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Author(s)

Toshihiro Nakanishi and Akira Ushijima

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Existence of exceptional points for discrete groups of isometries of the hyperbolic space

Toshihiro Nakanishi, Akira Ushijima

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Abstract

It is shown by Fera that there exists uncountably many exceptional points for cocompact Fuchsian groups. We generalise this result for cofinite discrete groups of isometries of the hyperbolic space with finite sided Dirichlet domain.

1 Introduction

Let G be a Fuchsian groups of finite covolume, or a cofinite Fuchsian groups acting on the hyperbolic plane $\mathbb{H}^2$. If G is of type (g, m), the number of the sides of the Dirichlet domains $\mathrm{D}(a)$ with center $a \in \mathbb{H}^2$ is at most $12g - 4m - 6$. A point a is called *regular* if the number of the sides of $\mathrm{D}(a)$ is the maximal number $12g-4m-6$ and called *exceptional* otherwise. It is known that a generic point is regular [Bea83, Theorem 9.4.5]. Joseph Fera studied in [Fer14] that the existence of exceptional points for cocompact Fuchsian groups. He showed that the set of exceptional points is an uncountable set. We generalise this result to cofinite Fuchsian groups. Our main theorem, Theorem 5.1, states that the set of exceptional points is uncountable, same as Fera's result for cocompact Fuchsian groups. But in order to prove the theorem, we needed several different arguments. A reason for that is there are cases which are not observed for cocompact Fuchsian groups. For example, if G is cofinite but not cocmpact, then, as Umemoto's result shows (see Example 4.7), there is a case where the Dirichlet domains $\mathrm{D}(a_n)$ for a converging sequence a_n to a have a common side-pairing element f and the sides s_n and $f(s_n)$ of $\mathrm{D}(a_n)$ paired by f eventually leave any compact set including a. It is also possible that the set of points having the same collection of side-pairing elements is unbounded, contrary to the case of cocompact Fuchsian groups.

2 Preliminaries

2.1 Dirichlet domains for discrete groups

We prepare notation and basic facts mainly following [Rat06].

Let $\mathbb{H}^n$ be an n-dimensional hyperbolic space with distance function d. Its sphere at infinity is denoted by $\partial_\infty \mathbb{H}^n$. For a subset $A \subset \mathbb{H}^n$, we denote its closure, complement and boundary of A in $\mathbb{H}^n$ by $\overline{A}$, $A^{\complement}$ and ∂A respectively. We also denote by $\mathrm{Int}(A)$ the interior of A with respect to its relative topology. The open ball in $\mathbb{H}^n$ with radius $r > 0$ and center $a \in \mathbb{H}^n$ is denoted by $\mathrm{B}_r(a) := \{ x \in \mathbb{H}^n \mid d(x,a) < r \}$.

Let G be a discrete group of isometries of $\mathbb{H}^n$. G acts on $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$. For a point x of $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$, $G(x) = \{g(x) : g \in G\}$ is the *G-orbit* of x. G acts discontinuously on $\mathbb{H}^n$; If $x \in \mathbb{H}^n$, then $G(x)$ has no accumulation points in $\mathbb{H}^n$. Then the quotient space $\mathbb{H}^n/G$ becomes a hyperbolic orbifold with complete hyperbolic metric induced from d. The canonical projection from $\mathbb{H}^n$ to $\mathbb{H}^n/G$ is denoted by π. For such a group G, we define a *Dirichlet domain* $\mathrm{D}(a)$ with respect to a point a as follows: for an isometry f on $\mathbb{H}^n$, we denote by $\mathrm{Fix}(f)$ a set of points in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ fixed by f:

$$\mathrm{Fix}(f) := \{ x \in \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n \mid f(x) = x \} .$$

The union of such sets for all non-trivial elements in a Fuchsian group G is denoted by F_G:

$$\mathrm{F}_G := \bigcup_{g \in G - \{\mathrm{id}\}} \mathrm{Fix}(g),$$

where id represents the identity mapping, which is also the trivial element of G. We denote by $\mathrm{H}_f(x)$ the set of points in $\mathbb{H}^n$ which are closer to x than $f(x)$, and its boundary by $\mathrm{L}_f(x)$:

$$\mathrm{H}_f(x) := \{ y \in \mathbb{H}^n \mid d(x,y) < d(f(x),y) \} ,$$
$$\mathrm{L}_f(x) := \partial \mathrm{H}_f(x) \Big(= \{ y \in \mathbb{H}^n \mid d(x,y) = d(f(x),y) \} \Big).$$

For any point $a \in \mathbb{H}^n - \mathrm{F}_G$, let $\mathrm{D}(a)$ be the *Dirichlet domain* for G with center a:

$$\mathrm{D}(a) := \bigcap_{g \in G - \{\mathrm{id}\}} \mathrm{H}_g(a).$$

It is known that Dirichlet domains are locally finite convex fundamental domains for G. See, for example, [Rat06, Theorem 6.6.13].

2.2 Dirichlet domains for Fuchsian groups

We consider the upper half plane model for $\mathbb{H}^2$. So let

$$\mathbb{H}^2 = \{ z = x + iy : y > 0 \} \quad \text{equipped with} \quad ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Let $d(z,w)$ denote the hyperbolic distance between two points z and w of $\mathbb{H}^2$. Then

$$\sinh \frac{d(z,w)}{2} = \frac{|z-w|}{2\sqrt{\mathrm{Im}[z]\mathrm{Im}[w]}}. \tag{1}$$

[Bea83, Theorem 7.2.1]. Let G be a discrete subgroup of orientation preserving isometries of $\mathbb{H}^2$. G is called a *Fuchsian group.* G is called *cofinite*, if the area of $\mathbb{H}^2/G$ is finite. If G is cofinite, G is said to be of *type* (g, m) if $(\mathbb{H}^2 - F_G)/G$ is a closed surface of genus g with m points deleted.

G be a Fuchsian group. We represent each element of G by a matrix in $SL(2, \mathbb{R})$. The group G acts also on $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. If G contains parabolic elements, then for each parabolic fixed point p, let $\mathscr{H}_p(r)$ be the *horocycle* of length e^{-r} centered at p and let $\mathscr{D}_p(r)$ be the *horodisk*, the disk bounded by $\mathscr{H}_p(r)$. They are defined as follows. By replacing G with a suitable conjugate of it, we suppose that $p = \infty$ and

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2}$$

is *primitive* in the sense that S generates the stabilizer

$$G_p = \{f \in G | f(p) = p\}$$

of p. Then

$$\mathscr{H}_p(r) = \{z | \operatorname{Im}[z] = e^r\}, \quad \mathscr{D}_p(r) = \{z | \operatorname{Im}[z] > e^r\}.$$

If $p = h(\infty)$ is a parabolic fixed point equivalent to ∞, where

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

is such that $ad - bc = 1$ and $c \neq 0$. Then $\mathscr{D}_p(r)$ is the Euclidean disk with diameter $|c|^{-2}e^{-r}$ tangent to the real line at $p = a/c$ and $\mathscr{H}_p(r)$ is the circle bounding $\mathscr{D}_p(r)$. Since $|c| \geq 1$ due to he Shimizu-Leutbecher inequality [Bea83, Theorem 5.2], if $r \geq 0$, then $\mathscr{D}_p(r)$ is *precisely invariant* under $G_p = \{f \in G : f(p) = p\}$, where $g = hSh^{-1}$: $g^m(\mathscr{D}_p(r)) = \mathscr{D}_p(r)$ for all integers m and

$$\mathscr{D}_p(r) \cap f(\mathscr{D}_p(r)) = \emptyset \text{ for } f \in G - G_p.$$

We have in general

Lemma 2.1. *Let G be a Fuchsian group. For each parabolic fixed point p of G, $f(\mathscr{D}_p(r)) = \mathscr{D}_{f(p)}(r)$ for any $f \in G$ and $r \in \mathbb{R}$, and if $r \geq 0$ and if distinct p and q are G-equivalent, then $\mathscr{D}_p(r)$ and $\mathscr{D}_q(r)$ are disjoint.* □

We consider a Dirichlet domain $\mathrm{D}(a)$ with center a. Let $\bar{\mathrm{D}}(a)$ and $\widetilde{\mathrm{D}}(a)$ denote the closure of $\mathrm{D}(a)$ in $\mathbb{H}^2$ and $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, respectively. A *side* of $\mathrm{D}(a)$ is a positive length geodesic segment of the form $\bar{\mathrm{D}}(a) \cap f(\bar{\mathrm{D}}(a))$ for some $f \in G - \{\mathrm{id}\}$. A *vertex* of $\mathrm{D}(a)$ is a single point of the form $\bar{\mathrm{D}}(a) \cap f(\bar{\mathrm{D}}(a)) \cap h(\bar{\mathrm{D}}(a))$ for some f, $h \in G - \{\mathrm{id}\}$. For each side s of $\mathrm{D}(a)$, there exists a unique isometry $f \in G$ such that $f(s)$ is another side of $\mathrm{D}(a)$. Such an isometry is called a *side-pairing transformation* of $\mathrm{D}(a)$. We denote by $\mathrm{S}(a)$ the set of all side-paring transformations of $\mathrm{D}(a)$. It is shown that $\mathrm{S}(a)$ generates G. See, for example, [Rat06, Theorem 6.8.3]. For side-pairing transformations, we have the following proposition.

Proposition 2.2 (See [Bea83, Section 9]). *Let G be a Fuchsian group and suppose that $f \in G$.*

(1) *Let s be a side of* $\mathrm{D}(a)$. *The side s is given by* $\bar{\mathrm{D}}(a) \cap f(\bar{\mathrm{D}}(a))$ *if and only if* $s \subset \mathrm{L}_f(a)$.

(2) *The element f is in* $\mathrm{S}(a)$ *if and only if* $\mathrm{D}(a)$ *has a side given by* $s = \bar{\mathrm{D}}(a) \cap f^{-1}(\bar{\mathrm{D}}(a))$.

(3) *If* $f \in \mathrm{S}(a)$, *then* $f^{-1} \in \mathrm{S}(a)$. □

Definition 2.3 (See [Bea83, Section 9.3]). Let E denote $\widetilde{\mathrm{D}}(a) \cap \partial_\infty \mathbb{H}^2$. A *free side* of $\mathrm{D}(a)$ is a maximal interval in E. If a point v of E is the endpoint of two sides of $\mathrm{D}(a)$, v is called a *proper vertex*, If v be the endpoint of a side and free side of $\mathrm{D}(a)$, v is called an *improper vertex.* If G is cofinite, then there are no improper vertices for any Dirichlet domain. Let v be a boundary point of $\widetilde{\mathrm{D}}(a)$. The *cycle* C of v is the intersection of $G(v)$ with $\widetilde{\mathrm{D}}(a)$. The number $|C|$ of points in C is called the *length* of C. Let v be a vertex of $\widetilde{\mathrm{D}}(a)$ and C the cycle of v. If $|C| \geq 3$, v is called an *accidental* vertex.

For a proof of the following theorem, see [Bea83, Theorem 9.4.5].

Theorem 2.4. *Let G be a Fuchsian group and let* $\mathrm{D}(a)$ *be the Dirichlet domain with center a. Then for almost all* $a \in \mathbb{H}^2$,

(1) *If v is a vertex of* $\mathrm{D}(a)$ *fixed by an elliptic element of G, then its cycle has length* 1*;*
(2) *If v is an accidental vertex of* $\mathrm{D}(a) \cap \mathbb{H}^2$, *then its cycle has length* 3*;*
(3) *If v is an improper vertex of* $\mathrm{D}(a)$ *and not a limit point of G , then then its cycle has length* 2*;*
(4) *If v is a proper vertex of* $\mathrm{D}(a)$, *then then its cycle has length* 1 *and is a parabolic fixed point;*
(5) *If v is a parabolic fixed point in* $\widetilde{\mathrm{D}}(a)$, *then its cycle has length* 2 *and is a proper vertex.*

Let $\mathrm{N}(G)$ be its *Nielsen domain* for G. This domain is the smallest non-empty G-invariant open convex subset of $\mathbb{H}^2$ [Bea83, §8.5]). If G is of cofinite, then $N(G) = \mathbb{H}^2$. If G is finitely generated, there are only finite G-equivalence classes of parabolic fixed points. Hence, if r_0 is sufficiently large, then $\mathscr{D}_p(r) \cap \mathscr{D}_q(r) = \emptyset$ if $p \neq q$ and $r \geq r_0$. For $r \geq r_0$, we define

$$\mathrm{K}_r(G) := \mathrm{N}(G) - \bigcup \mathscr{D}_p(r),$$

where p runs over all parabolic fixed points of G. Then the set $\pi(\overline{\mathrm{K}_r(G)})$ is a compact subset of the hyperbolic manifold $\mathbb{H}^2/G$ and hence has a finite diameter, say $M_G(r)$.

For each $a \in \mathbb{H}^n - \mathrm{F}_G$, we define a bounded set in $\mathbb{H}^n$, called the *truncated Dirichlet domain*, as

$$\mathrm{D}^*_r(a) := \mathrm{D}(a) \cap \mathrm{K}_r(G).$$

By definition, $\mathrm{D}_r^*(a)$ satisfies $\pi(\overline{\mathrm{K}_r(G)}) = \pi(\overline{\mathrm{D}_r^*(a)})$. We remark that the center a of $\mathrm{D}(a)$ is not assumed to be contained in $\mathrm{D}^*(a)$, for a could be in a horodisk $\mathscr{D}_p$. We also remark that $\mathrm{D}_r^*(a) = \mathrm{D}(a)$ when G is cocompact. There are two types of points in $\partial\mathrm{D}^*(a)$; whether it comes from the boundary of $\mathrm{D}(a)$ or not. The set of points of the former type is denoted by $\partial_{\mathrm{D}}\mathrm{D}_r^*(a)$:

$$\partial_{\mathrm{D}}\mathrm{D}_r^*(a) := \partial\mathrm{D}_r^*(a) \cap \partial\mathrm{D}(a).$$

Remark 2.5. Let s be a side of $\partial_{\mathrm{D}}\mathrm{D}_r^*(a)$. Let $\tilde{s}$ be the extension of s to a side of $\partial\mathrm{D}(a)$ and f be the side-pairing element of G which sends $\tilde{s}$ to another side. Since $\mathrm{K}_r(G)$ is invariant under f, $f(s)$ is also a side of $\partial_{\mathrm{D}}\mathrm{D}_r^*(a)$.

Lemma 2.6. *Let G be a cofinite Fuchsian group and let $\mathrm{D}(a)$ be the Dirichlet domain with center a. Then each proper vertex of $\mathrm{D}(a)$ is a parabolic fixed point.*

Proof. Let p be a proper vertex and be an end point of a side s of $\mathrm{D}(a)$. Let ζ be a point on s. As ζ approaches p, $d(\zeta, a)$ tends to ∞. Let $\pi : \mathbb{H}^2 \to \mathbb{H}^2/G$ be the natural projection. Since $\mathrm{D}(a)$ is a Dirichlet domain for G, $d(\zeta, a)$ is the shortest of the lengths of arcs from $\pi(a)$ to $\pi(\zeta)$. Hence $\pi(\zeta)$ tends to a puncture of $\mathbb{H}^2/G$. □

We assume again that S in (2) belongs G as a primitive element. Let $p = \infty$. By conjugating G with a translation $T(z) = z + h$ with a suitable real number h, we assume also that the center of a Dirichlet polygon is $a = ti$ with $t > 0$. Let $t = e^r$. In this case $\mathrm{L}_S(a) = \{\, 1/2 + iy \mid y > 0 \,\}$ and $\mathrm{L}_{S^{-1}}(a) = \{\, -1/2 + iy \mid y > 0 \,\}$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of G with $c \neq 0$. By the Shimizu-Leutbecher inequality $|c| \geq 1$. Since $A(\mathscr{D}_p(r))$ is a horodisk with Euclidean diameter $c^{-2}e^{-r} \leq e^{-r} = t^{-1}$ and $A(a)$ is on its boundary, we have $\mathrm{Im}[A(a)] \leq t^{-1}$. Now we assume that $t = e^t > 1$ and choose a point $z = x + iy$ with $y = e^\rho > 1$ and $-1/2 < x < 1/2$. Then $d(z, a) < d(z, S^m(a))$ for all non-zero integers m. If the distance $r + \rho$ between z and the horizontal line $\left\{\, x + it^{-1} \,\middle|\, -\infty < x < \infty \,\right\}$ is greater than $d(a, z)$, then

$$d(a, z) < d(A(a), z) \text{ for all } A \in G - \langle S \rangle \tag{3}$$

and hence $z \in \mathrm{D}(a)$. By (1), the inequality $d(a, z) < r + \rho$ is equivalent to

$$\frac{|a - z|}{2e^{(r+\rho)/2}} < \frac{e^{r+\rho} - 1}{2e^{(r+\rho)/2}} \text{ or } \sqrt{x^2 + (y - t)^2} < ty - 1.$$

Thus, since $|x| < 1/2$, if

$$\sqrt{\left(\frac{1}{2}\right)^2 + (t - y)^2} < ty - 1,$$

and thus, if z satisfies

$$y > \frac{\sqrt{t^2 - \frac{3}{4}}}{\sqrt{t^2 - 1}},$$

then (3) holds. The right-hand side of this inequality is decreasing for $t > 1$. So, if $a = ti \in \mathscr{D}_p(1/2)$, and $z = x + iy$ with $|x| < 1/2$ satisfies

$$y > \frac{\sqrt{e - \frac{3}{4}}}{\sqrt{e - 1}},$$

then $z \in \mathrm{D}(a)$. Let

$$C_0 = \frac{\sqrt{e - \frac{3}{4}}}{\sqrt{e - 1}} = 1.07028 \cdots < e^{1/2}.$$

We conclude that if $a = e^r i$ satisfies $e^r > e^{1/2}$, then

$$\mathscr{D}_p(1/2) \cap \left\{ z = x + iy \;\middle|\; |x| < \frac{1}{2} \right\} \subset \left\{ z = x + iy \;\middle|\; |x| < \frac{1}{2}, y > C_0 \right\} \subset \mathrm{D}(a).$$

Lemma 2.7. *Let G be a Fuchsan group and $a \in \mathbb{H}^2$. If $a \in \mathrm{D}(a) \cap \mathscr{D}_p(1/2)$ for a parabolic fixed point p of G. Then $\mathrm{D}(a) \cap \mathscr{D}_p(1/2)$ consists of two sides paired by S. Moreover, if $r \geq 1/2$ and $a \in \mathscr{D}_p(r)$, then the distance form a to $\mathrm{D}^*_r(a)$ is the distance from a to $\mathscr{H}_p(r)$.*

3 Hausdorff convergence of the boundary of truncated Dirichlet domains

We fix a large positive number $r_0 > 1/2$ and consider truncated Dirichlet domains in $\mathrm{K}_{r_0}(G)$. We shall omit the subscript r_0 throughout this section. Though the following fact may be well-known, we will provide its proof in this section:

Theorem 3.1 (See [Fer14, Proposition 2.6])**.** *Let G be a cofinite Fuchsian group. If a sequence of points $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}^2 - \mathrm{F}_G$ has the limit $a \in \mathbb{H}^2 - \mathrm{F}_G$ with respect to the distance d, then a sequence of sets $\{\partial_{\mathrm{D}}\mathrm{D}^*(a_n)\}_{n\in\mathbb{N}}$ converges to a set $\partial_{\mathrm{D}}\mathrm{D}^*(a)$ in the sense of Hausdorff topology.*

To prove this fact, we first prove the following lemma, which will be the key ingredient to prove not only Theorem 3.1 but also Lemma 4.5. We fix a positive number $r_0 > 1/2$ and consider truncated Dirichlet domains in $\mathrm{K}_{r_0}(G)$

Lemma 3.2. *Let G be a cofinite Fuchsian group. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{H}^2 - \mathrm{F}_G$ with limit $a \in \mathbb{H}^2 - \mathrm{F}_G$ with respect to the distance d. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{H}^2$ with $b_n \in \partial_{\mathrm{D}}\mathrm{D}^*(a_n)$ for any $n \in \mathbb{N}$. Let B be the set of accumulation points of $\{b_n\}_{n\in\mathbb{N}}$. Then the following holds:*

(i) *The set B is non-empty and contained in $\partial_{\mathrm{D}}\mathrm{D}^*(a)$.*

(ii) *For any $b \in B$, there are an element $f \in G$ and a subsequence $\{b_{n_j}\}_{j\in\mathbb{N}}$ converging to b such that both $f \in \mathrm{S}(a_{n_j})$ and $b_{n_j} \in \partial_{\mathrm{D}}\mathrm{D}^*(a_{n_j}) \cap \mathrm{L}_f(a_{n_j})$ hold for any $j \in \mathbb{N}$.*

Proof. We first see the non-emptyness of B, which is mentioned in (i). Finding a compact subset in $\mathbb{H}^2$ containing the sequence $\{b_n\}_{n\in\mathbb{N}}$ is enough to prove it, for $\mathbb{H}^2$ is a metric space. Since a_n converges to a, by dropping a finite number of terms, we may assume that $d(a, a_n) < 1/2$. Let r be the hyperbolic distance between a and $\mathrm{D}^*(a)$. If $r > 0$, then a is in the horodisk $\mathscr{D}_p = \mathscr{D}_p(r_0)$ for a parabolic fixed point p. By Lemma 2.7, r is the distance from a to $\mathscr{H}_p = \partial\mathscr{D}_p$. Likewise, if the distance r_n between a_n and $\mathrm{D}^*(a_n)$ is positive, then r_n is the distance from a_n to $\mathscr{H}_p$. Therefore we have $r_n < r + 1/2$. This inequality also holds when $r = 0$. Let $M = M_G + r + 1$. Since

$$b_n \in \overline{\mathrm{D}^*(a_n)} \subset \overline{\mathrm{B}_{M_G+r+1/2}(a_n)} \subset \overline{\mathrm{B}_{M_G+r+1}(a)} \tag{4}$$

for any $n \in \mathbb{N}$, The desired compact set can be taken as $\overline{\mathrm{B}_M(a)}$.

Before proving the remaining part of (i), we prove (ii). For $b \in B$, there is a subsequence $\{b_{n_j}\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} b_{n_j} = b$. Since $b_{n_j} \in \partial_{\mathrm{D}}\mathrm{D}^*(a_{n_j})$, we can find a side-pairing element $f_j \in \mathrm{S}(a_{n_j})$ which sends b_{n_j} to a point $f(b_{n_j})$ in $\partial_{\mathrm{D}}\mathrm{D}^*(a_{n_j})$. See Remark 2.5. So, we have $d(b_{n_j}, f(b_{n_j})) \le M_G$. From (4)

$$d(a, f_j(a)) \le d(a, b_{n_j}) + d(b_{n_j}, f_j(b_{n_j})) + d(f_j(a), f_j(b_{n_j})) \le 2M + M_G.$$

Since G is discrete, by replacing $\{n_j\}$ with a suitable subsequence, we can find an element $f \in G$ such that, $f = f_j$ and $b_{n_j} \in \partial_{\mathrm{D}}\mathrm{D}^*(a_{n_j}) \cap \mathrm{L}_f(a_{n_j})$ for all j.

We finally prove the remining part of (i): $B \subset \partial_{\mathrm{D}}\mathrm{D}^*(a)$. For a chosen $b \in B$, let $f \in \mathrm{S}(a_{n_i})$ and a subsequence $\{b_{n_i}\}_{i\in\mathbb{N}}$ of $\{b\}_{n\in\mathbb{N}}$ satisfying (ii). We first have $b \in \mathrm{L}_f(a)$ by the following calculation:

$$\begin{aligned} d(a, b) &= \lim_{i\to\infty} d(a_{n_i}, b_{n_i}) \\ &= \lim_{i\to\infty} d(f(a_{n_i}), b_{n_i}) \\ &= d(f(a), b). \end{aligned}$$

We next see that $b \in \partial\mathrm{D}(a)$, i.e., $f \in \mathrm{S}(a)$. As we have seen in (ii), we have $b_{n_i} \in \partial_{\mathrm{D}}\mathrm{D}^*(a_{n_i}) \cap \mathrm{L}_f(a_{n_i})$ for any $i \in \mathbb{N}$. So we have $d(a_{n_i}, b_{n_i}) \le d(g(a_{n_i}), b_{n_i})$ for any $g \in G$. Take the limits of the both sides and we have $d(a, b) \le d(g(a), b)$ for any $g \in G$, i.e., $b \in \overline{\mathrm{D}(a)}$. So we have $b \in \bar{\mathrm{D}}(a) \cap \mathrm{L}_f(a) \subset \partial\mathrm{D}(a)$.

Finally we prove that $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$. Since $\pi(b_{n_i}) \in \pi(\overline{\mathrm{K}(G)})$ and since $\pi(\overline{\mathrm{K}(G)})$ is compact, we have $\pi(b) \in \pi(\overline{\mathrm{K}(G)})$. Since $\overline{\mathrm{K}(G)} \cap \partial\mathrm{D}(a) = \partial_{\mathrm{D}}\mathrm{D}^*(a)$, we have $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$. □

Proof of Theorem 3.1. By the definition of Hausdorff topology, all we need to prove is that, for any chosen $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that the following hold for any $n \ge N$:

(1) $\mathrm{B}_\varepsilon(b) \cap \partial_{\mathrm{D}}\mathrm{D}^*(a) \neq \emptyset$ for any $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a_n)$, and

(2) $\mathrm{B}_\varepsilon(b) \cap \partial_{\mathrm{D}}\mathrm{D}^*(a_n) \neq \emptyset$ for any $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$.

Both of them will be proved by contradiction. We remark that the previous lemma is used only in the proof of (1), but not in the proof of (2).

In order to prove (1), we assume that there exists some $\varepsilon > 0$ such that, for any $N \in \mathbb{N}$, there exist $n \geq N$ and $b_n \in \partial_{\mathrm{D}}\mathrm{D}^*(a_n)$ satisfying $\mathrm{B}_\varepsilon(b_n) \cap \partial_{\mathrm{D}}\mathrm{D}^*(a) = \emptyset$, and find a contradiction. Lemma 3.2(i) provides an accumulation point b of $\{b_n\}_{n\in\mathbb{N}}$ with $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$. This is a contradiction for a large n such that $d(b, b_n) < \varepsilon$.

In order to prove (2), we assume that there exists some $\varepsilon > 0$ such that, for any $N \in \mathbb{N}$, there exist $n \geq N$ and $b_n \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$ satisfying

$$\mathrm{B}_\varepsilon(b_n) \cap \partial_{\mathrm{D}}\mathrm{D}^*(a_n) = \emptyset. \tag{5}$$

Since $\partial_{\mathrm{D}}\mathrm{D}^*(a)$ is compact, by passing to a subsequence, we can assume that the sequence b_n converges to a point b in $\partial_{\mathrm{D}}\mathrm{D}^*(a)$. We can assume also that $d(b, b_n) < \epsilon/2$ holds for all n. Since

$$d(\zeta, b) \geq d(\zeta, b_n) - d(b_n, b) > \epsilon/2,$$

for all $\zeta \in \partial_{\mathrm{D}}\mathrm{D}^*(a_n)$, $\partial_{\mathrm{D}}\mathrm{D}^*(a_n) \subset \mathrm{B}_{\varepsilon/2}(b)^{\complement}$. Since $\partial_{\mathrm{D}}\mathrm{D}^*(a)$ is a polygon with finitely many sides, there is an interior point b' of a side of $\partial_{\mathrm{D}}\mathrm{D}^*(a)$ with $d(b, b') < \epsilon/4$. We replace b by b' and ε by a smaller number $< \varepsilon/4$ so that $B_\varepsilon(b)$ is in the interior of $K(G)$ and disjoint from $\partial_{\mathrm{D}}\mathrm{D}^*(a_n)$ for all n. This condition implies also that $B_\varepsilon(b)$ is disjoint from $\partial\mathrm{D}(a_n)$. By passing to a subsequence, if necessary, we assume that either

(1) $\mathrm{B}_\varepsilon(b) \subset \mathrm{D}(a_n)$ for all n, or

(2) $\mathrm{B}_\varepsilon(b) \subset \overline{\mathrm{D}(a_n)}^{\complement}$ for all n.

If Case (1) occurs, for all $p \in \mathrm{B}_\varepsilon(b)$ and $f \in G - \{\mathrm{id}\}$, we have $d(p, a_n) < d(p, f(a_n))$ and then, by taking the limit, $d(p, a) \leq d(p, f(a))$. Hence, $\mathrm{B}_\varepsilon(b) \subset \overline{\mathrm{D}(a_n)}$. This contradicts that $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$. If Case (2) occurs, for all $p \in \mathrm{B}_\varepsilon(b)$, there is an $f_n \in G - \{\mathrm{id}\}$ such that $d(p, a_n) > d(p, f_n(a_n))$. Then

$$\begin{aligned} d(p, f_n(a)) &\leq d(p, f_n(a_n)) + d(f_n(a_n), f_n(a)) \\ &\leq d(p, a_n) + d(a_n, a) < d(p, a) + 2d(a, a_n), \end{aligned}$$

hence $d(p, f_n(a))$ is bounded. So, there is an $f \in G - \{\mathrm{id}\}$ such that $f = f_n$ for infinitely many n. By taking the limit in $d(p, a_n) > d(p, f(a_n))$, we have $d(p, a) \geq d(p, f(a))$. Thus $\mathrm{B}_\varepsilon(b) \subset \mathrm{D}(a)^{\complement}$. This contradicts also that $b \in \partial_{\mathrm{D}}\mathrm{D}^*(a)$. $\square$

4 Openness of the set of regular points

In this section, we establixh Propositions 3.1 and 3.3 of Fera's paper [Fer14] for cofinite Fuchsian groups. Let G be a cofinite Fuchsian group of type (g, m). The number of side-pairing elements of the Dirichlet domain centered at p satisfies

$$4g + 2m - 2 \leq |S(p)| \leq 12g + 4m - 6.$$

See [Bea83, Theorem 10.5.1]. Let S_G denote the maximum number $12g+4m-6$.

Definition 4.1. For a cofinite Fuchsian group G, a point $p \in \mathbb{H}^2 - \mathrm{F}_G$ is said to be *regular* if $|\mathrm{S}(p)| = \mathrm{S}_G$, and *exceptional* otherwise.

The sets of regular points and exceptional points are denoted as follows:

$$\mathrm{Reg}(G) := \left\{ p \in \mathbb{H}^2 \;\middle|\; p \text{ is regular} \right\}, \quad \mathrm{Exp}(G) := \left\{ p \in \mathbb{H}^2 \;\middle|\; p \text{ is exceptional} \right\}.$$

These sets together with F_G disjointly decompose $\mathbb{H}^2$. We also define the set of points having the same side paring transformations of given $\mathrm{D}(a)$ by $\mathrm{PS}(a)$:

$$\mathrm{PS}(a) := \left\{ p \in \mathbb{H}^2 \;\middle|\; \mathrm{S}(p) = \mathrm{S}(a) \right\}.$$

It is easily observed that $\mathrm{PS}(a) = \mathrm{PS}(b)$ if $b \in \mathrm{PS}(a)$.

Remark 4.2. Let G be a cofinite Fuchsian group of type (g, m) and $a \in \mathbb{H}^2$. By the proof of Theorem 10.5.1 of [Bea83], the Dirichlet domain $\mathrm{D}(a)$ has $S_G = 12g+4m-6$ sides if and only if all cycles of parabolic and elliptic fixed points on the vertices of $\mathrm{D}(a)$ have length 1 and all accidental cycles have length 3. Then, by Theorem 2.4, $\mathrm{Exp}(G)$ has measure 0, which implies that a generic point of $\mathbb{H}^2$ is a regular point. The idea of regular points is generalized to the case for Kleinian groups, that is, discrete groups of orientation-preserving isometries of $\mathbb{H}^3$ in [JM86]. Though they tried to show that the set of regular points is full measure in $\mathbb{H}^3$, the proof has not been completed. See [DU09, §1] for more detail.

The main result in this section is the following theorem, which will be used to prove Theorem 5.1:

Theorem 4.3 (See [Fer14, Proposition 3.3])**.** *For a cofinite Fuchsian group G and a point $a \in \mathrm{Reg}(G)$, the set $\mathrm{PS}(a)$ is open in $\mathbb{H}^2$.*

An immediate corollary, which will not be used to prove Theorem 5.1, is the openness of $\mathrm{Reg}(G)$, since it is the union of such $\mathrm{PS}(p)$.

Corollary 4.4 (See also [Nää85, Corollary 2.1] and [Fer14, Theorem 3.4])**.** *For a cofinite Fuchsian group G, the set $\mathrm{Reg}(G)$ is open in $\mathbb{H}^n$.* □

Our proof of Theorem 4.3 is based on a series of lemmata, which are prepared from now on.

Lemma 4.5 (See [Fer14, Proof of Proposition 3.1])**.** *For a cofinite Fuchsian G and any sequence of points $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}^2 - \mathrm{F}_G$ with limit $a \in \mathbb{H}^2 - \mathrm{F}_G$ has a subsequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ such that $\mathrm{S}(a) \subset \bigcap_{i\in\mathbb{N}} \mathrm{S}(a_{n_i})$.*

The ingredients of the proof are Lemma 3.2(ii) and Theorem 3.1(2).

Proof. Since G is cofinite, $\mathrm{D}(a)$ has finitely many sides and by Lemma 2.6, every ideal vertex is a parabolic fixed point. Therefore, we can choose a large r such that each ideal vertex p of $\mathrm{D}(a)$ has the following properties:

(1) $\partial\mathrm{D}(a)$ meets $\mathscr{D}_p(r)$ in two sides both ending at p;
(2) $\mathscr{H}_p(r) = \partial\mathscr{D}_p$ cuts the above two sides in their interior points.

Let $\partial_{\mathrm{D}}\mathrm{D}^*(a) = \mathrm{D}(a) \cap K_r(G)$. Let $f \in \mathrm{S}(a)$ be a side-pairing element. There is a side $s \subset \partial\mathrm{D}(a)$ such that $s = \bar{\mathrm{D}}(a) \cap f(\overline{\mathrm{D}(a)}) \subset \mathrm{L}_f(a)$. For this s, choose any $b \in \mathrm{Int}(s) \cap \partial_{\mathrm{D}}\mathrm{D}^*(a)$. For this b, Theorem 3.1(2) provides a point $b_n \in \partial_{\mathrm{D}}\mathrm{D}^*(a_n)$ for each $n \in \mathbb{N}$ such that $\lim_{n\to\infty} b_n = b$ in $\mathbb{H}^n$. Apply Lemma 3.2(ii) to b, and we can find a subsequence $\{b_{n_i}\}_{i\in\mathbb{N}}$ and $h \in \bigcap_{i\in\mathbb{N}} \mathrm{S}(a_{n_i})$ such that $b_{n_i} \in \mathrm{L}_h(a_{n_i})$.

In order to see that the subsequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ is the desired one, it is enough to show that $h = f$, and it is done as follows: Since $b_{n_i} \in \mathrm{L}_h(a_{n_i})$, by the continuity of the bisector, we have $b \in \mathrm{L}_h(a)$. Since $b \in s$, we also have $b \in \mathrm{L}_f(a)$. So we have $b \in \mathrm{L}_f(a) \cap \mathrm{L}_h(a)$. Since b is taken from $\mathrm{Int}(s)$, it cannot be a vertex of $\mathrm{D}(a)$. So, by the uniqueness of the side paring transformation, we have $\mathrm{L}_f(a) = \mathrm{L}_h(a)$, which implies $f = h$. Since $|S(a)|$ is finite, by repeating this argument we obtain a desired subsequence $\{a_{n_i}\}$. □

Lemma 4.6 (See [Fer14, Proposition 3.1])**.** *For a cofinite Fuchsian group G and a point $a \in \mathbb{H}^2 - \mathrm{F}_G$, there is an open neighbourhood $U \subset \mathbb{H}^2 - \mathrm{F}_G$ of a such that $S(a) \subset \mathrm{S}(u)$ for any $u \in U$.*

Proof. We prove this lemma by contradiction. So we assume that there are some $a \in \mathbb{H}^2 - \mathrm{F}_G$ and $f \in \mathrm{S}(a)$ such that, for any open neighborhood U of a, there is some $u \in U$ with $f \notin \mathrm{S}(u)$. Since F_G is a discrete set of points, there is a positive number n_0 such that $B_{1/n_0}(a) \cap \mathrm{F}_G = \emptyset$. Then we can find a point $a_n \in \mathrm{B}_{1/n}(a)$ for all $n > n_0$ such that $f \notin \bigcap_{n>n_0} \mathrm{S}(a_n)$, which contradicts Lemma 4.5. So there is an open neighborhood $U_f \subset \mathbb{H}^2 - \mathrm{F}_G$ of a such that $f \in S(u)$ for all $u \in U_f$. Since $S(a)$ is a finite set, $U = \bigcap_{f\in S(a)} U_f$ is a desired neighborhood of a. □

Proof of Theorem 4.3. Let U be an open neighborhood of a obtained in Lemma 4.6. We show that $U \subset \mathrm{PS}(a)$. For any $u \in U$, the inclusion $\mathrm{S}(a) \subset \mathrm{S}(u)$ comes from the definition of U. Thus $|\mathrm{S}(p)| \leq |\mathrm{S}(u)|$. Since $a \in \mathrm{Reg}(G)$, $\mathrm{S}_G = |\mathrm{S}(p)|$ is the maximal number of sides. Now we conclude $\mathrm{S}(u) = \mathrm{S}(a)$. □

Example 4.7. [Ume, Theorem 24]. Let $\triangle$ be the ideal triangle bounded by $m_1 = \{z : |z| = 1\}$, $m_2 = \{z : \mathrm{Re}[z] = 1\}$ and $m_3 = \{z : \mathrm{Re}[z] = -1\}$. Let R_i be the reflection in m_i for $i = 1, 2, 3$. We define

$$T_1 = R_1R_3 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},\ T_2 = R_2R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T_3 = T_1T_2 = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}.$$

Let G be the group generated by T_1 and T_2. If a is an interior point of $\triangle$, then the Dirichlet domain $\mathrm{D}(a)$ for G is the geodesic hexagon bounded by the bisection lines

$$\mathrm{L}_{T_1^{\pm 1}}(a), \quad \mathrm{L}_{T_2^{\pm 1}}(a), \quad \mathrm{L}_{T_3^{\pm 1}}(a)$$

and $S(a) = \{T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}\}$. $\mathrm{D}(a)$ includes $\triangle$ as a proper subdomain. However, $\mathrm{D}(i)$ equals the quadrilateral $\triangle \cup m_1 \cup R_1(\triangle)$ and $S(i) = \{T_1^{\pm 1}, T_2^{\pm}\}$. So T_3 belongs to $S(a)$ for any $a \in \triangle$ but not to $S(i)$. The two sides of $\mathrm{D}(a)$ which are paired by T_3 leave any compact subset of $\mathbb{H}^2$ as a approaches i from the inside of $\triangle$. Theorem 24 of [Ume] shows that $\mathrm{PS}(a) = \triangle$ for all $a \in \triangle$, and hence $\mathrm{PS}(a)$ is unbounded. We remark that $\mathrm{PS}(a)$ is bounded for all $a \in \mathbb{H}^2$ if G is a cocompact Fuchsian group. See [Fer14, Thereom 4.2].

5 Existence of exceptional points

Let G be a cofinite Fuchsian group. Theorem 2.4 says that $\mathrm{Reg}(G)$ is full-measure in $\mathbb{H}^2$, or equivalently that $\mathrm{Exp}(G)$ has measure zero. We should remark here that this fact does not tell us whether the set $\mathrm{Exp}(G)$ is empty or not. For cocompact Fuchsian groups G, Fera showed in [Fer14, Theorem 4.3] that $\mathrm{Exp}(G)$ is non-empty and moreover is an uncountable set. We provide a generalization of this result to cofinie Fuchsian groups.

Theorem 5.1 (See [Fer14, Theorem 4.3])**.** *For a cofinite Fuchsian G, the set* $\mathrm{Exp}(G)$ *contains uncountably many points.*

As is mentioned in Remark 4.2, the non-emptyness of $\mathrm{Reg}(G)$ is not sure for discrete groups of isometries in $\mathbb{H}^n$ for $n \geq 3$.

From now on, we choose the unit disc model $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of $\mathbb{H}^2$. Let $S = \partial\mathbb{D}$ be the unit circle. A point $\xi \in S$ is a *transitive limit point* (or *Myrberg point*) of G if the projection to unit tangent bundle of the surface $\mathbb{D}/G$ of the unit tangent field along a geodesic ray from any point $z \in \mathbb{D}$ to ξ is dense. (Hence the projections to the surface $\mathbb{D}/G$ of all geodesic rays ending at ξ is dense.) Let $\mathcal{E}_G$ denote the set of all transitive limit points of G. If G is cofinite (or if, more generally, G is of divergence type), the measure of $\mathcal{E}_G$ is 2π [Shimada60].

Proof of Theorem 5.1. By replacing G by a suitable conjugate of it, we assume that 0 is a regular point for G. Let $\mathcal{E}_G$ denote the set of transitive limit points of G. Then $\mathcal{E}_G$ is an uncountable set. Let ξ be an arbitrary point of $\mathcal{E}_G$ and $L(\xi) = \{t\xi : 0 \leq t < 1\}$.

We shall show there exists an exceptional point in $L(\xi)$. We suppose to the contrary that $L(\xi) \subset \mathrm{Reg}(G)$. Since 0 is a regular point, $L(\xi) \cap \mathrm{PS}(0)$ is an open subset of $L(\xi)$. We assume that $L(\xi) \subset \mathrm{PS}(0)$. Let $\pi : \mathbb{D} \to \mathbb{D}/G$ be the natural projection. Since ξ is a transitive limit point, $\pi(L(\xi))$ visits the open set $\pi(\mathrm{PS}(0))$ infinitely many times. So, there exist a sequence $\zeta_n = t_n\xi$ with $t_n \to 1$

as $n \to \infty$ and a sequence of distinct elements $h_n \in G$ such that $\zeta_n \in h_n(\mathrm{PS}(0))$. Since

$$\zeta_n \in \mathrm{PS}(0) \cap h_n(\mathrm{PS}(0)),$$

and $h_n(\mathrm{D}(0))$ is the Dirichlet domain with center $h_n(0)$, we have

$$S(0) = S(h_n(0)) = \{h_n f h_n^{-1} : f \in S(0)\}.$$

Let $S(0) = \{f_1, f_2, ..., f_N\}$, $N = S_G$. Then

$$S(h_n(0)) = \{h_n f_1 h_n^{-1}, h_n f_2 h_n^{-1}, ..., h_n f_N h_n^{-1}\}.$$

By passing to a subsequence, we can find a permutation σ of $\{1, 2, ..., N\}$ such that $h_n f_i h_n^{-1} = f_{\sigma(i)}$ for $1, 2, ..., N$. Then

$$h_n f_i h_n^{-1} = h_{n+1} f_i h_{n+1}^{-1}.$$

Since $S(0)$ generates G, this means that $h_{n+1}^{-1} h_n$ commutes with all elements of G. Since G is non-elementary, $h_{n+1}^{-1} h_n = \mathrm{id}$, and hence $h_n = h_{n+1}$. This is a contradiction. We conclude that there is a point $a \in L(\xi) - \mathrm{PS}(0)$. By our assumption, $L(\xi) \subset \mathrm{Reg}(G)$. Then, by Theorem 5.1, $L(\xi)$ can be covered more than one disjoint open sets of the form $\mathrm{PS}(b)$. However, this contradicts the connectivity of $L(\xi)$. Therefore there is an exceptional point on $L(\xi)$ for each $\xi \in \mathcal{E}_G$. □

References

[Bea83] Alan F Beardon. The geometry of discrete groups, volume 91 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1983.

[Bea77] Alan F Beardon. The Geometry of Discrete Groups. Discrete groups and automorphic functions (Proc. Conf., Cambridge, 1975), pages 47–72, 1977.

[DU09] Raquel Díaz and Akira Ushijima. On the properness of some algebraic equations appearing in Fuchsian groups. Topology Proceedings, pages 1–26, 2009.

[Fer14] Joseph Fera. Exceptional points for cocompact Fuchsian groups. Annales Academiae Scientiarum Fennicae Mathematica, 39:463–472, 2014.

[Gre77] L. Greenberg. Finiteness theorems for Fuchsian and Kleinian groups. Discrete groups and automorphic functions (Proc. Conf., Cambridge, 1975), pages 199–257, 1977.

[JM86] T. Jørgensen and A. Marden. Generic fundamental polyhedra for Kleinian groups. Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986), pages 69–85, 1988.

[Kat92] Svetlana Katok. Fuchsian groups. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.

[Nää85] Marjatta Näätänen. On the stability of identification patterns for Dirichlet regions. Annales Academiae Scientiarum Fennicae Mathematica, 10:411–417, 1985.

[Rat06] John G Ratcliffe. Foundations of hyperbolic manifolds, volume 149 of Graduate Texts in Mathematics. Springer, New York, second edition, 2006.

[Shimada60] I.. Shimada,. On P. J. Myrberg's approximation theorem on Fuchsian groups. Mem. Coll. Sci. Kyoto Univ., Ser. A. 33, 231–241, 1960.

[Ume] Yuriko Umemoto. On Dirichlet fundamental domain for Fuchsian groups. Master Thesis, Graduate School of Science, Osaka City University,