Title

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Journal

SN Operations Research Forum Vol.1

Published

2020-2-21

URL

https://link.springer.com/article/10.1007/s43069-020-0005-x

**SN Operations Research Forum manuscript No.**
(will be inserted by the editor)

# Duality theorems for convex and quasiconvex set functions

**Satoshi Suzuki · Daishi Kuroiwa**

Received: date / Accepted: date

**Abstract** In mathematical programming, duality theorems play a central role. Especially, in convex and quasiconvex programming, Lagrange duality and surrogate duality have been studied extensively. Additionally, constraint qualifications are essential ingredients of the powerful duality theory. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications.

A typical example of mathematical programming is a minimization problem of a real-valued function on a vector space. This types of problems have been studied widely and have been generalized in several directions. Recently, the authors investigate set functions and Fenchel duality. However, duality theorems and its constraint qualifications for mathematical programming with set functions have not been studied yet. It is expected to study set functions and duality theorems.

In this paper, we study duality theorems for convex and quasiconvex set functions. We show Lagrange duality theorem for convex set functions and surrogate duality theorem for quasiconvex set functions under the Slater condition. As an application, we investigate an uncertain problem with motion uncertainty.

**Keywords** set function · Lagrange duality · surrogate duality · mathematical programming with uncertainty

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This work was partially supported by JSPS KAKENHI Grant Numbers 19K03620, 19K03637.

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**Mathematics Subject Classification (2010)** 90C30 · 90C46 · 26B25

## 1 Introduction

In mathematical programming, duality theorems play a central role. Especially, in convex and quasiconvex programming, Lagrange duality and surrogate duality have been studied extensively. Additionally, constraint qualifications are essential ingredients of the powerful duality theory. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications. As generalizations of such interior point conditions, research on necessary and sufficient constraint qualifications for some types of duality theorems have been studied, see [1,4,8,14,15,23,28–37].

In [27], Morris introduces a set function, which is defined on the class of measurable subsets of an atomless finite measure space satisfying a certain convexity condition. Although a set-valued function $f$ is defined on a vector space and the value $f(x)$ is a set, a set function $F$ is defined on a class of subsets and the value $F(A)$ is a real number. For set functions, various results in convex analysis have been generalized, for example see [5,6,12,13,20–22,25, 27,39]. However, the domain of Morris's set functions is complicated. Hence, in [38], the authors study convex set functions in a simple way. We introduce Fenchel conjugate for set functions, and study Fenchel duality in terms of convex analysis on an embedding normed space of compact convex subsets. However, duality theorems and its constraint qualifications for mathematical programming with set functions in [38] have not been studied yet. It is expected to study set functions and duality theorems in simple definition and setting.

In this paper, we study duality theorems for convex and quasiconvex set functions. Especially, we show Lagrange duality theorem for convex set functions and surrogate duality theorem for quasiconvex set functions under the Slater condition. As an application, we investigate an uncertain problem with motion uncertainty. We regard a decision variable set as an error caused by a motion, and investigate robust approach for the problem.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we study Lagrange duality theorem for convex set functions. In Section 4, we show surrogate duality theorem for quasiconvex set functions. In Section 5, we investigate an uncertain problem as an application of our results.

## 2 Preliminaries

Let $X$ be a vector space over $\mathbb{R}$. Given nonempty sets $A$, $B \subset X$, and $\Gamma \subset \mathbb{R}$, we define $A + B$ and $\Gamma A$ as follows:

$$A + B = \{x + y \in X \mid x \in A, y \in B\},$$
$$\Gamma A = \{\gamma x \in X \mid \gamma \in \Gamma, x \in A\}.$$

Also, we define $A + \emptyset = \Gamma\emptyset = \emptyset A = \emptyset$. A set $A$ is convex if for each $x$, $y \in A$, and $\alpha \in [0,1]$, $(1-\alpha)x + \alpha y \in A$. Let $\mathcal{A}_0$ be the following family of nonempty sets:

$$\mathcal{A}_0 = \{A \subset X \mid A : \text{nonempty}\}.$$

It is clear that $\mathcal{A}_0$ is closed under addition and multiplication by positive scalars. A subfamily $\mathcal{A} \subset \mathcal{A}_0$ is said to be convex if for each $A$, $B \in \mathcal{A}$, and $\alpha \in [0,1]$, $(1-\alpha)A + \alpha B \in \mathcal{A}$. There are so many examples of a convex subfamily as follows:

– the family of convex subsets of a vector space,
– the family of singletons of a vector space,
– the family of finite subsets of a vector space,
– the family of compact subsets of a topological vector space,
– the family of open subsets of a topological vector space.

We introduce the following elementary results.

**Theorem 1** *[38] Let $\mathcal{A}$, $\mathcal{B} \subset \mathcal{A}_0$. The following statements hold:*

*(i) $\mathcal{A}_0$ is convex.*
*(ii) If $\mathcal{A}$, $\mathcal{B}$ are convex, then $\mathcal{A} + \mathcal{B} = \{A + B \subset X \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ is convex.*
*(iii) If $\mathcal{A}$ is convex and $\alpha \in \mathbb{R}$, then $\alpha\mathcal{A} = \{\alpha A \mid A \in \mathcal{A}\}$ is convex.*
*(iv) Let $I$ be an index set, and $\mathcal{A}_i$ a convex subfamily of $\mathcal{A}_0$ for each $i \in I$. Then, $\cap_{i \in I}\mathcal{A}_i$ is convex.*

Let $f$ be a real-valued function on $\mathcal{A}_0$. $f$ is said to be convex if for each $A$, $B \in \mathcal{A}_0$, and $\alpha \in [0,1]$, $f((1-\alpha)A + \alpha B) \leq (1-\alpha)f(A) + \alpha f(B)$. Additionally, $f$ is said to be quasiconvex if for each $A$, $B \in \mathcal{A}_0$, and $\alpha \in [0,1]$, $f((1-\alpha)A + \alpha B) \leq \max\{f(A), f(B)\}$. The epigraph of $f$ is defined as $\text{epi}f = \{(A, \alpha) \in \mathcal{A}_0 \times \mathbb{R} \mid f(x) \leq \alpha\}$. Define the level sets of $f$ with respect to a binary relation $\diamond$ on $\mathbb{R}$ as

$$L(f, \diamond, \alpha) = \{A \in \mathcal{A}_0 \mid f(A) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. The following theorem is easy to prove and the proof will be omitted.

**Theorem 2** *Let $f$ be a real-valued function on $\mathcal{A}_0$. Then, the following statements hold:*

*(i) $f$ is convex if and only if $\text{epi}f$ is convex.*
*(ii) $f$ is quasiconvex if and only if $L(f, \leq, \alpha)$ is convex for any $\alpha \in \mathbb{R}$, if and only if $L(f, <, \alpha)$ is convex for any $\alpha \in \mathbb{R}$.*

A real-valued function $f$ on $\mathcal{A}_0$ is said to be interval upper semicontinuous (interval-usc) on a convex subfamily $\mathcal{A} \subset \mathcal{A}_0$, if for each $A$, $B \in \mathcal{A}$ satisfying $A \neq B$, the following function $h$ is upper semicontinuous on $[0,1]$:

$$h(t) = f((1-t)A + tB).$$

Let $\langle v, z \rangle$ denote the inner product of two vectors $v$ and $z$ in the $n$-dimensional Euclidean space $\mathbb{R}^n$. Given a set $S \subset \mathbb{R}^n$, we denote the closure and the interior of $S$, by $\text{cl}S$ and $\text{int}S$, respectively. We denote by $B(z, r)$ the open ball centered at $z \in \mathbb{R}^n$ with radius $r > 0$.

## 3 Lagrange duality

We consider the following minimization problem involving convex set functions:

$$\begin{cases} \text{minimize } f(A), \\ \text{subject to } A \in \mathcal{C},\ g_i(A) \le 0, \forall i \in I, \end{cases}$$

where $I = \{1, \ldots, m\}$, $f$ is a real-valued convex function on $\mathcal{A}_0$, $g_i$ is a real-valued convex function on $\mathcal{A}_0$ for each $i \in I$, and $\mathcal{C}$ is a convex subfamily of $\mathcal{A}_0$. Let $\mathcal{A} = \{A \in \mathcal{C} \mid g_i(A) \le 0, \forall i \in I\}$, and assume that $\mathcal{A}$ is nonempty. In this section, we study Lagrange duality for the above problem. Lagrange duality is one of the most well-known duality in convex programming and plays a central role. The best-known constraint qualifications are the interior point conditions, also known as the Slater-type constraint qualifications.

We show the following Lagrange duality theorem under the Slater-type condition.

**Theorem 3** *Let $I = \{1, \ldots, m\}$, $f$ a real-valued convex function on $\mathcal{A}_0$, $g_i$ a real-valued convex function on $\mathcal{A}_0$ for each $i \in I$, $\mathcal{C}$ a convex subfamily of $\mathcal{A}_0$, and $\mathcal{A} = \{A \in \mathcal{C} \mid g_i(A) \le 0, \forall i \in I\}$. Assume that there exists $A_1 \in \mathcal{C}$ such that $g_i(A_1) < 0$ for each $i \in I$.*

*Then,*

$$\inf_{A \in \mathcal{A}} f(A) = \max_{\lambda \in \mathbb{R}^m_+} \inf_{A \in \mathcal{C}} \left\{ f(A) + \sum_{i=1}^m \lambda_i g_i(A) \right\}.$$

*Proof* Let $\mu = \inf_{A \in \mathcal{A}} f(A)$. At first, we show Lagrange weak duality. Let $\lambda \in \mathbb{R}^m_+$. For each $A \in \mathcal{A}$, $\sum_{i=1}^m \lambda_i g_i(A) \le 0$ since $g_i(A) \le 0$ for all $i \in I$. Hence

$$\begin{aligned} \mu &= \inf_{A \in \mathcal{A}} f(A) \\ &\ge \inf_{A \in \mathcal{A}} \left\{ f(A) + \sum_{i=1}^m \lambda_i g_i(A) \right\} \\ &\ge \inf_{A \in \mathcal{C}} \left\{ f(A) + \sum_{i=1}^m \lambda_i g_i(A) \right\}, \end{aligned}$$

that is, Lagrange weak duality holds.

If $\mu = -\infty$, then putting $\lambda = 0$, the equation holds.

Assume that $\mu > -\infty$. Since $A_1 \in \mathcal{A}$ and $f$ is real-valued,

$$-\infty < \mu \le f(A_1) < \infty,$$

that is, $\mu \in \mathbb{R}$. Then, without loss of generality, we can assume that $\mu = 0$. Let

$$S = \left\{ z = (z_0, z_1, \ldots, z_m) \in \mathbb{R}^{m+1} \,\middle|\, \exists A \in \mathcal{C} \text{ s.t. } \begin{array}{l} g_i(A) \le z_i, \forall i \in I, \\ f(A) \le z_0 \end{array} \right\},$$

$$T = \left\{ z = (z_0, z_1, \ldots, z_m) \in \mathbb{R}^{m+1} \,|\, z_i < 0, \forall i \in \{0, 1, \ldots, m\} \right\}.$$

It is clear that $T$ is open convex, and cl $T$ is a closed convex cone. $S$ is similar to the set in the proof of Lemma 6.2.3 in [2], and we can prove easily that $S$ is convex. Additionally, we can prove that $S \cap T = \emptyset$ since $\mu = \inf_{A \in \mathcal{A}} f(A) = 0$.

Hence, by the separation theorem between $S$ and $T$ on $\mathbb{R}^{m+1}$, there exist $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_m) \in \mathbb{R}^{m+1} \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that for each $v \in S$ and $w \in T$,

$$\langle \lambda, v \rangle \geq \beta \geq \langle \lambda, w \rangle .$$

Since cl $T$ is a cone, we can show that $\beta = 0$. In addition, by the definition of $T$, we can prove that $\lambda \in \mathbb{R}^{m+1}_+ \setminus \{0\}$. Actually, if $\lambda_i < 0$ for some $i$, then there exists $w \in T$ such that $\langle \lambda, w \rangle > 0 = \beta$. This is a contradiction.

Next, we show that $\lambda_0 > 0$. Assume that $\lambda_0 = 0$. Since $A_1 \in \mathcal{A}$, $f(A_1) \geq \mu = 0$. Put

$$v = (f(A_1), g_1(A_1), \dots, g_m(A_1)),$$

and

$$w = (-1, g_1(A_1), \dots, g_m(A_1)).$$

Then, we can check that $v \in S$ and $w \in T$. Hence,

$$\sum_{i=1}^{m} \lambda_i g_i(A_1) = \sum_{i=0}^{m} \lambda_i v_i = \langle \lambda, v \rangle \geq 0 \geq \langle \lambda, w \rangle = \sum_{i=0}^{m} \lambda_i w_i = \sum_{i=1}^{m} \lambda_i g_i(A_1).$$

Since $g_i(A_1) < 0$ for each $i \in I$, $\lambda_i = 0$ for each $i \in I$. This means that $\lambda = 0$. This is a contradiction. Hence $\lambda_0 > 0$.

Put $\bar{\lambda} \in \mathbb{R}^m_+$ as follows:

$$\bar{\lambda} = \left( \frac{\lambda_1}{\lambda_0}, \frac{\lambda_2}{\lambda_0}, \dots, \frac{\lambda_m}{\lambda_0} \right).$$

For each $A \in \mathcal{C}$, let $v_A = (f(A), g_1(A), \dots, g_m(A)) \in S$. Then, $\langle \lambda, v_A \rangle \geq 0$, that is,

$$\mu = 0 \leq \frac{1}{\lambda_0} \langle \lambda, v_A \rangle = f(A) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(A).$$

Hence,

$$\begin{aligned} \mu &\leq \inf_{A \in \mathcal{C}} \left\{ f(A) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(A) \right\} \\ &\leq \sup_{\lambda \in \mathbb{R}^m_+} \inf_{A \in \mathcal{C}} \left\{ f(A) + \sum_{i=1}^{m} \lambda_i g_i(A) \right\} \\ &\leq \mu. \end{aligned}$$

This completes the proof.

## 4 Surrogate duality

We consider the following minimization problem involving quasiconvex and convex set functions:

$$\begin{cases} \text{minimize } f(A), \\ \text{subject to } A \in \mathcal{C},\ g_i(A) \leq 0, \forall i \in I, \end{cases}$$

where $I = \{1, \ldots, m\}$, $f$ is a real-valued, interval-usc, quasiconvex function on $\mathcal{A}_0$, $g_i$ a real-valued convex function on $\mathcal{A}_0$ for each $i \in I$, and $\mathcal{C}$ is a convex subfamily of $\mathcal{A}_0$. Let $\mathcal{A} = \{A \in \mathcal{C} \mid g_i(A) \leq 0, \forall i \in I\}$, and assume that $\mathcal{A}$ is nonempty. In this section, we study surrogate duality for the minimization problem involving quasiconvex and convex set functions. Surrogate duality have been studied for various types of mathematical programming problems, for example, zero-one integer programming problem, quasiconvex programming, robust optimization, and so on. It is worth noting that the Slater-type condition is one of the constraint qualification for surrogate duality. For more details, see [7,9–11,26,32,34] and references therein.

We show the following surrogate duality theorem for the above problem involving quasiconvex and convex set functions under the Slater-type condition.

**Theorem 4** *Let $I = \{1, \ldots, m\}$, $f$ a real-valued quasiconvex function on $\mathcal{A}_0$, $g_i$ a real-valued convex function on $\mathcal{A}_0$ for each $i \in I$, $\mathcal{C}$ is a convex subfamily of $\mathcal{A}_0$, and $\mathcal{A} = \{A \in \mathcal{C} \mid g_i(A) \leq 0, \forall i \in I\}$. Assume that $f$ is interval-usc on $\mathcal{C}$, and there exists $A_1 \in \mathcal{C}$ such that $g_i(A_1) < 0$ for each $i \in I$.*

*Then,*

$$\inf_{A \in \mathcal{A}} f(A) = \max_{\lambda \in \mathbb{R}^m_+} \inf \left\{ f(A) \,\middle|\, A \in \mathcal{C}, \sum_{i=1}^m \lambda_i g_i(A) \leq 0 \right\}.$$

*Proof* Let $\mu = \inf_{A \in \mathcal{A}} f(A)$. At first, we show surrogate weak duality. Let $\lambda \in \mathbb{R}^m_+$ and $A \in \mathcal{A}$, then $g_i(A) \leq 0$ for all $i \in I$. Hence, $\mathcal{A} \subset \{A \in \mathcal{C} \mid \sum_{i=1}^m \lambda_i g_i(A) \leq 0\}$. This shows that

$$\mu \geq \sup_{\lambda \in \mathbb{R}^m_+} \inf \left\{ f(A) \,\middle|\, A \in \mathcal{C}, \sum_{i=1}^m \lambda_i g_i(A) \leq 0 \right\},$$

that is, surrogate weak duality holds.

If $\mu = -\infty$, then putting $\lambda = 0$, the equation holds.

Assume that $\mu > -\infty$. Let

$$S = \left\{ z \in \mathbb{R}^m \,\middle|\, \exists A \in \mathcal{C} \text{ s.t. } \begin{array}{l} g_i(A) \leq z_i, \forall i \in I, \\ f(A) < \mu \end{array} \right\},$$
$$N = \{z \in \mathbb{R}^m \mid z_i \leq 0, \forall i \in I\}.$$

It is clear that $N$ is a closed convex cone. $S$ is similar to the set in the proof of Theorem 1 in [26], and we can prove easily that $S$ is convex. In addition, we can check that $S \cap N = \emptyset$ since $\mu = \inf_{A \in \mathcal{A}} f(A)$.

Hence, by the separation theorem between $S$ and $N$, there exist $\bar{\lambda} \in \mathbb{R}^m \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that for each $z \in S$ and $w \in N$,

$$\langle \bar{\lambda}, z \rangle \geq \beta \geq \langle \bar{\lambda}, w \rangle .$$

By the definition of the cone $N$, we can show that $\beta = 0$ and $\bar{\lambda} \in \mathbb{R}^m_+ \setminus \{0\}$.

Next, we show that for each $z \in S$, $\langle \bar{\lambda}, z \rangle > 0$. Assume that there exists $z \in S$ such that $\langle \bar{\lambda}, z \rangle = 0$. Then, there exists $A \in \mathcal{C}$ such that $g_i(A) \leq z_i$ for each $i \in I$, and $f(A) < \mu$. Let

$$\bar{z} = (g_1(A_1), g_2(A_1), \ldots, g_m(A_1)) \in \mathbb{R}^m,$$

then $\bar{z} \in N \cap S^c$ since $\mu = \inf_{A \in \mathcal{A}} f(A)$ and the Slater condition. For each $\alpha \in (0, 1]$,

$$\langle \bar{\lambda}, (1-\alpha)z + \alpha\bar{z} \rangle = (1-\alpha) \langle \bar{\lambda}, z \rangle + \alpha \langle \bar{\lambda}, \bar{z} \rangle = \alpha \langle \bar{\lambda}, \bar{z} \rangle < 0$$

since $\alpha > 0$, $\bar{z}_i = g_i(A_1) < 0$ for each $i \in I$, and $\bar{\lambda} \in \mathbb{R}^m_+ \setminus \{0\}$. By the above separation inequality, $(1-\alpha)z + \alpha\bar{z} \notin S$. On the other hand, for each $i \in I$,

$$g_i((1-\alpha)A + \alpha A_1) \leq (1-\alpha)g_i(A) + \alpha g_i(A_1) \leq (1-\alpha)z_i + \alpha\bar{z}_i.$$

This shows that

$$f((1-\alpha)A + \alpha A_1) \geq \mu$$

because $(1-\alpha)z + \alpha\bar{z} \notin S$. Since $f$ is interval-usc,

$$f(A) \geq \limsup_{\alpha \to 0} f((1-\alpha)A + \alpha A_1) \geq \mu.$$

This is a contradiction.

Hence, we can see that

$$z \in S \Longrightarrow \langle \bar{\lambda}, z \rangle > 0,$$

that is,

$$\langle \bar{\lambda}, z \rangle \leq 0 \Longrightarrow z \notin S.$$

For each $A \in \mathcal{C}$ with $\sum_{i=1}^m \bar{\lambda}_i g_i(A) \leq 0$, let

$$z = (g_1(A), \ldots, g_m(A)) \in \mathbb{R}^m.$$

Since $\langle \bar{\lambda}, z \rangle = \sum_{i=1}^m \bar{\lambda}_i g_i(A) \leq 0$, $z \notin S$, that is, $f(A) \geq \mu$. Hence,

$$\begin{aligned}
\inf_{A \in \mathcal{A}} f(A) &= \mu \\
&\leq \inf \left\{ f(A) \,\middle|\, A \in \mathcal{C}, \sum_{i=1}^m \bar{\lambda}_i g_i(A) \leq 0 \right\} \\
&\leq \sup_{\lambda \in \mathbb{R}^m_+} \inf \left\{ f(A) \,\middle|\, A \in \mathcal{C}, \sum_{i=1}^m \lambda_i g_i(A) \leq 0 \right\} \\
&\leq \mu.
\end{aligned}$$

This completes the proof.

## 5 Applications to uncertain problems

In this section, we study applications of our results to uncertain problems with motion uncertainty.

In [38], we study uncertain problems with motion uncertainty in terms of set functions. We regard a decision variable set as an error caused by a motion, and investigate robust approach for the problem.

Let $X = \mathbb{R}^n$, $I = \{1, \dots, m\}$, $f$ a real-valued convex or quasiconvex function on $\mathbb{R}^n$, $g_i$ a real-valued convex function on $\mathbb{R}^n$ for each $i \in I$. The following problem $(P)$ is a convex or quasiconvex programming problem on $\mathbb{R}^n$ without uncertainty:

$$(P)\begin{cases}\text{minimize } f(x),\\ \text{subject to } g_i(x) \leq 0, \forall i \in I.\end{cases}$$

For such a problem, we may not be able to choose an exact vector because of an error by a motion. Hence, in [38], we study the following a worst case approach with motion uncertainty.

Let $\mathcal{C} = \{A \subset \mathbb{R}^n \mid A : \text{compact convex}, \text{int}A \neq \emptyset\}$, and $F$ be the following function on $\mathcal{A}_0 = \{A \subset \mathbb{R}^n \mid A : \text{nonempty}\}$: for each $A \in \mathcal{A}_0$,

$$F(A) = \sup_{x \in A} f(x).$$

For constraint functions, we define $G_i$ similarly, that is, $G_i(A) = \sup_{x \in A} g_i(x)$. Then, we consider the following robust optimization problem $(RP)$ with motion uncertainty:

$$(RP)\begin{cases}\text{minimize } F(A),\\ \text{subject to } A \in \mathcal{C}, G_i(A) \leq 0, \forall i \in I.\end{cases}$$

In $(RP)$, $F$ and $G_i$ are set functions, and $A$ means an error caused by a motion. Since $F(A)$ is the supremum of the value of $f$ at $x \in A$, $(RP)$ is one of the worst-case approach.

We show some results for set functions defined by the supremum of real-valued functions.

**Theorem 5** *Let $f$ be a real-valued function on $\mathbb{R}^n$, and $F$ the following function $F$ on $\mathcal{A}_0$: for each $A \in \mathcal{A}_0$,*

$$F(A) = \sup_{x \in A} f(x).$$

*Then, the following statements hold:*

*(i) if $A$ is compact and $f$ is usc on $\mathbb{R}^n$, then $F(A) \in \mathbb{R}$,*
*(ii) if $f$ is convex on $\mathbb{R}^n$, then $F$ is convex on $\mathcal{A}_0$,*
*(iii) if $f$ is quasiconvex on $\mathbb{R}^n$, then $F$ is quasiconvex on $\mathcal{A}_0$,*
*(iv) if $f$ is usc on $\mathbb{R}^n$, then $F$ is interval-usc on $\mathcal{C}$.*

*Proof* It is clear that (i) holds.

We show the statement (ii). The proof of the statement (iii) is similar and will be omitted. Let $A$, $B \in \mathcal{A}_0$ and $\alpha \in (0,1)$. For each $x \in (1-\alpha)A + \alpha B$, there exist $a \in A$ and $b \in B$ such that $x = (1-\alpha)a + \alpha b$. Then,

$$\begin{aligned} f(x) &= f((1-\alpha)a + \alpha b) \\ &\leq (1-\alpha)f(a) + \alpha f(b) \\ &\leq (1-\alpha)F(A) + \alpha F(B). \end{aligned}$$

This show that

$$F((1-\alpha)A + \alpha B) \leq (1-\alpha)F(A) + \alpha F(B).$$

(iv) Let $A$, $B \in \mathcal{C}$ with $A \neq B$. We show that $h(t) = F((1-t)A + tB)$ is usc at any $t \in [0,1]$, that is, for each sequence $\{\alpha_k\} \subset [0,1]$ satisfying $\alpha_k \to t$,

$$F((1-t)A + tB) \geq \limsup_{k\to\infty} F((1-\alpha_k)A + \alpha_k B)).$$

Since $A$ and $B$ are compact, $(1-\alpha_k)A + \alpha_k B$ is compact for each $k \in \mathbb{N}$. By the upper semicontinuity of $f$, there exists $x_k \in (1-\alpha_k)A + \alpha_k B$ such that $F((1-\alpha_k)A + \alpha_k B) = f(x_k)$. Additionally, $x_k = (1-\alpha_k)a_k + \alpha_k b_k$ for some $a_k \in A$ and $b_k \in B$. Without loss of generality, we can assume that

$$\limsup_{k\to\infty} F((1-\alpha_k)A + \alpha_k B)) = \lim_{k\to\infty} F((1-\alpha_k)A + \alpha_k B)) = \lim_{k\to\infty} f(x_k).$$

Since $A$ and $B$ are compact, there exist subsequences $\{a_{k_j}\}$ and $\{b_{k_j}\}$ such that $a_{k_j}$ converges to some $a_0 \in A$ and $b_{k_j}$ converges to some $b_0 \in B$. Then,

$$x_{k_j} = (1-\alpha_{k_j})a_{k_j} + \alpha_{k_j} b_{k_j} \to (1-t)a_0 + tb_0 \in (1-t)A + tB.$$

Hence

$$\begin{aligned} F((1-t)A + tB) &= \sup_{x\in(1-t)A+tB} f(x) \\ &\geq f((1-t)a_0 + tb_0) \\ &\geq \lim_{j\to\infty} f(x_{k_j}) \\ &= \lim_{j\to\infty} F((1-\alpha_{k_j})A + \alpha_{k_j} B) \\ &= \lim_{k\to\infty} F((1-\alpha_k)A + \alpha_k B). \end{aligned}$$

This completes the proof.

For constraint functions, we show the following theorem.

**Theorem 6** *Let $I = \{1, \ldots, m\}$, $g_i$ a real-valued convex function on $\mathbb{R}^n$ for each $i \in I$, $C$ a closed convex subset of $\mathbb{R}^n$, $G_i(A) = \sup_{x\in A} g_i(x)$, and $\mathcal{C} = \{A \subset C \mid A : \text{compact convex}, \operatorname{int} A \neq \emptyset\}$.*

*Then the following statements holds:*

*(i)* *$\mathcal{C}$ is convex,*
*(ii)* $\mathcal{A} = \{A \in \mathcal{C} \mid \forall i \in I, G_i(A) \leq 0\}$ *is convex,*
*(iii)* *if there exists* $x_0 \in \text{int } C$ *such that* $g_i(x_0) < 0$ *for each* $i \in I$*, then there exists* $A \in \mathcal{C}$ *such that* $G_i(A) < 0$ *for each* $i \in I$*.*

*Proof* It is clear that (i) and (ii) holds.

(iii) By the assumption, there exists $x_0 \in \text{int } C$ such that $g_i(x_0) < 0$ for each $i \in I$. Since $g_i$ is a real-valued convex function on $\mathbb{R}^n$, $g_i$ is continuous. Hence, there exists $r_i > 0$ such that $g(x) < 0$ for each $x \in A_i = \text{cl}B(x_0, r_i) \subset C$. Since $A_i$ is compact and $g_i$ is continuous, $G(A_i) < 0$. Let $A = \bigcap_{i \in I} A_i$, then $A$ is a compact convex subset of $C$ and int $A$ is nonempty. In addition, we can check that $G(A) < 0$.

For the robust problem $(RP)$, we show the following duality theorem as an application of our results for convex and quasiconvex set functions.

**Corollary 1** *Let* $I = \{1, \ldots, m\}$*,* $g_i$ *a real-valued convex function on* $\mathbb{R}^n$ *for each* $i \in I$*,* $C$ *a closed convex subset of* $\mathbb{R}^n$*,* $G_i(A) = \sup_{x \in A} g_i(x)$ *for each* $A \subset \mathbb{R}^n$*,* $\mathcal{C} = \{A \subset C \mid A : \textit{compact convex}, \text{int } A \neq \emptyset\}$*,* $\mathcal{A} = \{A \in \mathcal{C} \mid \forall i \in I, G_i(A) \leq 0\}$*,* $f$ *a real-valued function on* $\mathbb{R}^n$*, and* $F(A) = \sup_{x \in A} f(x)$ *for each* $A \subset \mathbb{R}^n$*. Assume that there exists* $x_0 \in \text{int } C$ *such that* $g_i(x_0) < 0$ *for each* $i \in I$*.*

*Then the following statements hold:*

*(i)* *if* $f$ *is convex, then,*

$$\inf_{A \in \mathcal{A}} F(A) = \max_{\lambda \in \mathbb{R}^m_+} \inf_{A \in \mathcal{C}} \left\{ F(A) + \sum_{i=1}^m \lambda_i G_i(A) \right\},$$

*(ii)* *if* $f$ *is usc quasiconvex, then,*

$$\inf_{A \in \mathcal{A}} F(A) = \max_{\lambda \in \mathbb{R}^m_+} \inf \left\{ F(A) \middle| A \in \mathcal{C}, \sum_{i=1}^m \lambda_i G_i(A) \leq 0 \right\}.$$

*Proof* By Theorem 6, there exists $A \in \mathcal{C}$ such that $G_i(A) < 0$ for each $i \in I$. In addition, if $f$ is usc, then $F$ is interval-usc on $\mathcal{C}$ by Theorem 5. Hence by Theorem 3 and Theorem 4, we can prove the theorem.

*Remark 1* In Corollary 1, we assume that $g_i$ are real-valued convex. This implies that $g = \max_{i \in I} g_i$ is convex and $g(x_0) < 0$. Hence $g$ is locally Lipschitz, and there exist $\rho$, $L > 0$ such that for each $x \in \mathbb{R}^n$ with $\|x - x_0\| \leq \rho$,

$$g(x) \leq g(x_0) + L\|x - x_0\|.$$

Let $r = \min\{\rho, -\frac{g(x_0)}{2L}\}$, then for each $x \in \mathbb{R}^n$ with $\|x - x_0\| \leq r$,

$$g_i(x) \leq g(x) \leq g(x_0) + L\frac{-g(x_0)}{2L} = \frac{g(x_0)}{2} < 0.$$

*Remark 2* Mathematical programming problems with data uncertainty are becoming important in optimization due to the reality of uncertainty in many real-world optimization problems. Various researchers study duality theory for mathematical programming problems under uncertainty with the worst-case approach, see [3, 16–19, 24, 34]. In these literatures, the following uncertain problem is studied:

$$\begin{array}{ll} \text{Minimize} & f(x,u), \\ \text{subject to} & x \in A,\ g_i(x,v) \leq 0, \end{array}$$

where $A$ is a closed convex subset of $\mathbb{R}^n$, $\mathcal{U}$ is a compact set, $f$ is a continuous function from $\mathbb{R}^n \times \mathcal{U}$ to $\mathbb{R}$, $\mathcal{V}$ is a set, and $g_i$ is a function from $\mathbb{R}^n \times \mathcal{V}$ to $\mathbb{R}$ for each $i \in I$. Because of the complexity of real-world optimization problems, measurement errors, and the other uncertainty, it is difficult to determine functions in the problem clearly. In the above problem, we cannot determine $u$ and $v$ clearly, however, we know that $u$ and $v$ are elements of the uncertainty sets $\mathcal{U}$ and $\mathcal{V}$. In order to solve such a problem robustly, robust optimization have been investigated. An error caused by a motion is not in consideration in previous works in robust optimization. In this paper, we can determine objective and constraint functions clearly. However, by an error caused by a motion, we may not be able to choose an exact decision vector. We regard a decision variable set as an error caused by a motion, and introduce robust approach for the uncertain problem. In future research, we try to consider the following uncertain problem with motion and data uncertainty:

$$\begin{array}{ll} \text{Minimize} & F(A,u), \\ \text{subject to} & A \in \mathcal{C},\ G_i(A,v) \leq 0. \end{array}$$

Furthermore, it is expected to study necessary and sufficient constraint qualification for Lagrange and surrogate duality for the problem in terms of recent advances in constraint qualifications.

**Acknowledgements** The authors are grateful to the anonymous referee for careful reading of the manuscript and many comments and suggestions improved the quality of the paper.

## Funding

This study was funded by Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research (grant numbers 19K03620, 19K03637).

## Conflict of interest

The authors declare that they have no conflict of interest.

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