Nonoscillation theorems for second-order linear difference equations via the Riccati-type transformation, II

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Abstract

The present paper deals with nonoscillation problem for the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \qquad n = 1, 2, \dots,$$

where $\{b_n\}$ and $\{c_n\}$ are positive sequences. All nontrivial solutions of this equation are nonoscillatory if and only if the Riccati-type difference equation

$$q_n z_n + \frac{1}{z_{n-1}} = 1$$

has an eventually positive solution, where $q_n = c_n^2/(b_n b_{n+1})$. Our nonoscillation theorems are proved by using this equivalence relation. In particular, it is focusing on the relation of the triple $(q_{3k-2}, q_{3k-1}, q_{3k})$ for each $k \in \mathbb{N}$. Our results can also be applied to not only the case that $\{b_n\}$ and $\{c_n\}$ are periodic but also the case that $\{b_n\}$ or $\{c_n\}$ is non-periodic. To compare the obtained results with previous works, we give some concrete examples and those simulations.

Key words: Linear difference equations; Nonoscillation; Riccati transformation; Sturm's separation theorem
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1. Introduction

The Riccati transformation is a very important tool for studying nonoscillation problem of second-order linear difference equations as well as ordinary differential equations. It is known that there are several types of Riccati transformations. For example, Hooker et al. [15, 16, 19] have presented three kinds of Riccati transformations for the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \qquad n = 1, 2, \dots, \tag{1.1}$$

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where $\{b_n\}$ and $\{c_n\}$ are sequences satisfying $b_n > 0$ for $n \in \mathbb{N}$ and $c_n > 0$ for $n \in \mathbb{N} \cup \{0\}$, respectively. Those Riccati transformations are expressed by $w_n = x_{n+1}/x_n$, $y_n = c_n x_{n+1}/x_n$ and $z_n = b_{n+1}x_{n+1}/(c_n x_n)$. Here, we assume that there exists an $M \in \mathbb{N}$ such that $x_n > 0$ for $n \geq M$. The transformations lead to the first-order non-linear difference equations

$$c_n w_n + \frac{c_{n-1}}{w_{n-1}} = b_n,$$

$$y_n + \frac{c_{n-1}^2}{y_n} = b_n$$

and

$$q_n z_n + \frac{1}{z_{n-1}} = 1, \qquad q_n = \frac{c_n^2}{b_n b_{n+1}} \tag{1.2}$$

with $n = M+1, M+2, \ldots$, respectively (see also the books [1, Chap. 6], [9, Chap. 7]). Although the transformation

$$z_n = \frac{b_{n+1}x_{n+1}}{c_n x_n}$$

is the most complicated one out of those three, equation (1.2) is easiest to use because the coefficient of (1.2) is only one.

It is clear that equation (1.1) has the trivial solution $\{x_n\}$; that is, $x_n = 0$ for $n \geq 0$. The others are called non-trivial solutions. A non-trivial solution of (1.1) is said to be *oscillatory* if, for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n x_{n+1} \leq 0$. Otherwise, it is said to be *nonoscillatory*. Hence, a nonoscillatory solution $\{x_n\}$ of (1.1) satisfies that $x_n > 0$ for n sufficiently large or $x_n < 0$ for n sufficiently large. Since equation (1.1) is linear, $\{x_n\}$ is a solution of (1.1) if and only if $\{-x_n\}$ is also a solution of (1.1). Hence, it is sufficient to consider that a nonoscillatory solution $\{x_n\}$ of (1.1) continues being positive for all large n.

As known well, Sturm's separation theorem holds for equation (1.1). About the proof of Sturm's separation theorem concerning linear difference equations, see [9, pp. 321–322] for example. From Sturm's separation theorem it follows that if one non-trivial solution of (1.1) is nonoscillatory, then all its non-trivial solutions are nonoscillatory. Hence, oscillatory solutions and nonoscillatory solutions do not coexist in equation (1.1).

Using equation (1.2) equivalent to (1.1), Hooker et al. [15] have proved the following results.

Theorem A. *If $q_n \geq 1/(4-\varepsilon)$ for some $\varepsilon > 0$ and for all sufficiently large n, then all non-trivial solutions of* (1.1) *are oscillatory.*

Theorem B. *If $q_n \leq 1/4$ for all sufficiently large n, then all non-trivial solutions of* (1.1) *are nonoscillatory.*

As can be seen from Theorems A and B, the constant $1/4$ is a critical value that divides oscillation and nonoscillation of solutions of (1.1). Such a value is called an *oscillation constant*. It seems to be appropriate that the constant $1/4$ appears in Theorems A and B, because it often becomes the oscillation constant for some ordinary differential equations. For example, it is well-known that all non-trivial solutions of the Euler differential equation

$$x'' + \frac{\gamma}{t^2}x = 0$$

are nonoscillatory if and only if $\gamma \le 1/4$ (for example, see [14, 18, 21, 26]). In this sense, it is not exaggeration even if we say that Theorems A and B have similarity between the results of ordinary differential equations. After that, Hooker et al. [16, 19] improved the sufficient condition was given in Theorem A which guarantees that all nontrivial solutions of (1.1) are oscillatory.

Equation (1.1) can be rewritten as the self-adjoint difference equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + p_n x_n = 0, \tag{1.3}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$ and

$$p_n = c_{n-1} + c_n - b_n$$

for $n \in \mathbb{N}$. The oscillation and nonoscillation of (1.3) and more generalized equations have been considered extensively by many authors. For example, see [1, 2, 3, 4, 5, 9, 12, 17] and the references cited therein. Chen and Erbe [4] discussed the oscillation and nonoscillation properties of (1.3) and obtained oscillation and nonoscillation criteria by using Riccati techniques. Their main assumptions were

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} p_j > -\infty \tag{1.4}$$

and others. Since the beginning of this century, oscillation and nonoscillation criteria are now being actively reported for the self-adjoint difference equation

$$\Delta(c_{n-1}\Phi(\Delta x_{n-1})) + p_n\Phi(x_n) = 0, \tag{1.5}$$

which is a generalization of (1.3). Here, $\Phi(z)$ is a real-valued nonlinear function defined by

$$\Phi(z) = \begin{cases} |z|^{p-2}z & \text{if } z \ne 0, \\ 0 & \text{if } z = 0 \end{cases}$$

for $z \in \mathbb{R}$ with $p > 1$ a fixed real number. For example, see [7, 10, 11, 13, 20, 22, 23, 24, 27]. Equation (1.4) is often called a *half-linear* difference equation. Most of these results emphasized similarity of difference equations (1.3) and (1.5) and the differential equation

$$\big(c(t)x'\big)' + p(t)x = 0$$

and its generalization

$$\big(c(t)\Phi(x')\big)' + p(t)\Phi(x) = 0,$$

where c, p $[a,\infty) \to \mathbb{R}$ are continuous functions, $c(t) > 0$ for $t \geq a$. The reader can refer to the book [8] as a very good monograph concerning half-linear differential equations and half-linear difference equations. In this book, along with the difficulty of the study of half-linear difference equations, many analogies can be found about the oscillation of half-linear differential equations and half-linear difference equations (see also [6]).

After a series of work of Hooker et al. [15, 16, 19], there were few studies that considered equation (1.2). Abu-Risha [3] gave the following result by focusing on the relation between the values of three successive coefficients of (1.2).

Theorem C. *All non-trivial solutions of* (1.1) *are nonoscillatory if there is an* $N \in \mathbb{N}$ *such that*

$$(\sqrt{q_{n+1}} + \sqrt{q_n})(\sqrt{q_n} + \sqrt{q_{n-1}}) \leq 1 \tag{1.6}$$

holds for $n \geq N$.

If $q_n \leq 1/4$ for n sufficiently large, then it is clear that condition (1.6) holds. Hence. Theorem C is superior to Theorem B. However, condition (1.6) imposes a fairly strong restraint in the coefficient sequence $\{q_n\}$. If there is an $m \in \mathbb{N}$ such that

$$(\sqrt{q_{m+1}} + \sqrt{q_m})(\sqrt{q_m} + \sqrt{q_{m-1}}) = 1,$$

then q_{n+3k} has to be equal to q_n for all $n \geq m-1$ and $k \in \mathbb{N}$.

By considering only the behavior of the pair (q_{2k-1}, q_{2k}) or (q_{2k}, q_{2k+1}) with $k \in \mathbb{N}$, the author and Tanaka [25] presented the following result.

Theorem D. *Suppose that there exists an* $N \in \mathbb{N}$ *such that either*

$$q_{2k-1} + q_{2k} \leq \frac{1}{2}$$

or

$$q_{2k} + q_{2k+1} \leq \frac{1}{2}$$

with $k \geq N$. *Then all non-trivial solutions of* (1.1) *are nonoscillatory.*

The purpose of this paper is to improve the nonoscillation theorems given in [3, 25]. To obtain desired results, we pay attention mainly to the relation of the triple $(q_{3k-2}, q_{3k-1}, q_{3k})$ with $k \in \mathbb{N}$.

Let α be a real number that is larger than 1 and let α^* be the conjugate number of p; namely,

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1.$$

Then α^* is also greater than 1.

Theorem 1.1. *Suppose that there exists an* $N \in \mathbb{N}$ *such that for any* $k \geq N$ *there is a sequence* $\{\alpha_k\}$ *with* $\alpha_k > 1$. *If*

$$\alpha_k^* q_{3k-2} < 1 \tag{1.7}$$

and

$$q_{3k-1} \leq (1 - \alpha_k^* q_{3k-2})(1 - \alpha_{k+1} q_{3k}), \tag{1.8}$$

then all non-trivial solutions of (1.1) *are nonoscillatory.*

Remark 1.1. Theorem 1.1 improves Theorems C and D in the sense that the weaknesses of Theorems C and D are overcome (see Section 4).

2. Proof of the main theorem

By virtue of Sturm's separation theorem, in order to prove Theorem 1.1, we have only to show that there exists an integer $N \geq M$ such that equation (1.2) has a solution $\{z_n\}$ satisfying $z_n > 0$ for all $n \geq N$.

Proof of Theorem 1.1. Consider a solution $\{z_n\}$ of (1.2) satisfying $z_{3N-3} \geq \alpha_N > 1$. Then we have

$$z_{3N-2} = \frac{1}{q_{3N-2}}\left(1 - \frac{1}{z_{3N-3}}\right) \geq \frac{1}{q_{3N-2}}\left(1 - \frac{1}{\alpha_N}\right) = \frac{\alpha_N - 1}{\alpha_N q_{3N-2}} > 0.$$

Hence, by (1.7) we obtain

$$z_{3N-1} = \frac{1}{q_{3N-1}}\left(1 - \frac{1}{z_{3N-2}}\right) \geq \frac{1 - \alpha_N^* q_{3N-2}}{q_{3N-1}} > 0,$$

and therefore,

$$z_{3N} = \frac{1}{q_{3N}}\left(1 - \frac{1}{z_{3N-1}}\right) \geq \frac{1}{q_{3N}}\left(1 - \frac{q_{3N-1}}{1 - \alpha_N^* q_{3N-2}}\right).$$

From (1.8) it follows that

$$1 - \frac{q_{3N-1}}{1 - \alpha_N^* q_{3N-2}} \geq \alpha_{N+1} q_{3N}.$$

Hence, we see that $z_{3N} \geq \alpha_{N+1}$. Similarly, we can easily check that

$$z_n \geq \begin{cases} \dfrac{\alpha_k - 1}{\alpha_k q_{3k-2}} & \text{if } n = 3k-2 \\ \dfrac{1 - \alpha_k^* q_{3k-2}}{q_{3k-1}} & \text{if } n = 3k-1 \\ \alpha_{k+1} & \text{if } n = 3k \end{cases}$$

with $k \geq N$. Hence, the solution $\{z_n\}$ of (1.2) is positive for $n \geq 3N - 3$. We therefore conclude that all non-trivial solutions of (1.1) are nonoscillatory. □

By the same way, we have the following results (we omit the proof).

Theorem 2.1. *Suppose that there exists an $N \in \mathbb{N}$ such that for any $k \geq N$ there is a sequence $\{\alpha_k\}$ with $\alpha_k > 1$. If*

$$\alpha_k^* q_{3k-1} < 1$$

and

$$q_{3k} \leq (1 - \alpha_k^* q_{3k-1})(1 - \alpha_{k+1} q_{3k+1}),$$

then all non-trivial solutions of (1.1) *are nonoscillatory.*

Theorem 2.2. *Suppose that there exists an $N \in \mathbb{N}$ such that for any $k \geq N$ there is a sequence $\{\alpha_k\}$ with $\alpha_k > 1$. If*

$$\alpha_k^* q_{3k} < 1$$

and

$$q_{3k+1} \leq (1 - \alpha_k^* q_{3k})(1 - \alpha_{k+1} q_{3k+2}),$$

then all non-trivial solutions of (1.1) *are nonoscillatory.*

3. Corollaries

To apply Theorem 1.1 to a concrete example, we need to find a suitable sequence $\{\alpha_k\}$ satisfying conditions (1.7) and (1.8) from the coefficient sequence $\{q_n\}$ of the Riccati-type difference equation (1.2). For each k sufficiently large, it will be natural to think that α_k is determined by q_{3k-2}, q_{3k-1} and q_{3k}. The following result provides a method of determining α_k.

Corollary 3.1. *Suppose that there exists an $N \in \mathbb{N}$ such that*

$$q_{3k-2} + q_{3k} < 1 \tag{3.1}$$

and

$$q_{3k-1} \leq \left(1 - q_{3k-2} - \sqrt{\frac{q_{3k}(1 - q_{3k-2})}{q_{3k-2}(1 - q_{3k})}}\, q_{3k-2}\right)\left(1 - q_{3k} - \sqrt{\frac{q_{3k+1}(1 - q_{3k+3})}{q_{3k+3}(1 - q_{3k+1})}}\, q_{3k}\right) \tag{3.2}$$

with $k \geq N$. Then all non-trivial solutions of (1.1) *are nonoscillatory.*

Proof. From (3.1) it follows that $q_{3k-2} < 1$ and $q_{3k} < 1$ for $k \geq N$. Hence, we can choose

$$\alpha_k = 1 + \sqrt{\frac{q_{3k-2}(1 - q_{3k})}{q_{3k}(1 - q_{3k-2})}}$$

for $k \geq N$. It is clear that $\alpha_k > 1$. Note that

$$\alpha_k^* = \frac{\alpha_k}{\alpha_k - 1} = 1 + \sqrt{\frac{q_{3k}(1 - q_{3k-2})}{q_{3k-2}(1 - q_{3k})}}.$$

By (3.1) again, we have

$$q_{3k-2}q_{3k} < (1 - q_{3k-2})(1 - q_{3k})$$

for $k \geq N$. Hence, we get

$$q_{3k-2} < \sqrt{\frac{q_{3k-2}(1 - q_{3k-2})(1 - q_{3k})}{q_{3k}}} = (1 - q_{3k-2})\sqrt{\frac{q_{3k-2}(1 - q_{3k})}{q_{3k}(1 - q_{3k-2})}}$$

for $k \geq N$. Using this estimation, we obtain

$$\alpha_k^* q_{3k-2} < 1;$$

namely, condition (1.7) holds. Since

$$\alpha_{k+1} = 1 + \sqrt{\frac{q_{3k+1}(1 - q_{3k+3})}{q_{3k+3}(1 - q_{3k+1})}},$$

we can rewrite (3.2) as

$$q_{3k-1} \leq (1 - \alpha_k^* q_{3k-2})(1 - \alpha_{k+1} q_{3k});$$

namely, condition (1.8). Hence, all non-trivial solutions of (1.1) are nonoscillatory by Theorem 1.1. □

Let us choose α_k as a fixed number $\alpha > 1$. Then we have the following corollary of Theorem 1.

Corollary 3.2. *Let α be a real number that is greater than 1. Suppose that there exists an $N \in \mathbb{N}$ such that*

$$\alpha^* q_{3k-2} < 1 \tag{3.3}$$

and

$$q_{3k-1} \leq (1 - \alpha^* q_{3k-2})(1 - \alpha\, q_{3k}) \tag{3.4}$$

with $k \geq N$. Then all non-trivial solutions of (1.1) *are nonoscillatory.*

Since $\alpha\alpha^* = \alpha + \alpha^*$, condition (3.4) can be rewritten as

$$q_{3k-1} + \alpha\, q_{3k}(1 - q_{3k-2}) + \alpha^* q_{3k-2}(1 - q_{3k}) \leq 1.$$

The following result is an immediate consequence of Corollary 3.2.

Corollary 3.3. *Suppose that there exists an $N \in \mathbb{N}$ such that*

$$q_{3k-2} < \frac{1}{2} \tag{3.5}$$

and

$$q_{3k-1} \leq (1 - 2q_{3k-2})(1 - 2q_{3k}) \tag{3.6}$$

with $k \geq N$. Then all non-trivial solutions of (1.1) *are nonoscillatory.*

If $q_n \leq 1/4$ for all sufficiently large n, then it is clear that conditions (3.5) and (3.6) are satisfied. Hence, Corollary 3.3 contains Theorem B completely.

4. Comparison between previous studies and the obtained results

To illustrate our results, we give some examples in this section. We first give an example that can be applied to Corollary 3.3 but cannot be applied to previous works.

Example 4.1. Let $c_0 = \sqrt{3}$ and let

$$c_n = \begin{cases} \sqrt{3} & \text{if } n = 9k-8, \\ 4 & \text{if } n = 9k-7, \\ 2 & \text{if } n = 9k-6, \\ 2 & \text{if } n = 9k-5, \\ 4 & \text{if } n = 9k-4, \\ \sqrt{2} & \text{if } n = 9k-3, \\ \sqrt{2} & \text{if } n = 9k-2, \\ 4 & \text{if } n = 9k-1, \\ \sqrt{3} & \text{if } n = 9k \end{cases} \quad \text{and} \quad b_n = \begin{cases} 6 & \text{if } n = 9k-8, \\ 5 & \text{if } n = 9k-7, \\ 5 & \text{if } n = 9k-6, \\ 8 & \text{if } n = 9k-5, \\ 5 & \text{if } n = 9k-4, \\ 5 & \text{if } n = 9k-3, \\ 4 & \text{if } n = 9k-2, \\ 5 & \text{if } n = 9k-1, \\ 5 & \text{if } n = 9k \end{cases}$$

with $k \in \mathbb{N}$. Then all non-trivial solutions of (1.1) are nonoscillatory.

Since

$$q_n = \frac{c_n^2}{b_n b_{n+1}} = \begin{cases} 0.1 & \text{if } n = 3k-2, \\ 0.64 & \text{if } n = 3k-1, \\ 0.1 & \text{if } n = 3k, \end{cases}$$

we see that

$$q_{3k-2} = 0.1 < \frac{1}{2}$$

and

$$q_{3k-1} = 0.64 = (1-0.2)(1-0.2) = (1-2q_{3k-2})(1-2q_{3k})$$

with $k \in \mathbb{N}$. Hence, conditions (3.5) and (3.6) are satisfied. Thus, by Corollary 3.3, all non-trivial solutions of (1.1) are nonoscillatory.

Theorem B cannot be applied to Example 4.1, because

$$q_{3k-1} = 0.64 > \frac{1}{4}$$

for $k \in \mathbb{N}$. Note that the assumption (1.4) of Chen and Erbe [4] is not satisfied. In fact, we can easily check that

$$p_n = c_{n-1} + c_n - b_n = \begin{cases} 2\sqrt{3} - 6 & \text{if } n = 9k-8, \\ \sqrt{3} - 1 & \text{if } n = 9k-7, \\ 1 & \text{if } n = 9k-6, \\ -4 & \text{if } n = 9k-5, \\ 1 & \text{if } n = 9k-4, \\ \sqrt{2} - 1 & \text{if } n = 9k-3, \\ 2\sqrt{2} - 4 & \text{if } n = 9k-2, \\ \sqrt{2} - 1 & \text{if } n = 9k-1, \\ \sqrt{3} - 1 & \text{if } n = 9k \end{cases}$$

with $k \in \mathbb{N}$. Hence, we obtain

$$\sum_{j=1}^{n} p_j = \begin{cases} 2\sqrt{3} - 6 = -2.535898384862246\dots & \text{if } n = 1, \\ 3\sqrt{3} - 7 = -1.803847577293368\dots & \text{if } n = 2, \\ 3\sqrt{3} - 6 = -0.803847577293368\dots & \text{if } n = 3, \\ 3\sqrt{3} - 10 = -4.803847577293368\dots & \text{if } n = 4, \\ 3\sqrt{3} - 9 = -3.803847577293368\dots & \text{if } n = 5, \\ \sqrt{2} + 3\sqrt{3} - 10 = -3.389634014920273\dots & \text{if } n = 6, \\ 3\sqrt{2} + 3\sqrt{3} - 14 = -4.561206890174082\dots & \text{if } n = 7, \\ 4\sqrt{2} + 3\sqrt{3} - 15 = -4.146993327800988\dots & \text{if } n = 8, \\ 4\sqrt{2} + 4\sqrt{3} - 16 = -3.41494252023211\dots & \text{if } n = 9. \end{cases}$$

Let n be an integer greater than 9. Then there exist an $m \in \mathbb{N}$ and an $\ell \in \mathbb{N}$ with $0 \le \ell \le 8$ such that $n = 9m + \ell$ and

$$\sum_{j=1}^{n} p_j = m(4\sqrt{2} + 4\sqrt{3} - 16) + \sum_{j=1}^{\ell} p_j < 0.$$

We conclude that

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} p_j &= p_1 + \frac{n-1}{n}p_2 + \frac{n-2}{n}p_3 + \cdots + \frac{2}{n}p_{n-1} + \frac{1}{n}p_n \\ &< p_1 + p_2 + p_3 + \cdots + p_{n-1} + p_n \end{aligned}$$

which tends to $-\infty$ as $n \to \infty$. Thus, the assumption (1.4) does not hold in Example 4.1. Since

$$\begin{aligned} (\sqrt{q_{3k}} + \sqrt{q_{3k-1}})(\sqrt{q_{3k-1}} + \sqrt{q_{3k-2}}) &= (\sqrt{0.1} + \sqrt{0.64})^2 \\ &= 1.245964425626941\cdots > 1 \end{aligned}$$

for $k \in \mathbb{N}$. Hence, Theorem C is also inapplicable to Example 4.1. Moreover, Theorem D is of no use, because

$$q_{6k-1} + q_{6k} = 0.64 + 0.1 > \frac{1}{2}$$

and

$$q_{6k-2} + q_{6k-1} = 0.1 + 0.64 > \frac{1}{2}$$

for $k \in \mathbb{N}$.

Let us denote by $\{x_n\}$ a solution of (1.1) with the sequences $\{b_n\}$ and $\{c_n\}$ that were given in Example 4.1 (see Figure 1). To make the motion of a solution of (1.1) more visible, we connect the dots x_{n-1} and x_n with a line segment and draw a line graph.

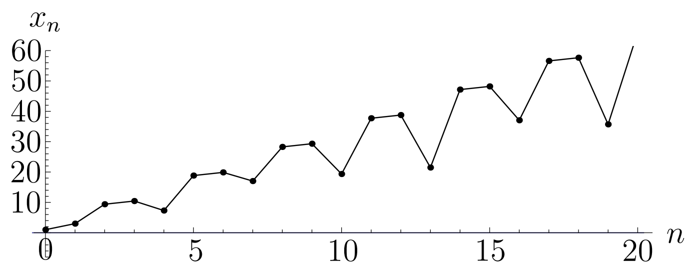

Figure 1: This line graph displays the motion of a solution $\{x_n\}$ of (1.1) given in Example 4.1. The initial condition of the solution is $(x_0, x_1) = (1, 3)$.

Figure 1 shows that $x_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, this solution $\{x_n\}$ is nonoscillatory. Recall that if equation (1.1) has a non-trivial solution which is nonoscillatory, then all non-trivial solutions are nonoscillatory.

Next, we give an example of Corollary 3.1.

Example 4.2. Let $c_0 = 2\sqrt{5}$ and let

$$c_n = \begin{cases} 2 & \text{if } n = 3k-2, \\ 1 & \text{if } n = 3k-1, \\ 2\sqrt{5} & \text{if } n = 3k \end{cases} \quad \text{and} \quad b_n = \begin{cases} 5 & \text{if } n = 6k-5, \\ 4 & \text{if } n = 6k-4, \\ 25 & \text{if } n = 6k-3, \\ 2 & \text{if } n = 6k-2, \\ 10 & \text{if } n = 6k-3, \\ 10 & \text{if } n = 6k-2 \end{cases}$$

with $k \in \mathbb{N}$. Then all non-trivial solutions of (1.1) are nonoscillatory.

It is easy to check that

$$q_n = \frac{c_n^2}{b_n b_{n+1}} = \begin{cases} 0.5 & \text{if } n = 3k-2 \\ 0.01 & \text{if } n = 3k-1 \\ 0.4 & \text{if } n = 3k. \end{cases}$$

Hence, we see that

$$q_{3k-2} + q_{3k} = 0.5 + 0.4 < 1$$

and

$$\begin{aligned}
&\left(1 - q_{3k-2} - \sqrt{\frac{q_{3k}(1-q_{3k-2})}{q_{3k-2}(1-q_{3k})}}\, q_{3k-2}\right)\left(1 - q_{3k} - \sqrt{\frac{q_{3k+1}(1-q_{3k+3})}{q_{3k+3}(1-q_{3k+1})}}\, q_{3k}\right) \\
&= \left(1 - 0.4 - \sqrt{\frac{0.5(1-0.4)}{0.4(1-0.5)}} \times 0.4\right)\left(1 - 0.5 - \sqrt{\frac{0.4(1-0.5)}{0.5(1-0.4)}} \times 0.5\right) \\
&= \frac{(3-\sqrt{6})^2}{30} = 0.01010205144336439\cdots > 0.01 = q_{3k-1}
\end{aligned}$$

with $k \in \mathbb{N}$; namely, conditions (3.1) and (3.2) hold. Thus, by Corollary 3.1, all non-trivial solutions of (1.1) are nonoscillatory.

Since

$$q_{3k-2} = 0.5 \geq \frac{1}{2}$$

and

$$q_{3k-1} = 0.01 > 0 = (1 - 2q_{3k-2})(1 - 2q_{3k})$$

with $k \in \mathbb{N}$, conditions (3.5) and (3.6) are not satisfied. Hence, Corollary 3.3 is not applicable to Example 4.2.

To make sure, we give a simulation of a solution $\{x_n\}$ of (1.1) with the sequences $\{b_n\}$ and $\{c_n\}$ that were given in Example 4.2 (see Figure 2).

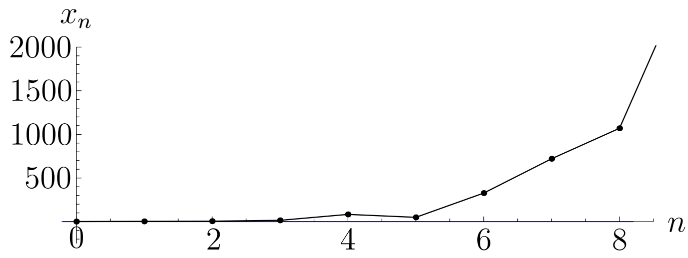

Figure 2: This line graph displays the motion of a solution $\{x_n\}$ of (1.1) given in Example 4.2. The initial condition of the solution is $(x_0, x_1) = (1, 3)$.

We can also verify Example 4.2 by using Corollary 3.2. In fact, we choose α as $56/25$. Then, $\alpha^* = 56/31$. Hence, we can check that

$$\alpha^* q_{3k-2} = \frac{56}{31} \times 0.5 = \frac{28}{31} < 1$$

and

$$\begin{aligned}
(1 - \alpha^* q_{3k-2})(1 - \alpha\, q_{3k}) &= \left(1 - \frac{28}{31}\right)\left(1 - \frac{56}{25} \times 0.4\right) \\
&= 0.01006451612903226\cdots > 0.01 = q_{3k-1}
\end{aligned}$$

with $k \in \mathbb{N}$. Hence, conditions (3.3) and (3.4) are satisfied, and therefore, by Corollary 3.2, all non-trivial solutions of (1.1) are nonoscillatory.

5. Relation between Corollary 3.1 and Corollary 3.2

As was verified in the preceding section, Example 4.2 can be applied to both Corollary 3.1 and Corollary 3.2. To tell the truth, Corollary 3.1 and Corollary 3.2 are equivalent when the coefficient sequence $\{q_n\}$ of the Riccati-type difference equation (1.2) is periodic with period 3. In this section, We will prove this equivalence relation.

For the sake of simplicity, we assume that $N = 1$ (if necessary, we have only to shift the suffix of q_n by a constant value). Since $\{q_n\}$ is periodic with period 3,

$$q_n = \begin{cases} q_1 & \text{if } n = 3k-2 \\ q_2 & \text{if } n = 3k-1 \\ q_3 & \text{if } n = 3k. \end{cases}$$

We first show that conditions (3.1) and (3.2) imply conditions (3.3) and (3.4) with

$$\alpha = 1 + \sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}.$$

From (3.1) it follows that $q_1 + q_3 < 1$. Hence, we have

$$q_1 q_3 < (1-q_1)(1-q_3).$$

Using this inequality, we can check that

$$\begin{aligned} \frac{1}{\alpha^*} - q_1 = \frac{\alpha - 1}{\alpha} - q_1 &= \frac{\sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}}{1+\sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}} - q_1 \\ &= \frac{1}{1+\sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}}\left\{\sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}} - q_1 - q_1\sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}\right\} \\ &= \frac{1}{1+\sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}}\left\{\sqrt{\frac{q_1(1-q_1)(1-q_3)}{q_3}} - q_1\right\} \\ &= \frac{\sqrt{q_1(1-q_1)}}{\sqrt{q_3(1-q_1)} + \sqrt{q_1(1-q_3)}}\left\{\sqrt{(1-q_1)(1-q_3)} - \sqrt{q_1 q_3}\right\} > 0. \end{aligned}$$

From (3.1) it follows that

$$q_2 \le \left(1 - q_1 - \sqrt{\frac{q_3(1-q_1)}{q_1(1-q_3)}}\, q_1\right)\left(1 - q_3 - \sqrt{\frac{q_1(1-q_3)}{q_3(1-q_1)}}\, q_3\right).$$

Since

$$\alpha^* = 1 + \sqrt{\frac{q_3(1-q_1)}{q_1(1-q_3)}},$$

we see that

$$q_2 \le (1-\alpha^* q_1)(1-\alpha\, q_3).$$

Thus, conditions (3.3) and (3.4) hold provided that $\{q_n\}$ is periodic with period 3.

We next show that conditions (3.3) and (3.4) imply conditions (3.1) and (3.2). From (3.3) and (3.4) it turns out that $q_1 < 1/\alpha^*$ and $q_3 < 1/\alpha$. Hence, we have

$$q_1 + q_3 < \frac{1}{\alpha^*} + \frac{1}{\alpha} = 1; \tag{5.1}$$

that is, condition (3.1) is satisfied. From (3.4) it follows that

$$q_2 \le (1-\alpha^* q_1)(1-\alpha\, q_3) < 1.$$

Hence, because $\alpha^* = \alpha/(\alpha-1)$, we obtain the quadratic inequality

$$q_3(1-q_1)\alpha^2 - (1-q_1-q_2+q_3)\alpha + 1 - q_2 \le 0. \tag{5.2}$$

Let

$$f(x) = q_3(1-q_1)x^2 - (1-q_1-q_2+q_3)x + 1 - q_2$$

for $x \in \mathbb{R}$. Since $f(0) = 1-q_2 > 0$ and $f(1) = q_1(1-q_3) > 0$, we see that there exists a real number $\alpha > 1$ satisfying (5.2) if and only if

$$\frac{1-q_1-q_2+q_3}{2q_3(1-q_1)} > 1$$

$$(1-q_1-q_2+q_3)^2 \ge 4q_3(1-q_1)(1-q_2). \tag{5.3}$$

Arranging these inequalities, we obtain

$$1 - q_1 - q_2 - q_3 + 2q_1q_3 > 0 \tag{5.4}$$

and

$$(1-q_1-q_2-q_3)^2 \ge 4q_1q_2q_3. \tag{5.5}$$

From (5.5) it turns out that there are two case to be considered:

(i) $1 - q_1 - q_2 - q_3 \ge 2\sqrt{q_1q_2q_3}$;

(ii) $1 - q_1 - q_2 - q_3 \le -2\sqrt{q_1q_2q_3}$.

However, case (ii) does not occur. In fact, from (ii) and (5.1) it follows that

$$q_2 - 1 \geq 2\sqrt{q_1q_2q_3} - (q_1 + q_3) > 2\sqrt{q_1q_2q_3} - 1.$$

Hence, we have

$$\sqrt{q_2} > 2\sqrt{q_1q_3}. \tag{5.6}$$

On the other hand, from from (ii) and (5.4) it follows that

$$-2q_1q_3 < 1 - q_1 - q_2 - q_3 \leq -2\sqrt{q_1q_2q_3},$$

and therefore,

$$\sqrt{q_2} \leq \sqrt{q_1q_3} < 2\sqrt{q_1q_3}.$$

This contradicts (5.6). Note that (5.4) is satisfied inevitably from (i). From (5.3) we obtain

$$q_2^2 - 2(1 - q_1 - q_3 + 2q_1q_3)q_2 + (1 - q_1 - q_2)^2 \geq 0.$$

This inequality leads to one of the following estimations:

$$\begin{aligned} q_2 &\leq 1 - q_1 - q_3 + 2q_1q_3 - \sqrt{(1 - q_1 - q_3 + 2q_1q_3)^2 - (1 - q_1 - q_3)^2} \\ &= 1 - q_1 - q_3 + 2q_1q_3 - 2\sqrt{q_1q_3(1 - q_1)(1 - q_3)}; \\ q_2 &\geq 1 - q_1 - q_3 + 2q_1q_3 + 2\sqrt{q_1q_3(1 - q_1)(1 - q_3)}. \end{aligned} \tag{5.7}$$

However, the latter is not true, because

$$q_2 \geq 1 - q_1 - q_3 + 2q_1q_3 + 2\sqrt{q_1q_3(1 - q_1)(1 - q_3)} > 1 - q_1 - q_3,$$

which contradicts (i). From (5.7) it turns out that

$$\begin{aligned} q_2 &\leq 1 - q_1 - q_3 + 2q_1q_3 - \sqrt{\frac{q_3(1 - q_1)}{q_1(1 - q_3)}}q_1(1 - q_3) - \sqrt{\frac{q_1(1 - q_3)}{q_3(1 - q_1)}}q_3(1 - q_1) \\ &= 1 - q_1 - \sqrt{\frac{q_3(1 - q_1)}{q_1(1 - q_3)}}\,q_1 - \left(1 - q_1 - \sqrt{\frac{q_3(1 - q_1)}{q_1(1 - q_3)}}\,q_1\right)q_3 \\ &\quad - \left(1 - q_1 - \sqrt{\frac{q_3(1 - q_1)}{q_1(1 - q_3)}}\,q_1\right)\sqrt{\frac{q_1(1 - q_3)}{q_3(1 - q_1)}}\,q_3 \\ &= \left(1 - q_1 - \sqrt{\frac{q_3(1 - q_1)}{q_1(1 - q_3)}}\,q_1\right)\left(1 - q_3 - \sqrt{\frac{q_1(1 - q_3)}{q_3(1 - q_1)}}\,q_3\right). \end{aligned}$$

Hence, condition (3.2) is satisfied.

Thus, we could conclude that Corollary 3.1 and Corollary 3.3 are equivalent in the case that $\{q_n\}$ is periodic with period 3.

To clarify the difference between Corollary 3.1 and Corollary 3.3, we give an example in which $\{q_n\}$ is periodic with period 6.

Example 5.1. Let $c_0 = 2$ and let

$$c_n = \begin{cases} 5 & \text{if } n = 6k-5, \\ 3 & \text{if } n = 6k-4, \\ 4 & \text{if } n = 6k-3, \\ 4 & \text{if } n = 6k-2, \\ \sqrt{2} & \text{if } n = 6k-1, \\ 2 & \text{if } n = 6k \end{cases} \quad \text{and} \quad b_n = \begin{cases} 2 & \text{if } n = 6k-5 \\ 25 & \text{if } n = 6k-4 \\ 20 & \text{if } n = 6k-3 \\ 2 & \text{if } n = 6k-2 \\ 20 & \text{if } n = 6k-1 \\ 5 & \text{if } n = 6k \end{cases}$$

with $k \in \mathbb{N}$. Then all non-trivial solutions of (1.1) are nonoscillatory.

In Example 5.1, the sequence $\{q_n\}$ satisfies

$$q_n = \frac{c_n^2}{b_n b_{n+1}} = \begin{cases} 0.5 & \text{if } n = 6k-5 \\ 0.018 & \text{if } n = 6k-4 \\ 0.4 & \text{if } n = 6k-3 \\ 0.4 & \text{if } n = 6k-2 \\ 0.02 & \text{if } n = 6k-1 \\ 0.4 & \text{if } n = 6k. \end{cases}$$

Since

$$q_{6k-5} + q_{6k-3} = 0.5 + 0.4 < 1$$

and

$$q_{6k-2} + q_{6k} = 0.4 + 0.4 < 1$$

for $k \in \mathbb{N}$, condition (3.1) is satisfied. We also check that

$$\begin{aligned} &\left(1 - q_{6k-5} - \sqrt{\frac{q_{6k-3}(1-q_{6k-5})}{q_{6k-5}(1-q_{6k-3})}}\, q_{6k-5}\right)\left(1 - q_{6k-3} - \sqrt{\frac{q_{6k-2}(1-q_{6k})}{q_{6k}(1-q_{6k-2})}}\, q_{6k-3}\right) \\ &= \left(1 - 0.5 - \sqrt{\frac{0.4(1-0.5)}{0.5(1-0.4)}} \times 0.5\right)\left(1 - 0.4 - \sqrt{\frac{0.4(1-0.4)}{0.4(1-0.4)}} \times 0.4\right) \\ &= \frac{3-\sqrt{6}}{30} = 0.0183503419072274\cdots > 0.018 = q_{6k-4} \end{aligned}$$

and

$$\begin{aligned} &\left(1 - q_{6k-2} - \sqrt{\frac{q_{6k}(1-q_{6k-2})}{q_{6k-2}(1-q_{6k})}}\, q_{6k-2}\right)\left(1 - q_{6k} - \sqrt{\frac{q_{6k+1}(1-q_{6k+3})}{q_{6k+3}(1-q_{6k+1})}}\, q_{6k}\right) \\ &= \left(1 - 0.4 - \sqrt{\frac{0.4(1-0.4)}{0.4(1-0.4)}} \times 0.4\right)\left(1 - 0.4 - \sqrt{\frac{0.5(1-0.4)}{0.4(1-0.5)}} \times 0.4\right) \\ &= \frac{3-\sqrt{6}}{25} = 0.0220204102886729\cdots > 0.02 = q_{6k-1} \end{aligned}$$

for $k \in \mathbb{N}$. Hence, condition (3.2) is satisfied. Thus, by Corollary 3.1, all non-trivial solutions of (1.1) are nonoscillatory.

The following figure is a simulation of a solution $\{x_n\}$ of (1.1) with the sequences $\{b_n\}$ and $\{c_n\}$ that were given in Example 5.1.

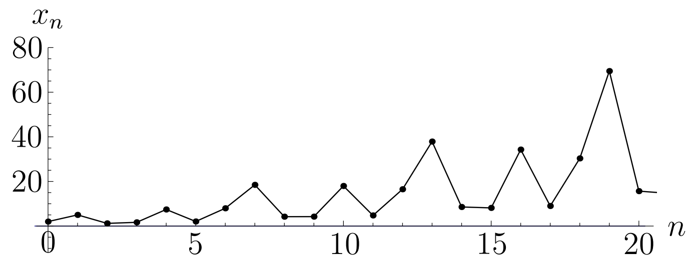

Figure 3: This line graph displays the motion of a solution $\{x_n\}$ of (1.1) given in Example 4.1. The initial condition of the solution is $(x_0, x_1) = (2, 5)$.

Finally, we show that condition (3.4) does not hold in Example 5.1. For this reason, we cannot use Corollary 3.2 to Example 5.1. To verify that condition (3.4) holds, we have to find a real number $\alpha > 1$ satisfying

$$q_{6k-4} \le (1 - \alpha^* q_{6k-5})(1 - \alpha\, q_{6k-3})$$

and

$$q_{6k-1} \le (1 - \alpha^* q_{6k-2})(1 - \alpha\, q_{6k}).$$

Taking into account that $\alpha^* = \alpha/(\alpha - 1)$, we obtain

$$100\alpha^2 - 441\alpha + 491 \le 0.$$

from the first inequality. However, there are no real numbers which satisfy this quadratic inequality.

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References

[1] R.P. Agarwal, Difference Equations and Inequalities: Theory, Methods, and Applications, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, **228**, Marcel Dekker, New York, 2000.

[2] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 2000.

[3] M.H. Abu-Risha, Oscillation of second-order linear difference equations, Appl. Math. Lett. 13 (2000), 129–135.

[4] S. Chen, L.H. Erbe, Riccati techniques and discrete oscillations, J. Math. Anal. Appl. 142 (1989), 468–487.

[5] S. Chen, L.H. Erbe, Disconjugacy, disfocality, and oscillation of second order difference equations, J. Differential Equations 107 (1994), 383–394.

[6] O. Došlý, Half-linear differential equations, in: Handbook of Differential Equations, Ordinary Differential Equations, vol. I, pp. 161–357, A. Cañada, P. Drábek, A. Fonda (eds.), Elsevier, Amsterdam, 2004.

[7] O. Došlý, P. Řehák, Nonoscillation criteria for half-linear second-order difference equations, Comput. Math. Appl. 42 (2001), 453–464.

[8] O. Došlý, P. Řehák, Half-linear Differential Equations, North-Holland, Mathematics Studies 202, Elsevier Sci. B.V., Amsterdam, 2005.

[9] S. Elaydi, An Introduction to Difference Equations, 3rd ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2005.

[10] H.A. El-Morshedy, Oscillation and nonoscillation criteria for half-linear second order difference equations, Dynam. Systems Appl. 15 (2006), 429–450.

[11] S. Fišnarová, Oscillatory properties of half-linear difference equations: two-term perturbations, Adv. Difference Equ. 2012:101, 16 pages (2012)

[12] S.R. Grace, A.A. Abadeer, H.A. El-Morshedy, Summation averaging technique for the oscillation of second order linear difference equations, Publ. Math. Debrecen 55 (1999), 333–347.

[13] P. Hasil, M. Veselý, Oscillation constants for half-linear difference equations with coefficients having mean values, Adv. Difference Equ. 2015, 2015:210, 18 pages.

[14] E. Hille, Non-oscillation theorems, Trans. Amer. Math. Soc. 64 (1948), 234–252.

[15] J.W. Hooker, M.K. Kwong, W.T. Patula, Oscillatory second order linear difference equations and Riccati equations, SIAM J. Math. Anal. 18 (1987), 54–63.

[16] J.W. Hooker, W.T. Patula, Riccati type transformations for second-order linear difference equations, J. Math. Anal. Appl. 82 (1981), 451–462.

[17] W.G. Kelley, A.C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, Boston, 1991.

[18] A. Kneser, Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen, Math. Ann. 42 (1893), 409–435.

[19] M.K. Kwong, J.W. Hooker, W.T. Patula, Riccati type transformations for second-order linear difference equations, II, J. Math. Anal. Appl. 107 (1985), 182–196.

[20] M. Ma, Dominant and recessive solutions for second order self-adjoint linear difference equations, Appl. Math. Lett. 18 (2005), 179–185.

[21] Z. Nehari, Oscillation criteria for second-order linear differential equations, Trans. Amer. Math. Soc. 85 (1957), 428–445.

[22] P. Řehák, Hartman-Wintner type lemma, oscillation, and conjugacy criteria for half-linear difference equations, J. Math. Anal. Appl. 252 (2000), 813–827.

[23] P. Řehák, Generalized discrete Riccati equation and oscillation of half-linear difference equations, Math. Comput. Modelling 34 (2001), 257–269.

[24] P. Řehák, Comparison theorems and strong oscillation in the half-linear discrete oscillation theory, Rocky Mountain J. Math. 33 (2003), 333–352.

[25] J. Sugie, M. Tanaka, Nonoscillation theorems for second-order linear difference equations via the Riccati-type transformation, Proc. Amer. Math. Soc. to appear.

[26] C.A. Swanson, Comparison and Oscillation Theory of Linear Differential Equations, Mathematics in Science and Engineering, **48**, New York and London, Academic Press, 1968.

[27] M. Veselý, P. Hasil, Oscillation and nonoscillation of asymptotically almost periodic half-linear difference equations, Abstr. Appl. Anal. 2013, Art. ID 432936, 12 pages.