bounded degree, and it converges to $h(z)$ coefficient-wise. The result follows from Lemma 3.2. \square

Lemma 3.4 ([18, Lemma 3.32]). *Let*

$$\tilde{A} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$$

be a Hermitian matrix, and suppose D is positive definite. Then \tilde{A} is positive semidefinite if and only if $A - BD^{-1}B^*$ is positive semidefinite.

Lemma 3.5. *Let $s \geq \sqrt{2}$, and let $L \in M_{m \times n}(\mathbb{C})$. Let $D \in M_n(\mathbb{C})$ be a positive definite Hermitian matrix. If ℓ is the largest eigenvalue of L^*L , then*

$$\lambda_{\min} \left(\begin{bmatrix} 0 & sL \\ sL^* & (s^2 - 1)D \end{bmatrix} \right) > -2\ell.$$

Proof. Let λ_1 denote the left-hand side. If $L = 0$, then the inequality holds trivially, so assume $L \neq 0$. Then the matrix in question contains a 2×2 matrix with negative determinant. Thus $\lambda_1 < 0$. Let μ^1 be the smallest eigenvalue of D . For $\lambda < 0$, Lemma 3.4 implies

$$\begin{aligned} \begin{bmatrix} -\lambda I & sL \\ sL^* & (s^2 - 1 - \lambda)D \end{bmatrix} \succeq 0 &\iff (s^2 - 1 - \lambda)D + \frac{s^2}{\lambda}L^*L \succeq 0 \\ &\iff 0 \succeq s^2L^*L + \lambda(s^2 - 1 - \lambda)D \\ &\iff 0 \geq \ell s^2 + \lambda(s^2 - 1 - \lambda)\mu^1 \end{aligned}$$

$$\iff \lambda^2 - (s^2 - 1)\lambda - \frac{\ell s^2}{\mu^1} \geq 0.$$

Thus

$$\begin{aligned} \lambda_1 &= \max\{\lambda \in \mathbb{R} \mid \lambda < 0, \begin{bmatrix} -\lambda I & sL \\ sL^* & (s^2 - 1 - \lambda)I \end{bmatrix} \succeq 0\} \\ &= \max\{\lambda \in \mathbb{R} \mid \lambda < 0, \lambda^2 - (s^2 - 1)\lambda - \frac{\ell s^2}{\mu^1} \geq 0\} \\ &= \frac{s^2 - 1 - \sqrt{(s^2 - 1)^2 + 4\ell s^2/\mu^1}}{2} \\ &= -\frac{2\ell s^2}{s^2 - 1 + \sqrt{(s^2 - 1)^2 + 4\ell s^2/\mu^1}} \\ &> -\frac{2\ell s^2}{s^2 - 1 + \sqrt{(s^2 - 1)^2}} \\ &= -\frac{\ell s^2}{s^2 - 1} \\ &\geq -2\ell, \end{aligned}$$

since $s^2 \geq 2$. □

Theorem 3.6. *Let $A \in M_m(\mathbb{C})$ and $D \in M_n(\mathbb{C})$ be Hermitian matrices, $L \in M_{m \times n}(\mathbb{C})$. Assume D is positive definite. For $t \in \mathbb{N}$, denote by $\mathbf{1}_t$ the row vector of dimension t all of whose entries are 1, and define*

$$(3) \quad A_t = \begin{bmatrix} A & L \otimes \mathbf{1}_t \\ L^* \otimes \mathbf{1}_t^\top & D \otimes (J_t - I_t) \end{bmatrix}.$$

Then

$$\lim_{t \rightarrow \infty} \lambda_{\min}(A_t) = \lambda_{\min}(A - LD^{-1}L^*).$$

Proof. Clearly, A_t is a principal submatrix of A_{t+1} . Thus

$$\lambda_{\min}(A_t) \geq \lambda_{\min}(A_{t+1}).$$

Next we show that the sequence $(\lambda_{\min}(A_t))_{t \in \mathbb{N}}$ is bounded from below. Indeed, let

$$(4) \quad g(t, z) = \det \begin{bmatrix} A - zI & tL \\ L^* & (t-1)D - zI \end{bmatrix} \in \mathbb{C}[t, z].$$

Since

$$g(t, z) = \det \left(\begin{bmatrix} A & \sqrt{t}L \\ \sqrt{t}L^* & (t-1)D \end{bmatrix} - zI \right),$$

the univariate polynomial $g(t, z)$ in z is real-rooted for all $t \in \mathbb{N}$. For $t \geq 2$, we have

$$\begin{aligned}
\min\{\lambda \in \mathbb{R} \mid g(t, \lambda) = 0\} &= \lambda_{\min} \left(\begin{bmatrix} A & \sqrt{t}L \\ \sqrt{t}L^* & (t-1)D \end{bmatrix} \right) \\
&\geq \lambda_{\min}(A) + \lambda_{\min} \left(\begin{bmatrix} 0 & \sqrt{t}L \\ \sqrt{t}L^* & (t-1)D \end{bmatrix} \right) \\
(5) \qquad \qquad \qquad &> \lambda_{\min}(A) - 2\ell \qquad \qquad \qquad \text{(by Lemma 3.5),}
\end{aligned}$$

where ℓ denotes the largest eigenvalue of L^*L . Let μ^1 be the smallest eigenvalue of D . By the assumption, we have $\mu^1 > 0$. Using the block decomposition (3) (see [3, Sect. 2.3]), we obtain

$$\begin{aligned}
\lambda_{\min}(A_t) &= \min \left\{ \text{Spec} \begin{bmatrix} A & tL \\ L^* & (t-1)D \end{bmatrix} \cup \text{Spec} \begin{bmatrix} A & 0 \\ 0 & D \otimes (J_t - I_t) \end{bmatrix} \right\} \\
&= \min \{ \min\{\lambda \in \mathbb{R} \mid g(t, \lambda) = 0\}, \lambda_{\min}(A), -\mu^1 \} \qquad \text{(by (4))} \\
&\geq \min\{\lambda_{\min}(A) - 2\ell, -\mu^1\} \qquad \qquad \qquad \text{(by (5)).}
\end{aligned}$$

Now, we have shown that the limit

$$\delta = \lim_{t \rightarrow \infty} \min\{\lambda \in \mathbb{R} \mid g(t, \lambda) = 0\}$$

exists. Since

$$\begin{aligned}
g(t, \lambda) &= t^n \det \begin{bmatrix} A - \lambda I & L \\ L^* & (1 - \frac{1}{t})D - \frac{\lambda}{t}I \end{bmatrix} \\
&= \det \begin{bmatrix} A - \lambda I & L \\ L^* & D \end{bmatrix} t^n + \sum_{i=0}^{n-1} h_i(z) t^i
\end{aligned}$$

for some polynomials $h_i(z)$, Lemma 3.3 implies

$$\begin{aligned}
\delta &= \min\{\lambda \in \mathbb{R} \mid \det \begin{bmatrix} A - \lambda I & L \\ L^* & D \end{bmatrix} = 0\} \\
&= \min\{\lambda \in \mathbb{R} \mid \det(A - \lambda I - LD^{-1}L^*) = 0\} \\
&= \lambda_{\min}(A - LD^{-1}L^*).
\end{aligned}$$

□

Corollary 3.7 ([9, Lemma 2.2]). *Let G be a simple graph with m vertices, and let A be the adjacency matrix of G . Let L be an $m \times n$ matrix with entries in $\{1, -1, 0\}$. For $t \in \mathbb{N}$, define A_t by (3), where $D = I_n$. Then*

$$\lim_{t \rightarrow \infty} \lambda_{\min}(A_t) = \lambda_{\min}(A - LL^*).$$

4. PROOF OF THEOREM 1.2

Definition 4.1. A **Hoffman signed graph** is a pair $\mathfrak{h} = (H, \mu)$ where $H = (V, E^+, E^-)$ is a signed graph with vertex set V and a labeling map $\mu : V \rightarrow \{f, s\}$, satisfying the following conditions:

- (i) every vertex with label f is adjacent to at least one vertex with label s ;
- (ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a **slim vertex**, and a vertex with label f a **fat vertex**. We denote by $V_s = V_s(\mathfrak{h})$ (resp. $V_f(\mathfrak{h})$) the set of slim (resp. fat) vertices of \mathfrak{h} . If $E^- = \emptyset$, then we call \mathfrak{h} an **unsigned Hoffman graph**, or simply, a **Hoffman graph**.

For a Hoffman signed graph \mathfrak{h} , let A be its adjacency matrix,

$$(6) \quad A = \begin{bmatrix} A_s & C \\ C^T & O \end{bmatrix}$$

in a labeling in which the fat vertices come last. The signed graph with adjacency matrix A_s is called the **slim subgraph** of \mathfrak{h} . **Eigenvalues** of \mathfrak{h} are the eigenvalues of the real symmetric matrix $B(\mathfrak{h}) = A_s - CC^T$. It is easy to see that, if two Hoffman signed graph are switching equivalent as signed graphs, then they have the same set of eigenvalues as Hoffman signed graphs. Let $\lambda_{\min}(\mathfrak{h})$ denote the smallest eigenvalue of \mathfrak{h} .

For a Hoffman signed graph \mathfrak{h} and a positive integer t , we denote by $G(\mathfrak{h}, t)$ the signed graph obtained by replacing every fat vertex of \mathfrak{h} by K_t consisting of $(+)$ -edges.

Theorem 4.2. *Let $\mathfrak{h} = (H, \mu)$ be a Hoffman signed graph. Then*

$$\lim_{t \rightarrow \infty} \lambda_{\min}(G(\mathfrak{h}, t)) = \lambda_{\min}(\mathfrak{h}).$$

Proof. If the adjacency matrix of \mathfrak{h} is given by (6), then the adjacency matrix of $G(\mathfrak{h}, t)$ is (3), where $D = I$. Thus, the result is immediate from Theorem 3.6. \square

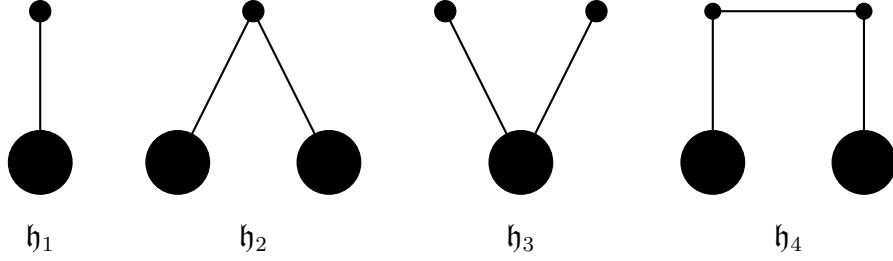
Following [16], let $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4$ denote the (unsigned) Hoffman graph defined in Fig. 1. Then $\lambda_{\min}(\mathfrak{h}_1) = -1$, and $\lambda_{\min}(\mathfrak{h}_i) = -2$ for $i = 2, 3, 4$.

For the remainder of this section, we fix a real number λ with $-2 < \lambda < -1$. By Theorem 4.2, there exist $n_0 \in \mathbb{N}$ such that

$$(7) \quad \lambda > \lambda_{\min}(G(\mathfrak{h}_i, n_0)) \quad (i = 2, 3, 4).$$

Lemma 4.3. *Let S be a signed graph. If $\lambda_{\min}(S) \geq \lambda$ and S is switching equivalent to an unsigned graph G , then*

$$(8) \quad G \not\supseteq G(\mathfrak{h}_i, n_0) \quad (i = 2, 3, 4).$$

FIGURE 1. The Hoffman graphs \mathfrak{h}_i ($i = 1, 2, 3, 4$)

Proof. Since $\lambda_{\min}(S) = \lambda_{\min}(G)$, the assertion is immediate from (7). \square

Lemma 4.4. *Let S be a signed graph. If $\lambda \leq \lambda_{\min}(S) < -1$ and S is switching equivalent to the line graph $\mathfrak{L}(H)$ of some connected graph H , then $\delta(S) < 2n_0$.*

Proof. Let $G = \mathfrak{L}(H)$. Since $\lambda_{\min}(S) < -1$, S is not switching equivalent to a complete graph. This implies that G is not complete, and hence H is not a claw. Since H is not a claw, there exists an edge $x = \{u, v\} \in E(H)$ with $\deg_H(u), \deg_H(v) > 1$.

Suppose, to the contrary, that $\delta(S) \geq 2n_0$. Then $\delta(G) \geq 2n_0$, and therefore $\deg_H(u) + \deg_H(v) \geq 2n_0 + 2$. If $\min\{\deg_H(u), \deg_H(v)\} \geq n_0 + 1$, then $N_G(x)$ consists of two connected components each of which contains K_{n_0} . This contradicts (8). Thus, we may assume without loss of generality that $\deg_H(v) < n_0 + 1$. Then $\deg_H(u) > n_0 + 1$. Since $\deg_H(v) > 1$, there exists a vertex $w \in N_H(v) \setminus \{u\}$. Let y denote the edge $\{v, w\}$. Since $\deg_G(y) \geq 2n_0$, we have $\deg_H(w) > n_0 + 1$. This implies that the subgraph of G induced by $\{x, y\} \cup N_G(x) \cup N_G(y)$ contains $G(\mathfrak{h}_2, n_0)$, contradicting (8). \square

Proof of Theorem 1.2. Since there are only finitely many exceptional graphs, there exists a positive integer d_0 such that every exceptional graph has minimum degree bounded by d_0 .

Recall that we have fixed $\lambda \in (-2, -1)$. We define the value $f(\lambda)$ of the function $f : (-2, -1) \rightarrow \mathbb{R}$ by

$$f(\lambda) = \max\{2n_0, d_0\} + 1.$$

Let S be a connected signed graph with $\lambda_{\min}(S) \geq \lambda$ and $\delta(S) \geq f(\lambda)$. Since $\delta(S) > d_0$, we see that S is not exceptional. This means that S is integrally represented. By Corollary 2.3, there exists a tree H such that $\mathfrak{L}(H)$ is switching equivalent to S or S with one vertex removed. In the former case, Lemma 4.4 implies that $\lambda_{\min}(S) = -1$, and hence S is

switching equivalent to a complete graph. Suppose $\mathfrak{L}(H)$ is switching equivalent to $S - u$ for some vertex u of S . Since $\lambda_{\min}(S - u) \geq \lambda_{\min}(S) \geq \lambda$ and $\delta(S - u) \geq \delta(S) - 1 \geq 2n_0$, Lemma 4.4 implies that $\lambda_{\min}(S - u) = -1$, and hence $S - u$ is switching equivalent to a complete graph. Since $\delta(S) > d_0$, the vertex u has degree greater than n_1 . If S is not switching equivalent to a complete graph, then there exists a non-neighbor u' of u in S . Then the subgraph induced on the common neighbors of u, u' together with u, u' themselves has smallest eigenvalue less than λ by (7). This implies $\lambda_{\min}(S) < \lambda$, contrary to the assumption. Therefore, S is switching equivalent to a complete graph. \square

Lemma 4.5. *Let S be an odd cycle with an odd number of $(-)$ -edges. Then S has smallest eigenvalue -2 .*

Proof. The signed graph S is switching equivalent to an odd cycle in which all edges are $(-)$ -edges. Since this is the negative of a 2-regular graph, it has smallest eigenvalue -2 . \square

Lemma 4.6. *Let S be a signed graph whose underlying graph $U(S)$ is complete. If $\lambda_{\min}(S) \geq -\sqrt{2}$, then S is switching equivalent to a complete graph.*

Proof. After switching, we may assume that there exists a vertex x of S such that all edges incident with x are positive. Since a triangle with one negative edge has smallest eigenvalue -2 while $\lambda_{\min}(S) \geq -\sqrt{2}$, it follows that S cannot contain such a triangle. This implies that all edges of S not containing x are positive. Therefore, S itself is a complete graph. \square

Proposition 4.7. *Let S be a connected signed graph with smallest eigenvalue greater than $-\sqrt{2}$. Then S is switching equivalent to a complete graph.*

Proof. If the underlying graph of S is not complete, then S contains a signed 2-path, which has smallest eigenvalue $-\sqrt{2}$. This contradiction shows that the underlying graph of S must be complete. The result then follows from Lemma 4.6. \square

As a consequence of Proposition 4.7, the values of the function f in Theorem 1.2 on the interval $(-\sqrt{2}, -1)$ can be arbitrary, since the conclusion of Theorem 1.2 holds for $\lambda \in (-\sqrt{2}, -1)$ without any assumption on the minimum degree $\delta(S)$.

A natural question is to determine the smallest possible value of $f(-\sqrt{2})$ so that $\delta(S) \geq f(-\sqrt{2})$ and $\lambda_{\min}(S) \geq -\sqrt{2}$ implies that S is

switching equivalent to a complete graph. In addition to the 2-path, there is another signed graph with smallest eigenvalue $-\sqrt{2}$, namely, a 4-cycle with one $(-)$ -edge. It has adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix},$$

and we have $A^2 = 2I$. Its underlying graph is regular of valency 2. So we must take $f(-\sqrt{2}) > 2$ to exclude this graph. In fact, $f(-\sqrt{2}) = 3$ does give the correct conclusion.

Proposition 4.8. *If S is a connected signed graph with $\lambda_{\min}(S) \geq -\sqrt{2}$ and $\delta(S) \geq 3$, then S is switching equivalent to a complete graph (and hence $\lambda_{\min}(S) = -1$).*

Proof. We note first that it suffices to show that the underlying graph $U(S)$ of S is complete, by Lemma 4.6. Since S has minimum degree at least 3, S must contain a cycle (since otherwise S is a tree, having a leaf, meaning minimum degree is 1).

Suppose first S contains a triangle. If $U(S)$ is a triangle, then $U(S)$ is a complete graph, so we are done by the first paragraph. Otherwise, S has at least 4 vertices, so $U(S)$ must contain a triangle with one pendant edge attached, or $K_{1,1,2}$. By Lemma 4.5, S can be switched to contain one of the two graphs with all positive edges. The former has smallest eigenvalue ≈ -1.48 , the latter ≈ -1.56 , both are strictly less than $-\sqrt{2}$. This is a contradiction. Thus S has no triangle, so S contains a cycle of length at least 4. S cannot contain a cycle of length at least 4 with all edges positive. So S contains a cycle of length at least 4 with odd number of negative edges. If the length is odd, then we get a contradiction by Lemma 4.5. So the length is even. If the length is at least 6, then it contains a path with 5 vertices. A path with 5 vertices has smallest eigenvalue $-\sqrt{3} < -\sqrt{2}$, a contradiction. So the only possible cycle is a 4-cycle with one negative edge. Since $\delta(S) \geq 3$, S strictly contains a 4-cycle with one negative edge. As S cannot contain a triangle with one pendant edge attached, or $K_{1,1,2}$, we see that S contains $K_{1,3}$ with smallest eigenvalue $-\sqrt{3} < -\sqrt{2}$, a contradiction. \square

In the original setting of Hoffman's Theorem 1.1, a much easier argument than the above shows that we may define $h(-\sqrt{2}) = 2$.

5. A SIGNED ANALOGUE OF GENERALIZED LINE GRAPHS

Definition 5.1. Given a signed graph $S = (V, E^+, E^-)$, the **line signed graph** $\mathfrak{L}(S)$ is the signed graph with vertex set $E^+ \cup E^-$, and two distinct vertices are joined by a signed edge if they are incident and the sign is the product of their signs.

The definition of line signed graph can be best understood in terms of signed incidence matrix. The **signed incidence matrix** B of a signed graph $S = (V, E^+, E^-)$ is the matrix whose rows and columns are indexed by V and $E^+ \cup E^-$ respectively, such that its (i, e) -entry is equal to the sign of e if $i \in e$, and otherwise 0. The adjacency matrix of $\mathfrak{L}(S)$ is then given by $B^\top B - 2I$. Note that for an unsigned graph S , $\mathfrak{L}(S)$ is nothing but the ordinary signed graph of the graph S .

Line signed graphs (or sometimes called signed line graphs) have been considered in [2, 5], but our definition is different from those introduced there. We list some properties of $\mathfrak{L}(S)$ together with comments pertaining to the corresponding properties of line graphs defined in [2, 5]. Note that, we denote by $-S$ the **negative** of a signed graph S , which is obtained by exchanging E^+ and E^- in S .

- (i) If S is an unsigned graph, that is, $E^- = \emptyset$, then $\mathfrak{L}(S)$ coincides with the ordinary line graph. This is not true in [2, 5]. Indeed, $\mathfrak{L}(S)$ is the line graph of $-S$ in the sense of [2], while it is the negative of the line graph of $-S$ in the sense of [5].
- (ii) The line signed graph $\mathfrak{L}(S)$ has smallest eigenvalue at least -2 . This is true in [2] but not in [5].
- (iii) The line signed graph $\mathfrak{L}(S)$ is uniquely determined by S . This is not true in [2, 5], where the signed line graphs depends on the choice of an orientation, and defined only up to switching equivalence.
- (iv) If S and S' are switching equivalent, so are $\mathfrak{L}(S)$ and $\mathfrak{L}(S')$. This is true in [2, 5].

To see the property (ii), recall that the adjacency matrix of $\mathfrak{L}(S)$ is given by $B^\top B - 2I$, where B is the signed incidence matrix of S . Since $B^\top B$ is positive semidefinite, (ii) holds.

The property (iv) follows from a stronger claim that $\mathfrak{L}(S)$ is switching equivalent to the line graph of the underlying graph $U(S)$ of S . To see this, observe that every cycle in $\mathfrak{L}(S)$ contains an even number of $(-)$ -edges, and then invoke [17, Prop. 3.2]. This stronger statement indicates that spectral consideration on line signed graphs in our sense reduces to that of line (unsigned) graphs. However, such a reduction

will not occur in the generalization to follow (see Definition 5.3 and comments after that).

Given a signed graph $S = (V(S), E^+(S), E^-(S))$, for convenience, we denote by $\sigma(e)$ the sign of an edge $e \in E^+(S) \cup E^-(S)$. We can construct a Hoffman signed graph $\mathfrak{h} = (H, \mu)$, where $H = (V(H), E^+(H), E^-(H))$ is a signed graph, as follows. Define

$$\begin{aligned} V(H) &= V(S) \cup E^+(S) \cup E^-(S), \\ E^\epsilon(H) &= \{\{i, e\} \mid i \in V(S), e \in E^\epsilon(S), i \in e\} \\ &\quad \cup \{\{e, e'\} \mid e, e' \in E^+(S) \cup E^-(S), |e \cap e'| = 1, \sigma(e)\sigma(e') = \epsilon\}, \end{aligned}$$

where $\epsilon = \pm 1$, and $\mu : V(H) \rightarrow \{f, s\}$ by

$$\mu(x) = \begin{cases} f & \text{if } x \in V(S), \\ s & \text{otherwise.} \end{cases}$$

Then the slim subgraph of H coincides with the line signed graph of S defined in Definition 5.1. Note that every slim vertex of H has exactly two fat neighbors, joined by edges of the same sign. In other words, H is obtained by gluing Hoffman graphs \mathfrak{h}_2 (see Figure 1) and \mathfrak{h}_2^- (see Figure 2) appropriately. The adjacency of two slim vertices occurs exactly when they have a fat neighbor in common, and the sign of the edge connecting them is also determined by the sign of edges connecting them to fat neighbors. This observation motivates the definitions to follow.

Let $\mathfrak{h} = (H, \mu)$ be a Hoffman signed graph, where $H = (V, E^+, E^-)$. For a slim vertex x of \mathfrak{h} , the **representing vector** of x is the vector $\varphi(x)$ indexed by the set of fat vertices, defined by

$$\varphi(x)_z = \begin{cases} 1 & \text{if } \{x, z\} \in E^+, \\ -1 & \text{if } \{x, z\} \in E^-, \\ 0 & \text{otherwise.} \end{cases}$$

If the Hoffman signed graph $\mathfrak{h} = (H, \mu)$ is obtained from a signed graph S as above, then for two distinct slim vertices x and y of H , the sign of the edge $\{x, y\}$ is the inner product $(\varphi(x), \varphi(y))$. We will axiomatize this to define decompositions of Hoffman signed graph. Note that this concept has already been considered by Woo and Neumaier [16] for unsigned Hoffman graphs.

For a vertex x of \mathfrak{h} we define $N^f(x) = N_{\mathfrak{h}}^f(x)$ (resp. $N^s(x) = N_{\mathfrak{h}}^s(x)$) the set of fat (resp. slim) neighbors of x in \mathfrak{h} . The set of all neighbors of x is denoted by $N(x) = N_{\mathfrak{h}}(x)$, that is $N(x) = N^f(x) \cup N^s(x)$. In a

similar fashion, for vertices x and y we define $N^f(x, y) = N_{\mathfrak{h}}^f(x, y)$ to be the set of common fat neighbors of x and y .

A **decomposition** of a Hoffman signed graph \mathfrak{h} is a family $\{\mathfrak{h}^i\}_{i=1}^n$ of non-empty induced Hoffman subgraphs of \mathfrak{h} satisfying the following conditions:

- (i) $V(\mathfrak{h}) = \bigcup_{i=1}^n V(\mathfrak{h}^i)$;
- (ii) $V^s(\mathfrak{h}^i) \cap V^s(\mathfrak{h}^j) = \emptyset$ if $i \neq j$;
- (iii) For each $x \in V^s(\mathfrak{h}^i)$, $N_{\mathfrak{h}}^f(x) \subseteq V^f(\mathfrak{h}^i)$
- (iv) If $x \in V^s(\mathfrak{h}^i)$, $y \in V^s(\mathfrak{h}^j)$, and $i \neq j$, the inner product $(\varphi(x), \varphi(y))$ is $1, -1, 0$, according as $\{x, y\}$ is a (+)-edge, (-)-edge, or non-edge.

A Hoffman signed graph \mathfrak{h} is said to be **decomposable** if \mathfrak{h} has a decomposition $\{\mathfrak{h}^i\}_{i=1}^n$ with $n \geq 2$, and \mathfrak{h} is said to be **indecomposable** if \mathfrak{h} is not decomposable.

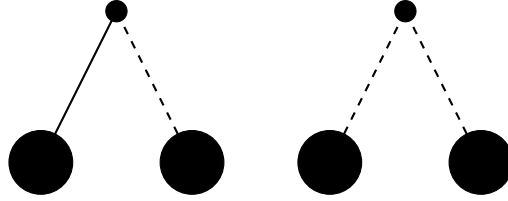
Definition 5.2. Two Hoffman signed graphs \mathfrak{h} and \mathfrak{h}' are **switching equivalent** if \mathfrak{h}' can be obtained from \mathfrak{h} by switching with respect to a subset of slim vertices of \mathfrak{h} .

For example, \mathfrak{h}_2 is switching equivalent to \mathfrak{h}_2^- , but not to \mathfrak{h}_2^{--} .

Definition 5.3. Let \mathfrak{H} be a family of switching classes of Hoffman signed graphs. An **\mathfrak{H} -line signed graph** is an induced Hoffman subgraph of a Hoffman signed graph which has a decomposition $\{\mathfrak{h}^i\}_{i=1}^n$ such that the switching class of \mathfrak{h}^i belongs to \mathfrak{H} for all $i = 1, \dots, n$.

It is clear from the definition that the line signed graphs are precisely the slim subgraphs of a Hoffman signed graph which admits a decomposition all of whose components are isomorphic to \mathfrak{h}_2 or \mathfrak{h}_2^- . Since \mathfrak{h}_2^{--} is switching equivalent to \mathfrak{h}_2 , this means that every line signed graph is an \mathfrak{H} -line signed graph, where \mathfrak{H} is the singleton set consisting of the switching class of \mathfrak{h}_2 . This is precisely the stronger claim mentioned in the proof of the property (iv) of line signed graphs.

We note that, however, some \mathfrak{H} -line signed graph are not switching equivalent to an unsigned graph for some other family \mathfrak{H} . For, let \mathfrak{h} be the Hoffman graph with one fat vertex having two slim neighbors connected by a (-)-edge, where the edges connecting the fat vertex and slim vertices are (+)-edges. Let $\mathfrak{H} = \{[\mathfrak{h}_1], [\mathfrak{h}]\}$ (see Fig. 1 for the definition of \mathfrak{h}_1). Then the slim graph of an \mathfrak{H} -line signed graph obtained by identifying the fat vertices of \mathfrak{h}_1 and \mathfrak{h} is the triangle with only one (-)-edge. Thus, it is not switching equivalent to an unsigned graph.

FIGURE 2. The Hoffman signed graphs \mathfrak{h}_2^- , \mathfrak{h}_2^{--}

Proposition 5.4. *Let \mathfrak{H} consist of three switching classes $[\mathfrak{h}_2]$, $[\mathfrak{h}_2^-]$, $[\mathfrak{h}_3]$. Then every \mathfrak{H} -line signed graph is switching equivalent to an $\{[\mathfrak{h}_2], [\mathfrak{h}_2^-]\}$ -line signed graph.*

Proof. Suppose that a signed Hoffman graph \mathfrak{h} is an \mathfrak{H} -line signed graph. If a decomposition of \mathfrak{h} contains a summand \mathfrak{h}^i which is switching equivalent to \mathfrak{h}_3 , then we may apply switching with respect to one or both of slim vertices of \mathfrak{h}^i to make \mathfrak{h}^i isomorphic to \mathfrak{h}_3 . Having done this, we can add a new common fat neighbor to the two slim vertices of \mathfrak{h}^i , where one of the edge connecting a slim vertex to the new fat neighbor is a $(-)$ -edge, the other is a $(+)$ -edge. Doing this process for each summand isomorphic to \mathfrak{h}_3 , we can convert \mathfrak{h}_3 to the sum of \mathfrak{h}_2 and \mathfrak{h}_2^- . \square

The conclusion of Proposition 5.4 cannot be changed to claim the switching equivalence to an $\{[\mathfrak{h}_2], [\mathfrak{h}_3]\}$ -line signed graph, or to an $\{[\mathfrak{h}_2^-], [\mathfrak{h}_3]\}$ -line signed graph. Indeed, a triangle consisting of three $(-)$ -edges is an $\{[\mathfrak{h}_2^-]\}$ -line signed graph which is not switching equivalent to an $\{[\mathfrak{h}_2], [\mathfrak{h}_3]\}$ -line signed graph, and a pentagon consisting of five $(+)$ -edges is an $\{[\mathfrak{h}_2]\}$ -line signed graph which is not switching equivalent to an $\{[\mathfrak{h}_2^-], [\mathfrak{h}_3]\}$ -line signed graph.

The meaning of Proposition 5.4 for unsigned slim graphs is as follows. If G is an unsigned slim graph which is an $\{[\mathfrak{h}_2], [\mathfrak{h}_2^-], [\mathfrak{h}_3]\}$ -line signed graph, then it is an $\{[\mathfrak{h}_2], [\mathfrak{h}_3]\}$ -line graph, and hence a generalized line graph by [16, Example 2.1]. Generalized line graphs are also known [4] as graphs represented by the root system

$$D_\infty = \{\pm \mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i < j < \infty\}.$$

This fact becomes transparent by Proposition 5.4 since clearly, $\{[\mathfrak{h}_2], [\mathfrak{h}_2^-]\}$ -line signed graphs are precisely the graphs represented by D_∞ , in terms of representing vectors. Such a representation shows that every slim

$\{[\mathfrak{h}_2], [\mathfrak{h}_2^-]\}$ -line signed graph has smallest eigenvalue at least -2 . Alternatively, this is a consequence of the following proposition which generalizes the more general fact about decomposition of Hoffman signed graphs.

Proposition 5.5. *Suppose a Hoffman signed graph \mathfrak{h} has a decomposition $\{\mathfrak{h}^i\}_{i=1}^n$. Let*

$$A = \begin{bmatrix} A_s & C \\ C^T & O \end{bmatrix} \text{ and } A_i = \begin{bmatrix} A_s^{(i)} & C_i \\ C_i^T & O \end{bmatrix}$$

be the adjacency matrices of \mathfrak{h} and \mathfrak{h}^i in a labeling in which the fat vertices come last. Then $A_s - CC^T$ is the direct sum of matrices $A_s^{(i)} - C_i C_i^T$ for $i = 1, \dots, n$. In particular,

$$\lambda_{\min}(\mathfrak{h}) = \min\{\lambda_{\min}(\mathfrak{h}^i) \mid 1 \leq i \leq n\}.$$

Proof. Let $x \in V^s(\mathfrak{h}^i)$, $y \in V^s(\mathfrak{h}^j)$. Suppose $i \neq j$. Then by the condition (iv) of decomposition, the (x, y) -entry of A_s coincides with the inner product $(\varphi(x), \varphi(y))$ which is the (x, y) -entry of CC^T . Thus the (x, y) -entry of $A_s - CC^T$ is 0, which is the same as the corresponding entry of the diagonal join.

Suppose $i = j$. Since \mathfrak{h}^i is an induced subgraph of \mathfrak{h} , the submatrix of A_s corresponding to the $V^s(\mathfrak{h}^i)$ is exactly $A_s^{(i)}$. By the condition (iii) of decomposition, $\varphi(x)$ has support contained in $V^f(\mathfrak{h}^i)$. Thus, the inner product $(\varphi(x), \varphi(y))$ coincides with the (x, y) -entry of $C_i C_i^T$. Therefore, the (x, y) -entry of $A_s - CC^T$ is the same as the corresponding entry of $A_s^{(i)} - C_i C_i^T$. \square

Acknowledgements. The authors thank Zoran Stanić for valuable comments.

REFERENCES

- [1] F. Belardo, S. Cioabă, J. Koolen and J. Wang, Open problems in the spectral theory of signed graphs, *Art Discrete Appl. Math.* 1 (2018), #P2.10, 23 pp.
- [2] F. Belardo and S. K. Simić, On the Laplacian coefficients of signed graphs, *Linear Algebra and Appl.*, 475 (2015), 94–113.
- [3] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, 2012.
- [4] P.J. Cameron, J.M. Goethals, E.E. Shult and J.J. Seidel. Line graphs, root systems, and elliptic geometry. *J. Algebra* 43(1):305–327, 1976.
- [5] K.A. Germina, S. Hameed K and T. Zaslavsky, On products and line graphs of signed graphs, their eigenvalues and energy, *Linear Algebra Appl.* 435 (2011), 2432–2450.
- [6] G. Greaves, J. Koolen, A. Munemasa, Y. Sano and T. Taniguchi, Edge-signed graphs with smallest eigenvalue greater than -2 , *J. Combin. Theory, Ser. B.* 110 (2015) 90–111.

- [7] A. J. Hoffman. On graphs whose least eigenvalue exceeds $-1 - \sqrt{2}$. *Linear Algebra and Appl.*, 16(2):153–165, 1977.
- [8] A. J. Hoffman, The change in the least eigenvalue of the adjacency matrix of a graph under embedding, *SIAM J. Appl. Math.* 17 (1969), 664–671.
- [9] A. J. Hoffman, On signed graphs and Gramians, *Geometriae Dedicata* 6 (1977), 455–470.
- [10] H. J. Jang, J. Koolen, A. Munemasa, and T. Taniguchi: On fat Hoffman graphs with smallest eigenvalue at least -3 , *Ars Mathematica Contemporanea* 7 (2014) 105–121.
- [11] J. H. Koolen M.-Y. Cao and Q. Yang, Recent progress on graphs with fixed smallest eigenvalue, preprint arXiv:2011.11935.
- [12] J. H. Koolen, J. Y. Yang and Q. Yang, On graphs with smallest eigenvalue at least -3 and their lattices, *Adv. Math.* 338 (2018), 847–864.
- [13] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*. Oxford University Press, Oxford, 2002.
- [14] Z. Stanic, Notes on exceptional signed graphs, *Ars Math. Contemp.* 18 (2020), 105–115.
- [15] G. R. Vijayakumar, Signed graphs represented by D_∞ , *Europ. J. Combin.* 8 (1987), 103–112.
- [16] R. Woo and A. Neumaier, On graphs whose smallest eigenvalue is at least $-1 - \sqrt{2}$, *Linear Algebra Appl.* 226-228:577–591 (1995).
- [17] T. Zaslavsky, Signed graphs, *Discrete Applied Mathematics* 4 (1982), 47–74.
- [18] X. Zhan, *Matrix Theory*, Amer. Math. Soc., Providence, Rhode Island, 2013.

PUSAN NATIONAL UNIVERSITY

Email address: gavrilyuk@riko.shimane-u.ac.jp

GRADUATE SCHOOL OF INFORMATION SCIENCES, TOHOKU UNIVERSITY

Email address: munemasa@math.is.tohoku.ac.jp

FACULTY OF ENGINEERING, INFORMATION AND SYSTEMS, UNIVERSITY OF TSUKUBA

Email address: sano@cs.tsukuba.ac.jp

HIROSHIMA INSTITUTE OF TECHNOLOGY

Email address: t.taniguchi.t3@cc.it-hiroshima.ac.jp