Constraint qualifications and characterizations of solutions in convex optimization

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Introduction

A mathematical optimization problem is described by the following form:

$$(P)\begin{cases}\text{Minimize } f(x)\\ \text{subject to } x \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ \forall i \in I\}.\end{cases}$$

In particular, optimality conditions and duality theorems have been investigated by many researchers in convex optimization problem.

Constraint qualifications of the following inequality system of (P):

$$\sigma = \{g_i(x) \leq 0, i \in I\},$$

are important technical assumptions for solving (P), which have been studied by many researchers, see [3, 4, 8, 9, 10, 16]. One of the most important constraint qualification is the basic constraint qualification (BCQ, for short). Recall that σ satisfies BCQ at $\bar{x} \in S$ if

$$N_S(\bar{x}) = \text{coneco} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

and it is well-known that BCQ is a necessary and sufficient condition to ensure the equivalence between the optimality and the existence of Lagrange multipliers for convex optimization problems, so that candidates for optimal solutions may be found by using the existence of Lagrange multipliers, see [9, 16] for details. The other important constraint qualification is the Farkas Minkowski property (FM, for short), which is defined as follows

$$\text{coneco} \bigcup_{i \in I} \text{epi} g_i^* + \{0\} \times [0, +\infty) \text{ is closed},$$

is a necessary and sufficient constraint qualification for the Lagrange duality theorem, and a sufficient condition of BCQ at every points of S, see [8, 3]. Moreover the conical epigraph hull property (conical EHP, for short), which was defined as follows

$$\text{coneco} \bigcup_{i \in I} \text{epi} g_i^* \text{ is closed},$$

is well known as a sufficient condition of FM.

In this thesis, we deal with constraint qualifications and characterizations of solutions in convex optimization. Especially, we consider the following topics mainly:

(I) Checking methods of BCQ.

(II) Constraint qualifications for locally Lipschitz inequality systems.

In order to check BCQ at $\bar{x} \in S$, we may calculate the characteristic cones of conical EHP and FM instead of $N_S(\bar{x})$, $I(\bar{x})$ and $\partial g_i(\bar{x})$, because BCQ holds at every points of S when one of these cones is closed. However, when both cones are not closed, it is unknown whether BCQ holds or not at a given point of S, and methods of checking BCQ by using these cones have not been observed as far as we know. This fact is a motivation for (I). Recently, the KKT optimality conditions for a convex optimization problem, whose constraint functions are not necessarily convex, was studied. In 2010, a convex optimization problem, whose objective function is differentiable convex and constraint functions are differentiable but not necessarily convex, was discussed and a constraint qualification for the optimality condition was given by Lasserre, see [15]. In 2013, a convex optimization problem, whose objective function is convex not necessarily differentiable and constraint functions are locally Lipschitz but not necessarily convex or differentiable, was discussed, and a constraint qualification for the optimality condition was given by Dutta and Lalitha, see [5]. However, the constraint qualification is not necessarily constraint qualification. This fact is a motivation for (II).

This thesis consists of four chapters. Chapter 1 deals with notation and preliminaries in convex analysis. Chapter 2 deals with alternative theorems for a separable convex inequality system. We show two alternative theorems for separable convex inequality system. In Section 2.1, we show a certain condition is a necessary and sufficient one for an alternative theorem of separable convex functions, and we give an interesting example. Based on the example, we prove another alternative theorems in Section 2.2, Chapter 3 deals with checking methods of BCQ. We give a theorem which gives a method of checking BCQ via the characteristic cones of conical EHP and FM. Also we give some examples of the theorem are given with figures. Chapter 4 deals with constraint qualifications for a locally Lipschitz inequality system. We give several constraint qualifications for the KKT optimality condition, which are modifications of well-known constraint qualifications of convex or nonlinear optimization, the Basic constraint qualification (BCQ), Guignard's constraint qualification, Abadie's constraint qualification, Cottle's constraint qualification and the linearly independent constraint qualification. We discuss all relations among these constraint qualifications, especially, we show that two of them are necessary and sufficient constraint qualifications for the KKT optimality condition. In addition, we remark that the Slater condition is not a constraint qualification for the optimality in this convex optimization problem.

Contents

Chapter 1

Preliminaries

In this chapter, we introduce some notation and preliminaries in convex analysis. In this thesis, we deal with functions and sets on $\mathbb{R}^n$. In section 1.1, we introduce notions of convex set, convex function, and these properties. In section 1.2, we introduce properties of locally Lipschitz function. In section 1.3, we introduce important constraint qualifications and previous results in convex optimization.

1.1 Convex sets and functions

Definition 1.1. Let C be a subset of $\mathbb{R}^n$,

(i) C is said to be convex if for each x, $y \in C$ and $\alpha \in (0,1)$, $(1-\alpha)x+\alpha y \in C$,

(ii) C is said to be a cone if C is non-empty set, and for each $\lambda \geq 0$ and $x \in C$, $\lambda x \in C$.

Let C be a set in $\mathbb{R}^n$. We denote the closure, the interior, the conical hull and the convex hull of C by $\mathrm{cl}C$, $\mathrm{int}C$, $\mathrm{cone}C$ and $\mathrm{co}C$, respectively. Also, we denote $A + B = \{a + b \mid a \in A, b \in B\}$, $\lambda A = \{\lambda a \mid a \in A\}$ and $\Lambda a = \{\lambda a \mid \lambda \in \Lambda\}$ for any $A, B \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ $\Lambda \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$.

The following separation theorem has important roles in convex analysis.

Theorem 1.1. Let C be non-empty convex subset of $\mathbb{R}^n$, and $x \notin \mathrm{cl}C$. Then there exist $a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that for each $y \in C$, $\langle a, x\rangle < \alpha \leq \langle a, y\rangle$

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}\cup\{+\infty\}$. The effective domain of f, denoted by $\mathrm{dom}f$, is defined by

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

f is said to be convex if for any $x, y \in \mathbb{R}^n$ and for any $\lambda \in (0,1)$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y),$$

and f is said to be strictly convex if for any $x, y \in \mathrm{dom} f$ with $x \neq y$ and for any $\lambda \in (0,1)$,

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y).$$

Also, f is said to be quadratic if f is written by the following form:

$$f(x) = \frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \alpha, \ \forall x \in \mathbb{R}^n,$$

where $A \in S^n = \{B \subseteq \mathbb{R}^{n\times n} \mid B \text{ is a symmetric matrix}\}$, $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. In addition, a convex quadratic function f has the following property:

(i) f is convex if and only if A is positive semidefinite, and

(ii) f is strictly convex if and only if A is positive definite.

Also f is said to be separable if f is written by the following form:

$$f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n), \forall x_1, \ldots, x_n \in \mathbb{R},$$

where $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$. f is convex if and only if $f_1, \ldots, f_n$ are convex. The epigraph of f, denoted by $\mathrm{epi} f$, is defined by

$$\mathrm{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}.$$

$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be proper and lower semicontinuous (lsc, for short) if $\mathrm{epi} f$ is non-empty and closed set, respectively. In addition f is convex if and only if $\mathrm{epi} f$ a convex set. The conjugate function of f, $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(u) = \sup\{\langle u, x\rangle - f(x) \mid x \in \mathbb{R}^n\},$$

where $\langle u, x\rangle$ denotes the inner product of two vectors u and x. The following inequality always holds:

$$\langle u, x\rangle - f(x) \leq f^*(u),$$

which is called the Young-Fenchel inequality. Also, if f is separable convex, that is, $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n), \forall x_1, \ldots, x_n \in \mathbb{R}$, where $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$, then

$$f^*(y_1 \ldots, y_n) = f_1^*(y_1) + \cdots + f_n^*(y_n), \forall y_1, \ldots, y_n \in \mathbb{R}.$$

The subdifferential of f at $x \in \mathbb{R}^n$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid f(x) + \langle \xi, y - x\rangle \leq f(y), \forall y \in \mathbb{R}^n\}.$$

From the Young-Fenchel inequality, it is clear that $\xi \in \partial f(x)$ if and only if $\langle \xi, x\rangle - f(x) = f^*(\xi)$. For non-empty convex set $S \subseteq \mathbb{R}^n$, the indicator function of S, denoted by $\delta_S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$\delta_S(x) = \begin{cases} 0, & \text{if} \quad x \in S, \\ +\infty, & \text{if} \quad x \notin S. \end{cases}$$

For proper lsc convex functions g, $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the infimal convolution of g with h, denoted by $g \oplus h$, is defined by

$$(g \oplus h)(x) := \inf_{x_1+x_2=x} \{g(x_1) + h(x_2)\}.$$

It is well known that if $\mathrm{dom} g \cap \mathrm{dom} h \neq \emptyset$, then

$$(g \oplus h)^* = g^* + h^* \text{ and } (g+h)^* = \mathrm{cl}(g^* \oplus h^*). \tag{1.1}$$

If one of g and h is continuous at some $a \in \mathrm{dom} g \cap \mathrm{dom} h$, the closure operation in the second equation of (1.1) is superfluous,

$$\mathrm{epi}(g+h)^* = \mathrm{epi} g^* + \mathrm{epi} h^*, \text{ and} \tag{1.2}$$

$$\partial(g+h)(x) = \partial g(x) + \partial h(x), \text{ for each } x \in \mathrm{dom} g \cap \mathrm{dom} h, \tag{1.3}$$

see Theorem 2.8.7 in [27]. Let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function for each $i \in I$, and let $\lambda \in \mathbb{R}_+^{(I)}$, that is, $\lambda = (\lambda_i)_{i \in I}$ such that $\lambda_i \geq 0$ for each $i \in I$, and with only finitely many λ_i different from zero. Assume that one of g_i, $i \in I$, is continuous at some $a \in \bigcap_{i \in I} \mathrm{dom}\, g_i$. Then

$$\partial \left(\sum_{i \in I} \lambda_i g_i \right)(x) = \sum_{i \in I} \lambda_i \partial\, g_i(x), \forall x \in \bigcap_{i \in I} \mathrm{dom}\, g_i, \tag{1.4}$$

where $0 \times (+\infty) = 0$. Let C be a set in $\mathbb{R}^n$. The negative polar cone of C, denoted by C^-, is defined by

$$C^- = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0, \forall x \in C\}.$$

It is well-known that C^- is a closed convex cone, and

$$C^{--} = (C^-)^- = \mathrm{clconeco} C.$$

For any $x \in C$, the tangent cone of C at x, denoted by $T_C(x)$, is defined by

$$T_C(x) = \{y \in \mathbb{R}^n \mid \exists \{(x_k, \alpha_k)\} \subseteq C \times \mathbb{R}_+ \text{ s.t. } x_k \to x, \alpha_k(x_k - x) \to y\},$$

where $\mathbb{R}_+ = [0, +\infty)$. The set $T_C(\bar{x})$ is a closed cone. The normal cone of C at x, denoted by $N_C(x)$, is defined by $N_C(x) = (T_C(x))^-$. When C is a convex set, it is well-known that

$$T_C(x) = \mathrm{clcone}(C - x) = N_C(x)^-, \text{ and}$$

$$N_C(x) = (C - x)^- = \{\xi \in \mathbb{R}^n \mid \langle \xi, y - x \rangle \leq 0, \forall y \in C\}.$$

1.2 Locally Lipschitz functions

A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for each $x \in \mathbb{R}^n$, there exist $M > 0$ and $r > 0$ such that $|g(y) - g(z)| \leq M\|y - z\|$ for each $y, z \in B(x, r)$, where $B(x, r) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$.

Definition 1.2. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function,

(i) the Clarke directional derivative of g at $x \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$, denoted by $g^\circ(x, d)$, is given by

$$g^\circ(x, d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y + td) - g(y)}{t},$$

(ii) the Clarke subdifferential of g at x, denoted by $\partial^\circ g(x)$, is defined by

$$\partial^\circ g(x) = \{\xi \in \mathbb{R}^n \mid \langle \xi, d \rangle \leq g^\circ(x, d), \forall d \in \mathbb{R}^n\}.$$

For each $x \in \mathbb{R}^n$, the function $g^\circ(x, \cdot)$ is a positively homogeneous convex function. The set $\partial^\circ g(x)$ is a non-empty, convex and compact subset of $\mathbb{R}^n$. Moreover the Clarke directional derivative is the support function of the Clarke subdifferential, that is,

$$g^\circ(x, d) = \max_{\xi \in \partial^\circ g(x)} \langle \xi, d \rangle .$$

When g is convex, then g is locally Lipschitz, $g^\circ(x, \cdot) = g'(x, \cdot)$ and $\partial^\circ g(x) = \partial g(x)$ for each $x \in \mathbb{R}^n$, where

$$g'(x, d) = \lim_{t \downarrow 0} \frac{g(x + td) - g(x)}{t}.$$

In general, a locally Lipschitz function g is said to be regular at x if g is directionally differentiable at x in the all directions d and $g^\circ(x, \cdot) = g'(x, \cdot)$, see [2].

1.3 Convex optimization

In this section, we consider a given infinite convex inequality system:

$$\sigma := \{g_i(x) \leq 0, i \in I\},$$

where I is an arbitrary, possibly infinite, index set, and $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous (lsc) proper convex functions for all $i \in I$. Let S be the solution set of σ, that is,

$$S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I\}.$$

Throughout the thesis, we assume the following general assumption

$$\text{(H):} \begin{cases} S \neq \emptyset, \\ \text{for each } i \in I, \text{ there exists } x_i \in S \text{ such that } g_i \text{ is continuous at } x_i. \end{cases}$$

Constraint qualifications have important roles to solve convex optimization problems. The most famous constraint qualification is the Slater constraint qualification as follows:

Definition 1.3. Assume that I is finite and g_i are real-valued convex. The inequality system σ is said to satisfy the Slater constraint qualification if

$$\text{there exists } x_0 \in S \text{ such that for each } i \in I, g_i(x_0) < 0.$$

The following useful conditions (I) and (II) of Theorem 1.2 are assured by the Slater constraint qualification.

Theorem 1.2. Let I be finite set, g_i be real-valued convex on $\mathbb{R}^n$, $i \in I$, and $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I\}$. Assume that σ satisfies the Slater constraint qualification. Then the following statements hold:

(I) for each real-valued convex function f on $\mathbb{R}^n$, the following statements are equivalent:

(a) $\bar{x}$ is a minimizer of the following optimization problem:

$$\begin{cases} \min f(x) \\ \text{ s.t. } g_i(x) \leq 0, \; i \in I, \end{cases}$$

(b) there exists $\lambda \in \mathbb{R}^I_+$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial g_i(\bar{x})$ and $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$.

(II) for each real-valued convex function f on $\mathbb{R}^n$,

$$\inf_{x \in S} f(x) = \max_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i \in I} g_i(x) \right\}.$$

Condition (b) of (I) is called the Karush-Kuhn-Tucker (KKT, for short) optimality condition. Condition (II) is a called the Lagrange duality theorem. The Slater constraint qualification is a sufficient constraint qualification for the optimality condition and the duality theorem in convex optimization problem. It is easy to check whether the Slater constraint qualification holds or not. However, the Slater constraint qualification is often not satisfied for many problems. Constraint qualifications are have been studied by many researchers, see [3, 4, 8, 9, 10, 16].

First, as study of (I), we introduce the basic constraint qualification (BCQ, for short) and a previous result of BCQ.

Definition 1.4. ([9, 16]) σ is said to satisfy the basic constraint qualification (BCQ) at $\bar{x} \in S$ if

$$N_S(\bar{x}) = \text{coneco} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$.

Theorem 1.3. ([9, 16]) Let $\bar{x} \in S$. Then the following statements are equivalent:

(i) σ satisfies BCQ at $\bar{x}$,

(ii) for each lsc proper convex function f on $\mathbb{R}^n$ such that $\text{dom} f \cap S \neq \emptyset$ and $\text{epi}\delta_S^* + \text{epi} f^*$ is closed, the following statements are equivalent:

(a) $\bar{x}$ is a minimizer of the following optimization problem:

$$\begin{cases} \min \ f(x) \\ \text{ s.t. } \ g_i(x) \leq 0, \ i \in I, \end{cases}$$

(b) there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial g_i(\bar{x})$ and $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$.

By Theorem 1.3, BCQ is a necessary and sufficient condition for the optimality condition.

Second, as study of (II), we introduce constraint qualifications for the Lagrange duality and previous result of these.

Definition 1.5. ([8, 16]) Assume that σ is satisfying (H). σ is said to satisfy Farkas-Minkowski (FM) if

$$\text{coneco} \bigcup_{i \in I} \text{epi} g_i^* + \{0\} \times [0, +\infty) \text{ is closed.}$$

σ is said to satisfy the conical epigraph hull property (conical EHP) if

$$\text{coneco} \bigcup_{i \in I} \text{epi} g_i^* \text{ is closed.}$$

Especially, FM is a well-known necessary and sufficient constraint qualification for the Lagrange duality theorem as follows:

Theorem 1.4. ([8]) Assume that σ is satisfying (H). Then the following statements are equivalent:

(i) σ satisfies FM,

(ii) for each lsc proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $S \cap \mathrm{dom} f \neq \emptyset$ and $\mathrm{epi} f^* + \mathrm{epi} \delta_S^*$ is closed, strong duality holds, that is,

$$\inf_{x \in S} f(x) = \max_{\lambda \in \mathbb{R}_+^{(I)}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i \in I} \lambda_i g_i(x)\}.$$

The relationship of the constraint qualifications for convex optimality is shown by the following proposition.

Proposition 1.1. ([8, 4] and Theorem 4.1 in [16]) Assume that σ is satisfying (H). Then (i)⇒(ii)⇒(iii)⇒(iv).

(i) σ satisfies Slater constraint qualification,

(ii) σ satisfies conical EHP,

(iii) σ satisfies FM,

(iv) σ satisfies BCQ at every point of S.

Finally, the following two results are used in our results.

Theorem 1.5. (Theorem 4.1 in [3]) Let I be an arbitrary index set. For each $i \in I$, let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Let $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Assume that $\{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I\}$ is non-empty. Then the following statements are equivalent:

(i) $\{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I\} \subseteq \{x \in \mathbb{R}^n \mid \langle u, x \rangle \leq \alpha\}$,

(ii) $(u, \alpha) \in \mathrm{clconeco} \bigcup_{i \in I} \mathrm{epi} g_i^*$.

Theorem 1.6. ([19]) Let f be a real-valued convex function on $\mathbb{R}^n$. If there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) < 0$, then we have $\{x \in \mathbb{R}^n \mid f(x) < 0\} = \mathrm{int}\{x \in \mathbb{R}^n \mid f(x) \leq 0\}$.

Proof. The proof is shown by using Theorem 11 and Remark 1 in [19]. □

Chapter 2

Alternative theorems for separable convex functions

In this chapter, we consider the following type alternative theorem: exactly one of the following two statements is true:

(i) There exists $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1(x) \leq 0, \ldots, f_m(x) \leq 0, \\ f_0(x) < 0. \end{cases}$$

(ii) There exist $\lambda_1, \ldots, \lambda_m \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}.$, $i = 0, 1, \ldots, m$. In 1902, Farkas established an alternative theorem when f_i, $i = 0, 1, \ldots, m$, are linear functions. This alternative theorem is well-known as the Farkas Lemma and plays very important roles to have duality results in mathematical programming problems. In 2009, Jeyakumar and Li proved the following alternative theorem:

Theorem 2.1. ([12]) Let $f_0 : \mathbb{R}^n \to \mathbb{R}$ be a sublinear function and let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be separable sublinear functions. Then exactly one of the following two statements is true:

(i) there exist $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1(x) \leq 0, \ldots, f_m(x) \leq 0, \\ f_0(x) < 0, \end{cases}$$

(ii) there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$ such that for each $x \in \mathbb{R}^n$,

$$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0.$$

Clearly, this result is a generalization of Farkas Lemma because linear function is separable sublinear function.

On the other hand, Tseng showed some Lagrange duality theorem for separable convex programming problems in 2009. If f_i, $i = 0, 1, \ldots, m$, are separable convex function, then

$$\inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, \ldots, m\} = \sup_{\mu \in \mathbb{R}_+^m} \inf_{x \in \mathbb{R}^n} \left(f_0(x) + \sum_{i=1}^{m} \mu_i f_i(x) \right),$$

where $\mathbb{R}_+ = [0, \infty)$, see [20]. In 2010, Jeyakumar and Li proved another Lagrange strong duality theorem for separable convex programming problems under certain constraint qualification, see [13]. In this chapter, we show two alternative theorems for separable convex functions. One is a generalization of Theorem2.1, and the proof is given by using a result of [13] in Section 2.1. The other is a generalization of the original Farkas Lemma, which is motivated from example of Section 2.1, and the proof is given in Section 2.2. All results of this chapter is based on [22].

2.1 A necessary and sufficient condition for an alternative theorem of separable convex functions

In this section, we give a necessary and sufficient condition for an alternative theorem of separable convex functions.

Theorem 2.2. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be separable convex functions. Then (A) and (B) are equivalent:

(A) $\operatorname{epi} \inf_{\lambda_i \geq 0} \left(\sum_{i=1}^{m} \lambda_i f_i \right)^* = \bigcup_{\lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i f_i \right)^*,$

(B) for each convex function $f_0 : \mathbb{R}^n \to \mathbb{R}$, exactly one of the following two statements is true:

(i) there exists $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1(x) \leq 0, \ldots, f_m(x) \leq 0, \\ f_0(x) < 0, \end{cases}$$

(ii) there exist $\lambda_i \geq 0$, $i =, 1, \ldots, m$ such that for each $x \in \mathbb{R}^n$,

$$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0.$$

Proof. We show that the following (I) and (II) are equivalent:

(I) for each convex function $f_0 : \mathbb{R}^n \to \mathbb{R}$, $\inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, \ldots, m\} = \max_{\mu \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left(f_0(x) + \sum_{i=1}^{m} \mu_i f_i(x) \right)$,

(II) for each convex function $f_0 : \mathbb{R}^n \to \mathbb{R}$, exactly one of the following two statements is true:

(i) there exists $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1(x) \leq 0, \ldots, f_m(x) \leq 0, \\ f_0(x) < 0, \end{cases}$$

(ii) there exist $\lambda_i \geq 0$, $i =, 1, \ldots, m$ such that for each $x \in \mathbb{R}^n$,

$$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0.$$

First we assume (I). Let f_0 be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. It is clear that (i) and (ii) do not hold simultaneously. If (i) does not hold, then $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$ implies $f_0(x) \geq 0$. This shows

$$\inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, \ldots, m\} \geq 0,$$

we have

$$\max_{\mu \in \mathbb{R}^m_+} \inf_{x \in \mathbb{R}^n} \left(f_0(x) + \sum_{i=1}^{m} \mu_i f_i(x) \right) \geq 0.$$

So, there exist $\mu \in \mathbb{R}^m_+$ such that for each $x \in \mathbb{R}^n$

$$f_0(x) + \sum_{i=1}^{m} \mu_i f_i(x) \geq 0.$$

Therefore (ii) holds, and then (II) holds.

Next we assume (II). Let f_0 be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$, and put

$$p := \inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, \ldots, m\}.$$

It is clear that $p < +\infty$. When $p = -\infty$, (I) holds for any $\mu \in \mathbb{R}^m_+$ by using the weak duality. When p is finite, put $\hat{f}_0 = f_0 - p$, then

$$f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \text{ implies } \hat{f}_0(x) \geq 0,$$

that is, (i) does not hold, and then (ii) holds. So, there exist $\hat{\mu} \in \mathbb{R}^m_+$ such that for each $x \in \mathbb{R}^n$

$$\hat{f}_0(x) + \sum_{i=1}^{m} \hat{\mu}_i f_i(x) \geq 0, \text{ that is, } f_0(x) + \sum_{i=1}^{m} \hat{\mu}_i f_i(x) \geq p.$$

Therefore

$$\sup_{\mu\in\mathbb{R}^n_+}\inf_{x\in\mathbb{R}^n}\left(f_0(x)+\sum_{i=1}^m \mu_i f_i(x)\right)\geq \inf_{x\in\mathbb{R}^n}\left(f_0(x)+\sum_{i=1}^m \hat{\mu}_i f_i(x)\right)\geq p.$$

From this and the weak duality, (I) holds. □

This theorem is a generalization of Theorem 2.1. It is impossible to find any weaker conditions than (A) where (B) holds. However, the following example shows us possibility of another alternative theorems.

Example 2.1. Let $f_1(x_1,x_2)=f_{11}(x_1)+f_{12}(x_2)$ be a separable function satisfying

$$f_{11}(x_1)=\begin{cases}\frac{1}{2}(x_1+1)^2 & (x_1<-1)\\ 0 & (-1\leq x_1\leq 1)\\ \frac{1}{2}(x_1-1)^2 & (x_1>1)\end{cases},\text{ and } f_{12}(x_2)=|x_2|.$$

Then we can calculate

$$f_1^*(y_1,y_2)=\frac{1}{2}y_1^2+|y_1|+\delta_{[-1,1]}(y_2)$$

and

$$(\lambda_1 f_1)^*(y_1,y_2)=\begin{cases}\frac{y_1^2}{2\lambda_1}+|y_1|+\delta_{[-\lambda_1,\lambda_1]}(y_2) & (\lambda_1>0),\\ \delta_{\{(0,0)\}}(y_1,y_2) & (\lambda_1=0).\end{cases}$$

Thus

$$\operatorname{epi}\inf_{\lambda_1\geq 0}(\lambda_1 f_1)^*=\{(x_1,x_2,\alpha)\mid |x_1|\leq\alpha\},\text{ but}$$

$$\bigcup_{\lambda_1\geq 0}\operatorname{epi}(\lambda_1 f_1)^*=\{(x_1,x_2,\alpha)\mid |x_1|<\alpha\}\bigcup\{(0,0,0)\}.$$

Thus (A) of Theorem 2.2 does not hold.

Now, we consider linear functions $f_0(x_1,x_2)=ax_1+bx_2,\ a,b\in\mathbb{R}$. In this case, the alternative holds, that is, exactly one of the following two statements is true:

(i) there exists $x\in\mathbb{R}^2$ such that $f_1(x)\leq 0$ and $f_0(x)<0$,

(ii) there exist $\lambda_1\geq 0$ such that for each $x\in\mathbb{R}^2$, $f_0(x)+\lambda_1 f_1(x)\geq 0$.

Because $a\neq 0$ whenever (i) holds, and $a=0$ whenever (ii) holds.

2.2 Another alternative theorems of separable convex functions

By inspiring Example 2.1, we have another alternative theorems.

Theorem 2.3. Let $f_0 : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $f_0(0) = 0$, and let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ be separable convex functions such that $f_i(0) = 0$. Then (C) implies (D):

(C) there exists $\delta > 0$ such that for each $x \in B(0, \delta)$ and $i = 1, \ldots, m$,

$$f_i'(0; x) = f_i(x),$$

where $B(0, \delta) = \{x \in \mathbb{R}^n \mid \|x\| < \delta\}$,

(D) exactly one of the following two statements is true:

(i) there exists $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1(x) \leq 0, \ldots, f_m(x) \leq 0, \\ f_0(x) < 0, \end{cases}$$

(ii) there exist $\lambda_1, \ldots, \lambda_m \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0.$$

Proof. It can be checked easily that $f_0'(0; \cdot)$ is sublinear and $f_i'(0; \cdot)$, $i = 1, \ldots, m$ are separable sublinear. By Theorem 2.1, exactly one of the following two statements is true:

(i′) there exist $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1'(0; x) \leq 0, \ldots, f_m'(0; x) \leq 0, \\ f_0'(0; x) < 0, \end{cases}$$

(ii′) there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$ such that for each $x \in \mathbb{R}^n$,

$$f_0'(0; x) + \sum_{i=1}^{m} \lambda_i f_i'(0; x) \geq 0.$$

First, we prove that (i′) implies (i). Suppose that (i′) holds. Clearly, $x \neq 0$. For any $i = 1, 2, \ldots, m$ and $t \in (0, \frac{\delta}{2\|x\|}]$, since $tx \in B(0; \delta)$,

$$f_i(tx) = f'(0; tx) = t f_i'(0; x) \leq 0.$$

From $f_0'(0;x) < 0$, there exists $t_0 > 0$ such that for any $t \in (0, t_0]$,

$$\frac{f_0(0+tx) - f_0(0)}{t} < 0, \text{ that is } f_0(tx) < 0.$$

Put $\mu = \min\left\{\frac{\delta}{2||x||}, t_0\right\}$, we have $f_i(\mu x) \leq 0$ for each $i = 1, \ldots, m$ and $f_0(\mu x) < 0$. Thus (i) holds.

Next, we prove that (ii′) implies (ii). Since f_0 is convex and $f_0(0) = 0$, $f_0'(0, \cdot) \leq f_0$ holds because $t \mapsto \frac{f_0(0+tx)-f_0(0)}{t}$ is non-increasing when $t \downarrow 0$. In the same reason, $f_i'(0, \cdot) \leq f_i$ holds for each $i = 1, \ldots, m$. So we have (ii).

Hence, the conclusion now follows as (i) and (ii) do not hold simultaneously. □

Remark 2.1. A family of functions f_i in Example 2.1 holds condition (C).

We showed an example (C) holds but (A) does not hold. That is, condition (C) does not imply condition (A). Next we show an example (A) holds but (C) does not hold.

Example 2.2. Let $f_1(x_1, x_2) = f_{11}(x_1) + f_{12}(x_2)$ be a separable function satisfying $f_{1j}(x_j) = \frac{1}{2}x_j^2 + |x_j|$. Then we can verify that $f_1^*(y_1, y_2) = f_{11}^*(y_1) + f_{12}^*(y_2)$, and

$$f_{1j}^*(y_j) = \begin{cases} \frac{1}{2}(y_j+1)^2 & (y_j \in (-\infty, -1)), \\ 0 & (y_j \in [-1, 1]), \\ \frac{1}{2}(y_j-1)^2 & (y_j \in (1, \infty)). \end{cases}$$

We can check that

$$\operatorname{epi}\left(\inf_{\lambda \geq 0} (\lambda f_1)^*\right) = \bigcup_{\lambda \geq 0} \operatorname{epi}(\lambda f_1)^* = \mathbb{R} \times [0, \infty).$$

That is, (A) holds. But (C) does not hold. Indeed, for each $\delta > 0$, $(\frac{1}{2}\delta, 0) \in B((0,0), \delta)$ and $f_1'((0,0); (\frac{1}{2}\delta, 0)) = \frac{1}{2}\delta < \frac{1}{8}\delta^2 + \frac{1}{2}\delta = f_1(\frac{1}{2}\delta, 0)$.

Finally, we have the following alternative theorem:

Corollary 2.1. Let $\bar{x} \in \mathbb{R}^n$, $f_0 : \mathbb{R}^n \to \mathbb{R}$ be convex such that $f_0(\bar{x}) = 0$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, be separable convex such that $f_i(\bar{x}) = 0$. Then (E) implies (D):

(E) there exists $\delta > 0$ such that for each $x \in B(0, \delta)$, and $i = 1, \ldots, m$,

$$f_i'(\bar{x}; x) = f_i(x + \bar{x}) - f_i(\bar{x}),$$

(D) exactly one of the following two statements is true:

(i) there exists $x \in \mathbb{R}^n$ such that

$$\begin{cases} f_1(x) \leq 0, \ldots, f_m(x) \leq 0, \\ f_0(x) < 0, \end{cases}$$

(ii) there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$ such that for each $x \in \mathbb{R}^n$,

$$f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \geq 0.$$

Proof. For each $i = 0, 1, \ldots, m$, define g_i a function from $\mathbb{R}^n$ to $\mathbb{R}$ by $g_i = f_i(\cdot + \bar{x})$. Then we can verify that $g_i(0) = f_i(\bar{x}) = 0$ and $g_i'(0; x) = f_i'(\bar{x}; x)$ hold for each $i = 0, 1, \ldots, m$. This and Theorem 2.3 completes the proof. □

Chapter 3

Alternative characterization of BCQ

In convex optimization problem, in order to check BCQ at $\bar{x} \in S$, we may calculate the characteristic cones of conical EHP and FM instead of $N_S(\bar{x})$, $I(\bar{x})$ and $\partial g_i(\bar{x})$. When one of these cones is closed, BCQ holds at every points of S. However, when these cones are not closed, it is unknown whether BCQ holds or not at a given point of S, and methods of checking BCQ by using these cones have not been observed as far as we know.

In this chapter, we show a theorem which gives a method of checking BCQ via the characteristic cones of conical EHP and FM. In addition, we studied application for a specific class of functions. All results of this chapter is based on [23, 24, 25].

3.1 Characterization of BCQ via the characteristic cone of the conical EHP

In this section, let I be an index set. For each $i \in I$, let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, be lsc proper convex. Let $S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\}$. For each $\bar{x} \in S$, let $V(\bar{x}) = \{(y, \langle y, \bar{x} \rangle) \in \mathbb{R}^{n+1} \mid y \in \mathbb{R}^n\}$. Before our main result, we give the following definition:

Definition 3.1. ([1]) Let U and V be subset of $\mathbb{R}^n$. We say that U is closed regarding the set V if

$$(\mathrm{cl}U) \cap V = U \cap V.$$

Now, we give the characterization of BCQ at a point via the characteristic cones of conical EHP and FM, by observing closedness regarding of these cones; this theorem suggests another usage of these characteristic cones.

Theorem 3.1. ([25]) Let $\bar{x} \in S$. Assume that σ is satisfying (H). Then the following statements are equivalent:

(i) $\{g_i(x) \le 0, i \in I\}$ satisfies BCQ at $\bar{x}$,

(ii) $\text{coneco} \bigcup_{i\in I} \text{epi} g_i^*$ is closed regarding the set $V(\bar{x})$,

(iii) $\text{coneco} \bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)$ is closed regarding the set $V(\bar{x})$.

Proof. First we prove (i) $\Rightarrow$ (ii). Let $(y, \beta) \in \left(\text{clconeco} \bigcup_{i\in I} \text{epi} g_i^*\right) \cap V(\bar{x})$. From $(y, \beta) \in V(\bar{x})$, $\beta = \langle y, \bar{x}\rangle$, that is, $(y, \langle y, \bar{x}\rangle) \in \text{clconeco} \bigcup_{i\in I} \text{epi} g_i^*$. By Theorem 1.5,

$$\langle y, x\rangle \le \langle y, \bar{x}\rangle \text{ for each } x \in S,$$

that is $y \in N_S(\bar{x})$. From (i), $y \in \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})$ and then there exist a finite subset $J \subseteq I(\bar{x})$, $\lambda_i \ge 0$ and $y_i \in \partial g_i(\bar{x})$ $(i \in J)$ such that $y = \sum_{i\in J} \lambda_i y_i$. For each $i \in J$, since $i \in I(\bar{x})$ and $y_i \in \partial g_i(\bar{x})$, we have

$$g_i^*(y_i) = g_i(\bar{x}) + g_i^*(y_i) = \langle y_i, \bar{x}\rangle,$$

then $(y_i, \langle y_i, \bar{x}\rangle) \in \text{epi} g_i^*$. Therefore,

$$\begin{aligned} (y, \beta) &= \textstyle\sum_{i\in J} \lambda_i (y_i, \langle y_i, \bar{x}\rangle) \\ &\in \text{coneco} \textstyle\bigcup_{i\in J} \text{epi} g_i^* \\ &\subseteq \text{coneco} \textstyle\bigcup_{i\in I} \text{epi} g_i^*. \end{aligned}$$

Since $(y, \beta) = (y, \langle y, \bar{x}\rangle) \in V(\bar{x})$, we have $(y, \beta) \in \left(\text{coneco} \bigcup_{i\in I} \text{epi} g_i^*\right) \cap V(\bar{x})$, and consequently $\text{coneco} \bigcup_{i\in I} \text{epi} g_i^*$ is closed regarding the set $V(\bar{x})$.

Next we prove (ii)$\Rightarrow$(iii). Let $(y, \beta) \in \left(\text{cl}\left(\text{coneco} \bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)\right)\right) \cap V(\bar{x})$. From $(y, \beta) \in V(\bar{x})$, $\beta = \langle y, \bar{x}\rangle$, that is, $(y, \langle y, \bar{x}\rangle) \in \text{cl}\left(\text{coneco} \bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)\right)$. When $y = 0$, $(y, \beta) = (0, 0) \in \left(\text{coneco} \bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)\right) \cap V(\bar{x})$. When $y \ne 0$, there exists $\{(y_k, \beta_k)\} \subseteq \text{coneco} \bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)$ such that $(y_k, \beta_k) \to (y, \beta)$ and $y_k \ne 0$ for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, there exist a non-empty finite set $J_k \subseteq I$, $\lambda_k^i > 0$, $(x_k^i, \alpha_k^i) \in \text{epi} g_i^*$, $i \in J_k$, and $r_k \ge 0$ such that $(y_k, \beta_k) = \sum_{i\in J_k} \lambda_k^i (x_k^i, \alpha_k^i) + (0, r_k)$ and $\sum_{i\in J_k} \lambda_k^i > 0$. We have

$$\begin{aligned} (y_k, \beta_k) &= \textstyle\sum_{i\in J_k} \lambda_k^i (x_k^i, \alpha_k^i + \frac{1}{\sum_{i\in J_k} \lambda_k^i} r_k) \\ &\in \text{coneco} \textstyle\bigcup_{i\in J_k} \text{epi} g_i^* \\ &\subseteq \text{coneco} \textstyle\bigcup_{i\in I} \text{epi} g_i^*. \end{aligned}$$

Thus, $(y, \beta) \in \left(\text{clconeco} \bigcup_{i\in I} \text{epi} g_i^*\right) \cap V(\bar{x})$. From (ii),

$$\begin{aligned} (y, \beta) &\in \left(\text{coneco} \textstyle\bigcup_{i\in I} \text{epi} g_i^*\right) \cap V(\bar{x}) \\ &\subseteq \left(\text{coneco} \textstyle\bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)\right) \cap V(\bar{x}), \end{aligned}$$

that is, $\text{coneco} \bigcup_{i\in I} \text{epi} g_i^* + \{0\} \times [0, +\infty)$ is closed regarding the set $V(\bar{x})$.

Finally, we prove (iii)$\Rightarrow$(i). Let $y \in N_S(\bar{x})$. When $y = 0$, $y = 0 \in \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})$. When $y \ne 0$, by using Theorem 1.5,

$$(y, \langle y, \bar{x}\rangle) \in \text{clconeco} \bigcup_{i\in I} \text{epi} g_i^*,$$

and then $(y, \langle y, \bar{x}\rangle) \in \mathrm{cl}\left(\mathrm{coneco}\bigcup_{i\in I} \mathrm{epi}g_i^* + \{0\} \times [0, +\infty)\right)$. Since $\mathrm{coneco}\bigcup_{i\in I} \mathrm{epi}g_i^* + \{0\} \times [0, +\infty)$ is closed regarding the set $V(\bar{x})$ from (iii), $(y, \langle y, \bar{x}\rangle) \in \mathrm{coneco}\bigcup_{i\in I} \mathrm{epi}g_i^* + \{0\} \times [0, +\infty)$. So, there exist a non-empty finite set $J \subseteq I$, $\lambda_i > 0$, $(x_i, \alpha_i) \in \mathrm{epi}g_i^*$, $i \in J$, and $r \geq 0$ such that $(y, \langle y, \bar{x}\rangle) = \sum_{i\in J} \lambda_i(x_i, \alpha_i) + (0, r)$. For each $i \in J$ and $x \in \mathbb{R}^n$, we have

$$\begin{array}{ccccc} \langle x_i, x\rangle - g_i(x) + g_i(\bar{x}) & \leq & \langle x_i, x\rangle - g_i(x) & \leq & \alpha_i \\ \langle x_i, x - \bar{x}\rangle + g_i(\bar{x}) & \leq & \langle x_i, x - \bar{x}\rangle & \leq & g_i(x) + \alpha_i - \langle x_i, \bar{x}\rangle . \end{array}$$

Thus, for each $x \in \mathbb{R}^n$,

$$\begin{array}{ccccc} \sum\limits_{i\in J} \lambda_i(\langle x_i, x - \bar{x}\rangle + g_i(\bar{x})) & \leq & \sum\limits_{i\in J} \lambda_i \langle x_i, x - \bar{x}\rangle & \leq & \sum\limits_{i\in J} \lambda_i(g_i(x) + (\alpha_i - \langle x_i, \bar{x}\rangle)) \\ \langle y, x - \bar{x}\rangle + \sum\limits_{i\in J} \lambda_i g_i(\bar{x}) & \leq & \langle y, x - \bar{x}\rangle & \leq & \sum\limits_{i\in J} \lambda_i g_i(x) - r \leq \sum\limits_{i\in J} \lambda_i g_i(x). \end{array}$$

Hence, $y \in \partial(\sum_{i\in J} \lambda_i g_i)(\bar{x})$ and $0 = \sum_{i\in J} \lambda_i g_i(\bar{x})$ by putting $x = \bar{x}$. From $\lambda_i > 0$ and $g_i(\bar{x}) \leq 0$ for each $i \in J$, we have $J \subseteq I(\bar{x})$. Therefore,

$$\begin{array}{cll} y & \in & \partial(\sum_{i\in J} \lambda_i g_i)(\bar{x}) \\ & = & \sum_{i\in J} \lambda_i \partial g_i(\bar{x}) \qquad\qquad \text{(from (1.4))} \\ & \subseteq & \mathrm{coneco}\bigcup_{i\in J} \partial g_i(\bar{x}) \\ & \subseteq & \mathrm{coneco}\bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}). \end{array}$$

This completes the proof. □

By using Theorem 3.1, we can check BCQ holds or not at every $x \in S$ by using the characteristic cone of conical EHP. Especially, when $n \leq 2$, the figure of the cone is useful and effective for the purpose, see the following examples.

Example 3.1. Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a function as follows:

$$g_1(x) = \begin{cases} \frac{1}{2}x^2 - x & \text{if } x \in (-\infty, 0], \\ 0 & \text{if } x \in (0, 1), \\ \frac{1}{2}(x-1)^2 & \text{if } x \in [1, +\infty). \end{cases}$$

Then $S = [0, 1]$. We check whether BCQ holds or not at each point of S. The conjugate of g_1 is as follows:

$$g_1^*(x) = \begin{cases} \frac{1}{2}(x+1)^2 & \text{if } x \in (-\infty, -1], \\ 0 & \text{if } x \in (-1, 0), \\ \frac{1}{2}x^2 + x & \text{if } x \in [0, +\infty). \end{cases}$$

Also,

$$\mathrm{coneepi}g_1^* = \{(y, r) \in \mathbb{R}^2 \mid y \leq 0, r \geq 0\} \cup \{(y, r) \in \mathbb{R}^2 \mid y > 0, r > y\},$$

$$\mathrm{clconeepi}g_1^* = \{(y, r) \in \mathbb{R}^2 \mid y \leq 0, r \geq 0\} \cup \{(y, r) \in \mathbb{R}^2 \mid y > 0, r \geq y\}.$$

Then conical EHP does not hold; also FM does not hold. On the other hand, for each $\bar{x} \in S$,

$$\text{clconeepi}g_1^* \cap V(\bar{x}) = \begin{cases} (-\infty, 0] \times \{0\} & \text{if} \quad \bar{x} = 0, \\ \{(0,0)\} & \text{if} \quad \bar{x} \in (0,1), \\ \{(y,r) \mid 0 \le y, y \le r\} & \text{if} \quad \bar{x} = 1, \end{cases}$$

and

$$\text{coneepi}g_1^* \cap V(\bar{x}) = \begin{cases} (-\infty, 0] \times \{0\} & \text{if} \quad \bar{x} = 0, \\ \{(0,0)\} & \text{if} \quad \bar{x} \in (0,1]. \end{cases}$$

Therefore, BCQ holds at each point of $[0,1)$, but BCQ doesn't hold at 1 from Theorem 3.1, see Figure 3.1 which is reprinted from [25].

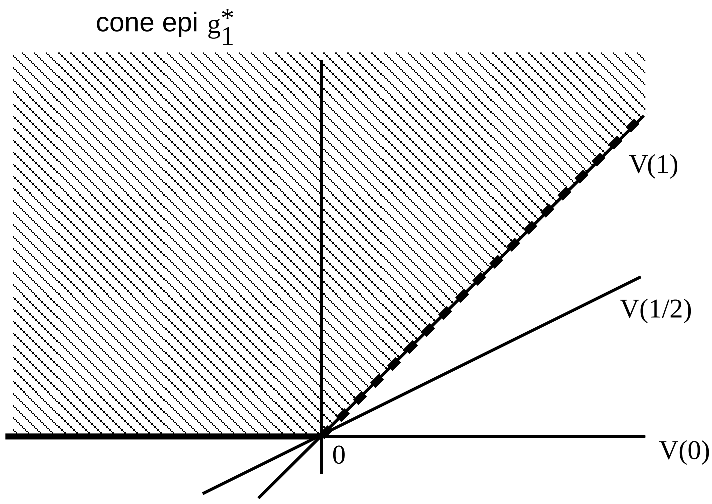


Figure 3.1: (Example 3.1)

Example 3.2. Let $g_2 : \mathbb{R}^2 \to \mathbb{R}$ be a function as follows: $g_2(x_1, x_2) = g_{21}(x_1) + g_{22}(x_2)$, where

$$g_{2j}(x_j) = \begin{cases} \frac{1}{2}(x_j + 1)^2 & \text{if} \quad x_j \in (-\infty, -1], \\ 0 & \text{if} \quad x_j \in (-1, 1), \\ \frac{1}{2}(x_j - 1)^2 & \text{if} \quad x_j \in [1, +\infty). \end{cases}$$

Then, $S = [-1,1]^2$. Since g_2 is a separable function, $g_2^*(y_1, y_2) = g_{21}^*(y_1) + g_{22}^*(y_2) = \frac{1}{2}y_1^2 + |y_1| + \frac{1}{2}y_2^2 + |y_2|$,

$$\text{epi}g_2^* = \{(y_1, y_2, r) \in \mathbb{R}^3 \mid (|y_1| + 1)^2 + (|y_2| + 1)^2 \le 2(r+1)\},$$

$$\text{coneepi}g_2^* = \{(y_1, y_2, r) \in \mathbb{R}^3 \mid |y_1| + |y_2| < r\} \cup \{(0,0,0)\},$$

and

$$\text{clconeepi}g_2^* = \{(y_1, y_2, r) \in \mathbb{R}^3 \mid |y_1| + |y_2| \le r\}.$$

Then conical EHP does not hold; also FM does not hold. It is easy to check that $\text{coneepi}g_2^* \cap V(\bar{x}) = \{(0,0,0)\}$ for each $\bar{x} \in S$, and $\bar{x} \in \text{int}S = (-1,1)^2$ if and only if $\mathrm{clconeepi}g_2^* \cap V(\bar{x}) = \{(0,0,0)\}$. On the other hand, $\mathrm{clconeepi}g_2^* \cap V(\bar{x}) \neq \{(0,0,0)\}$ for each $\bar{x} \in \mathrm{bd}S$. Indeed, for each $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathrm{bd}S$, there exists $i_0 \in \{1,2\}$ such that $|\bar{x}_{i_0}| = 1$. Put

$$y_i = \begin{cases} \bar{x}_i & \text{if} \quad |\bar{x}_i| = 1, \\ 0 & \text{if} \quad |\bar{x}_i| \neq 1. \end{cases}$$

Then $(y_1, y_2, \bar{x}_1 y_1 + \bar{x}_2 y_2) \in \mathrm{clconeepi}g_2^* \cap V(\bar{x}) \setminus \{(0,0,0)\}$, that is, $\mathrm{clconeepi}g_2^* \cap V(\bar{x}) \neq \{(0,0,0)\}$. By using Theorem 3.1, BCQ holds at each point of $\mathrm{int}S$, but BCQ doesn't hold at each point of $\mathrm{bd}S$, see Figure 3.2 which is reprinted from [25]. However, in this case, it is not easy whether BCQ holds or not at every point

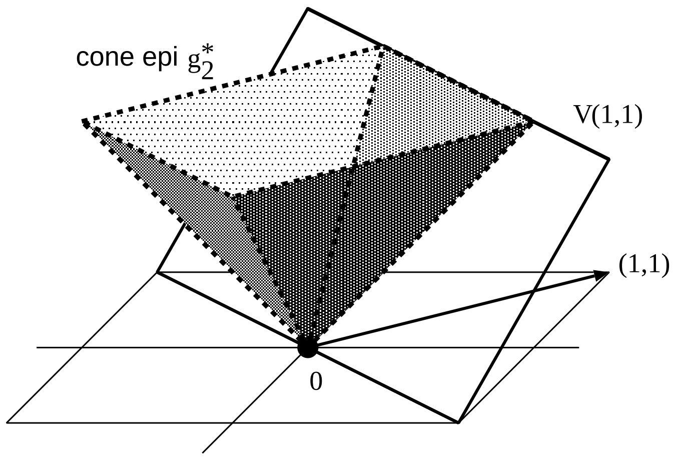


Figure 3.2: (Example 3.2)

of S from the definition of BCQ, because the calculation of $N_S(\bar{x})$ and $\partial g_2(\bar{x})$ for every $\bar{x} \in S$ often need time. Indeed, $\partial g_2(\bar{x}_1, \bar{x}_2) = \{(0,0)\}$ for each $\bar{x} \in S$, but

$$N_S(\bar{x}_1, \bar{x}_2) = \begin{cases} \{(0,0)\} & \text{if} \quad (\bar{x}_1, \bar{x}_2) \in \mathrm{int}S, \\ \mathbb{R}_+ \times \{0\} & \text{if} \quad \bar{x}_1 = 1, \bar{x}_2 \in [-1,1], \\ \mathbb{R}_+^2 & \text{if} \quad \bar{x}_1 = 1, \bar{x}_2 = 1, \\ \{0\} \times \mathbb{R}_+ & \text{if} \quad \bar{x}_1 \in [-1,1], \bar{x}_2 = 1, \\ -\mathbb{R}_+ \times \mathbb{R}_+ & \text{if} \quad \bar{x}_1 = -1, \bar{x}_2 = 1, \\ -\mathbb{R}_+ \times \{0\} & \text{if} \quad \bar{x}_1 = -1, \bar{x}_2 \in [-1,1], \\ -\mathbb{R}_+^2 & \text{if} \quad \bar{x}_1 = -1, \bar{x}_2 = -1, \\ \{0\} \times -\mathbb{R}_+ & \text{if} \quad \bar{x}_1 \in [-1,1], \bar{x}_2 = -1, \\ \mathbb{R}_+ \times -\mathbb{R}_+ & \text{if} \quad \bar{x}_1 = 1, \bar{x}_2 = -1. \end{cases}$$

In next example, we check BCQ in the special case of $n \geq 2$.

Example 3.3. Let $g_3 : \mathbb{R}^n \to \mathbb{R}$ be a function as follows:

$$g_3(x) = 1/8(\langle v, x \rangle - |\langle v, x \rangle|)^2 + \frac{1}{2}(|\langle w, x \rangle| - \langle w, x \rangle),$$

where v, $w \in \mathbb{R}^n$ with $\|v\| = \|w\| = 1$ and $\{v, w\}$ is linearly independent. Then, $S = \{x \in \mathbb{R}^n \mid \langle v, x\rangle \geq 0, \langle w, x\rangle \geq 0\}$. Put $f_1(x) = 1/8(\langle v, x\rangle - |\langle v, x\rangle|)^2$ and $f_2(x) = \frac{1}{2}(|\langle w, x\rangle| - \langle w, x\rangle)$, we have $g_3 = f_1 + f_2$ and

$$\text{epi}g_3^* \;=\; \left\{(sv + tv, r) \in \mathbb{R}^{n+1} \;\middle|\; \frac{1}{2}s^2 \leq r, s \in (-\infty, 0], t \in [-1, 0]\right\}$$

because $\text{epi}(f_1+f_2)^* = \text{epi}f_1^* + \text{epi}f_2^*$, $f_1^*(y) = \frac{1}{2}(\langle v, y\rangle)^2 + \delta_{(-\infty,0]v}(y)$, and $f_2^*(y) = \delta_{[-1,0]w}(y)$. Hence,

$$\text{coneepi}g_3^* \;=\; \{(-sv - tw, r) \in \mathbb{R}^{n+1} \mid t \geq 0, s, r > 0\} \cup \{(-tw, r) \in \mathbb{R}^{n+1} \mid t, r \geq 0\},$$

and

$$\text{clconeepi}g_3^* = \{(-sv - tw, r) \in \mathbb{R}^{n+1} \mid s, t, r \geq 0\}.$$

Then conical EHP does not hold; also FM does not hold. For each $\bar{x} \in S$,

$$\text{coneepi}g_3^* \cap V(\bar{x}) = \begin{cases} \{(0,0)\} & \text{if} \quad \langle v, \bar{x}\rangle \geq 0, \langle w, \bar{x}\rangle > 0, \\ (-\infty, 0]w \times \{0\} & \text{if} \quad \langle v, \bar{x}\rangle \geq 0, \langle w, \bar{x}\rangle = 0. \end{cases}$$

and

$$\text{clconeepi}g_3^* \cap V(\bar{x}) = \begin{cases} \{(0,0)\} & \text{if} \quad \langle v, \bar{x}\rangle > 0, \langle w, \bar{x}\rangle > 0, \\ (-\infty, 0]v \times \{0\} & \text{if} \quad \langle v, \bar{x}\rangle = 0, \langle w, \bar{x}\rangle > 0, \\ (-\infty, 0]w \times \{0\} & \text{if} \quad \langle v, \bar{x}\rangle > 0, \langle w, \bar{x}\rangle = 0, \\ ((-\infty, 0]v + (-\infty, 0]w) \times \{0\} & \text{if} \quad \langle v, \bar{x}\rangle = 0, \langle w, \bar{x}\rangle = 0. \end{cases}$$

Therefore, BCQ holds at each point of $\{x \in \mathbb{R}^n \mid \langle v, x\rangle > 0, \langle w, x\rangle \geq 0\}$, but BCQ doesn't hold at each point of $\{x \in \mathbb{R}^n \mid \langle v, x\rangle = 0, \langle w, x\rangle \geq 0\}$ from Theorem 3.1. In particular, if $n = 2$, $\langle v, w\rangle = 0$, then $\text{coneepi}g_3^*$ is as follows: Figure 3.3 is

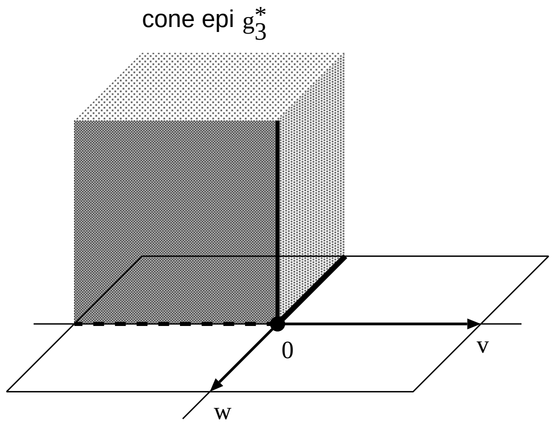


Figure 3.3: (Example 3.3)

reprinted from [25].

3.2 Application to convex quadratic functions

In this section, we consider the convex optimization problem:

$$(\mathrm{P}) \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, is a convex function, and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are convex quadratic functions. The purpose of this section is to study Theorem 3.1 by a specific class of functions which are presented by quadratic. We show results to check the BCQ without calculating g^*.

Lemma 3.1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and assume that $S = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ be a non-empty set. If $g(x) \geq 0$ for each $x \in \mathbb{R}^n$, then $g^*(0) = 0$ and $S = \partial g^*(0)$.

Proof. The proof is easy and omitted. □

Theorem 3.2. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex quadratic function that is not identically zero. Suppose that $S = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ is a non-empty set. Then the following statements are equivalent:

(i) for each $x \in \mathbb{R}^n$, $g(x) \geq 0$,

(ii) for each $x \in S$, $\{g\}$ doesn't satisfy the BCQ at x,

(iii) there exists $x_0 \in S$ such that $\{g\}$ doesn't satisfy the BCQ at x_0.

Proof. The implication (ii)⇒(iii) is clear. Since the Slater condition assures BCQ holds at every feasible point, see [11], then (iii)⇒(i) is clear.

Now, we turn to the proof of (i)⇒(ii), and assume (i). Let $u \in S$. By Lemma 3.1, $g^*(0) = 0$ and $S = \partial g^*(0)$. Put

$$B = \{(y, \beta) \in \mathbb{R}^n \times \mathbb{R} \mid \langle (u, -1), (y, \beta) \rangle = 0\}.$$

Then,

$$B \cap \operatorname{epi} g^* \subseteq \{(0, 0)\}. \tag{3.1}$$

In fact, assume that there exists $(\xi, \alpha) \in B \cap \operatorname{epi} g^*$ such that $(\xi, \alpha) \neq (0, 0)$. By $u \in \partial g^*(0)$ and $g^*(0) = 0$, for each $(\eta, \beta) \in \operatorname{epi} g^*$,

$$\langle u, \eta \rangle \leq g^*(\eta) \leq \beta. \tag{3.2}$$

Since $(\xi, \alpha) \in B \cap \operatorname{epi} g^*$, we have $\langle u, \xi \rangle \leq g^*(\xi) \leq \alpha$ and $\langle u, \xi \rangle = \alpha$, that is $\langle u, \xi \rangle = g^*(\xi) = \alpha$. It is clear that $\xi \neq 0$, if not $(\xi, \alpha) = (0, 0)$. From $g(u) = 0$, we have $g(u) + g^*(\xi) = \langle u, \xi \rangle$, so $u \in \partial g^*(\xi)$. Therefore $0, \xi \in \partial g(u)$. This shows

g^* is subdifferentiable on $\text{co}\{0,\xi\}$, then g^* is strictly convex on $\text{co}\{0,\xi\}$ by using Theorem 26.3 in [17]. For any $\lambda \in (0,1)$, we have

$$\begin{array}{rcl} g^*(\lambda\xi) & = & g^*((1-\lambda)0+\lambda\xi) \\ & < & (1-\lambda)g^*(0)+\lambda g^*(\xi) \\ & = & \lambda \langle u,\xi\rangle \\ & = & \langle u,\lambda\xi\rangle . \end{array}$$

From (3.2), $\langle u,\lambda\xi\rangle \le g^*(\lambda\xi)$ because $(\lambda\xi, g^*(\lambda\xi)) \in \text{epi}g^*$. This is a contradiction.

From (3.1), we have

$$\{y \in \mathbb{R}^n \mid (y,\langle y,u\rangle) \in \text{coneepi}g^*\} \subseteq \{0\}. \tag{3.3}$$

Actually, let $y \in \mathbb{R}^n$ satisfy $(y,\langle y,u\rangle) \in \text{coneepi}g^*$. There exist $\lambda \ge 0$ and $(x,\alpha) \in \text{epi}g^*$ such that $(y,\langle y,u\rangle) = \lambda(x,\alpha)$. If $\lambda = 0$, then $y = 0$. If $\lambda > 0$, we have

$$(\frac{1}{\lambda}y,\left\langle \frac{1}{\lambda}y,u\right\rangle) \in \text{epi}g^* \text{ and } \left\langle (u,-1),(\frac{1}{\lambda}y,\left\langle \frac{1}{\lambda}y,u\right\rangle)\right\rangle = 0.$$

Thus, $(\frac{1}{\lambda}y,\left\langle \frac{1}{\lambda}y,u\right\rangle) \in B \cap \text{epi}g^*$. From (3.1), we have $y = 0$.

Since g is a quadratic convex function, there exist $A \in S^n_+$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ such that $g(x) = \frac{1}{2}\langle x,Ax\rangle + \langle a,x\rangle + \alpha$ for each $x \in \mathbb{R}^n$, Put $L = \{x \in \mathbb{R}^n \mid Ax = 0\}^\perp$, then

$$L \subseteq \{y \in \mathbb{R}^n \mid (y,\langle y,u\rangle) \in \text{clconeepi}g^*\}. \tag{3.4}$$

Actually, let $z \in L$. There exists a unique symmetric positive semi-definite matrix $A^* \in S^n$ such that $AA^* = A^*A = P$, where P is the matrix of the linear transformation which projects $\mathbb{R}^n$ orthogonally onto L, and for this A^* one has

$$g^*(y) = \frac{1}{2}\langle y-a, A^*(y-a)\rangle - \alpha + \delta_L(y-a) \text{ for each } y \in \mathbb{R}^n,$$

see Section 12 in [17]. From $g^*(0) = 0$, we have

$$g^*(y) = \frac{1}{2}\langle y, A^*y\rangle - \frac{1}{2}\langle y, A^*a+a\rangle + \delta_L(y), \text{ for each } y \in \mathbb{R}^n.$$

For each $k \in \mathbb{N}$, $(\frac{1}{k}z, g^*(\frac{1}{k}z)), (-\frac{1}{k}z, g^*(-\frac{1}{k}z)) \in \text{epi}g^*$ because $z \in L$ and L is subspace of $\mathbb{R}^n$. From (3.2), we have $\left\langle u,\frac{1}{k}z\right\rangle \le g^*(\frac{1}{k}z)$ and $\left\langle u,-\frac{1}{k}z\right\rangle \le g^*(-\frac{1}{k}z)$, that is

$$\begin{array}{rcl} -g^*(-\frac{1}{k}z) & \le & \left\langle \frac{1}{k}z,u\right\rangle \le g^*(\frac{1}{k}z), \\ -kg^*(-\frac{1}{k}z) & \le & \langle z,u\rangle \le kg^*(\frac{1}{k}z). \end{array}$$

From $-kg^*(-\frac{1}{k}z) = -\frac{1}{2k}\langle z,A^*z\rangle - \frac{1}{2}\langle z,A^*a+a\rangle$ and $kg^*(\frac{1}{k}z) = \frac{1}{2k}\langle z,A^*z\rangle - \frac{1}{2}\langle z,A^*a+a\rangle$, we have $-kg^*(-\frac{1}{k}z) \to -\frac{1}{2}\langle z,A^*a+a\rangle$ and $kg^*(\frac{1}{k}z) \to -\frac{1}{2}\langle z,A^*a+a\rangle$. Thus $\langle z,u\rangle = -\frac{1}{2}\langle z,A^*a+a\rangle$. Therefore $(z,kg^*(\frac{1}{k}z)) \to (z,\langle z,u\rangle)$. From $(z,kg^*(\frac{1}{k}z)) \in \text{coneepi}g^*$, $(z,\langle u,z\rangle) \in \text{clconeepi}g^*$.

In addition, $L \neq \{0\}$ since g is not identically zero and (i). From this, (3.3) and (3.4),

$$\{y \in \mathbb{R}^n \mid (y, \langle y, u \rangle) \in \operatorname{coneepi} g^*\} \neq \{y \in \mathbb{R}^n \mid (y, \langle y, u \rangle) \in \operatorname{clconeepi} g^*\}.$$

By Theorem 3.1, $\{g\}$ doesn't satisfy the BCQ at u, and thus (ii) holds. This completes the proof. $\square$

By Theorem 1.3, Theorem 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex quadratic function that is not identitically zero. Suppose that $S = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ is a non-empty set. Then the following statements are equivalent:

(i) there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) < 0$,

(ii) there exists $x_1 \in S$ such that $\{g\}$ satisfies the BCQ at x_1,

(iii) for each $x \in S$, $\{g\}$ satisfies the BCQ at x,

(iv) for each $\bar{x}$ and $f : \mathbb{R}^n \to \mathbb{R}$, convex, the following statements are equivalent:

  (a) $\bar{x}$ is a minimizer of the following optimization problem:

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0, \end{cases}$$

  (b) there exists $\lambda \geq 0$ such that $0 \in \partial f(\bar{x}) + \lambda \partial g(\bar{x})$.

Example 3.4. Let $g(x) = x^2 + 2x$. Then $S = \{x \in \mathbb{R} \mid g(x) \leq 0\} = [-2, 0]$, and $g(-1) = -1 < 0$. Thus (i) of Corollary 3.1 holds. So, the BCQ holds at every point of S.

Example 3.5. Let $g(x_1, x_2) = \frac{1}{2}x_1$.Then, $S = \{0\} \times \mathbb{R}$, and for each $x \in \mathbb{R}^n$, $g(x) \geq 0$. Thus (i) of Corollary 3.1 doesn't hold. So, the BCQ doesn't hold at every point of S.

3.3 Application results of alternative criteria in convex composite functions

Throughout this section, constraint convex functions g_i are given as

$$g_i = h_i \circ v_i = h_i(v_i(\cdot)),$$

where $h_i : \mathbb{R} \to \mathbb{R}$ is a convex function for each $i \in I$ and we denote the inner product of $v_i \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ by $v_i(y)$ as a function of y for each fixed v_i. we observe σ by using the result of [25]

Theorem 3.3. Let g be a real-valued convex function on $\mathbb{R}^n$, h be a real-valued convex function on $\mathbb{R}$, and $v \in \mathbb{R}^n \setminus \{0\}$. Assume that $g = h \circ v$ and $\|v\| = 1$. Then $g^* = h^* \circ v + \delta_{[v]}$ where $[v] = \{\alpha v \mid \alpha \in \mathbb{R}\}$.

Proof. Let $y \in \mathbb{R}^n$. If $y \in [v]$, there exists $\alpha \in \mathbb{R}$ such that $y = \alpha v$. So, we have

$$\begin{aligned} g^*(y) &= \sup_{x\in\mathbb{R}^n}\{y(x) - g(x)\} \\ &= \sup_{x\in\mathbb{R}^n}\{\alpha v(x) - h(v(x))\} \\ &= \sup_{t\in\mathbb{R}} \sup_{v(x)=t}\{\alpha v(x) - h(v(x))\} \\ &= \sup_{t\in\mathbb{R}}\{\alpha t - h(t)\} \\ &= h^*(\alpha). \end{aligned}$$

From $\|v\| = 1$, $\alpha = v(y)$. Thus, $g^*(y) = (h^* \circ v)(y)$. If $y \notin [v]$, put $p = v(y)v$. For each $t \in \mathbb{R}$, $t = v(tv)$. For each $k \in \mathbb{N}$, put $x_k = tv + k(y - p)$, we have $v(x_k) = t$. For each $t \in \mathbb{R}$,

$$\begin{aligned} \sup_{v(x)=t} y(x) &\geq \sup_{k\in\mathbb{N}} y(x_k) \\ &= \sup_{k\in\mathbb{N}} y(tv + k(y-p)) \\ &= \sup_{k\in\mathbb{N}} ky(y-p) + tv(y) \\ &= \sup_{k\in\mathbb{N}} k\{y(y) - y(p)\} + tv(y) \\ &= \sup_{k\in\mathbb{N}} k\{\|y\|^2 - y(v(y)v)\} + tv(y) \\ &= \sup_{k\in\mathbb{N}} k\{\|y\|^2 - v(y)y(v)\} + tv(y) \\ &= \sup_{k\in\mathbb{N}} k\{\|y\|^2 - v(y)v(y)\} + tv(y) \\ &= \sup_{k\in\mathbb{N}} k(\|y\|^2\|v\|^2 - (v(y))^2) + tv(y). \end{aligned}$$

By Cauchy-Schwartz inequality and $y \notin [v]$, $0 < \|y\|^2\|v\|^2 - (v(y))^2$. Then $g^*(y) = +\infty$. Indeed,

$$\begin{aligned} g^*(y) &= \sup_{x\in\mathbb{R}^n}\{y(x) - g(x)\} \\ &= \sup_{x\in\mathbb{R}^n}\{y(x) - h(v(x))\} \\ &= \sup_{t\in\mathbb{R}} \sup_{v(x)=t}\{y(x) - h(t)\} \\ &\geq \sup_{v(x)=0} y(x) - h(0) \\ &\geq \sup_{k\in\mathbb{N}} \tfrac{k}{\|v\|^2}(\|y\|^2\|v\|^2 - (v(y))^2) - h(0) \\ &= +\infty. \end{aligned}$$

Hence,

$$g^* = h^* \circ v + \delta_{[v]}.$$

□

Example 3.6. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function as follows:

$$g(x) = \frac{1}{2}((v(x))^2 + v(x) + |v(x)|),$$

where $v \in \mathbb{R}^n \setminus \{0\}$ and $\|v\| = 1$. Put $h(t) = \frac{1}{2}(t^2 + t + |t|)$. Then $g = h \circ v$ and $S = \{x \in \mathbb{R}^n \mid v(x) = 0\}$.

$$h^*(s) = \begin{cases} \frac{1}{2}s^2 & \text{if} \quad s \in (-\infty, 0) \\ 0 & \text{if} \quad s \in [0, 1) \\ \frac{1}{2}(s-1)^2 & \text{if} \quad s \in [1, +\infty), \end{cases}$$

$$g^*(y) = \begin{cases} \frac{1}{2}(v(y))^2 & \text{if} \quad y \in (-\infty, 0)v \\ 0 & \text{if} \quad y \in [0, 1)v \\ \frac{1}{2}(v(y)-1)^2 & \text{if} \quad y \in [1, +\infty)v \\ +\infty & \text{if} \quad \text{otherwise.} \end{cases}$$

So,

$$\text{coneepi}g^* = \mathbb{R}v \times [0, +\infty) \setminus (-\infty, 0)v \times \{0\},$$

$$\text{clconeepi}g^* = \mathbb{R}v \times [0, +\infty).$$

σ does not satisfy FM. For each $\bar{x} \in S$,

$$\text{coneepi}g^* \cap V(\bar{x}) = [0, +\infty)v \times \{0\},$$

$$\text{clconeepi}g^* \cap V(\bar{x}) = \mathbb{R}v \times \{0\}.$$

Thus, σ does not satisfy BCQ at each $x \in S$.

Example 3.7. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function as follows:

$$g(x) = \begin{cases} -1 - v(x) & \text{if} \quad v(x) < -1 \\ 0 & \text{if} \quad -1 \le v(x) < 1 \\ \frac{1}{2}(v(x)-1)^2 & \text{if} \quad 1 \le v(x), \end{cases}$$

where $v \in \mathbb{R}^n \setminus \{0\}$ and $\|v\| = 1$. Put $h(t) = \begin{cases} -1 - t & \text{if} \quad t \in (-\infty, -1) \\ 0 & \text{if} \quad t \in [-1, 1) \\ \frac{1}{2}(t-1)^2 & \text{if} \quad t \in [1, +\infty). \end{cases}$
Then $g = h \circ v$, $S = \{x \in \mathbb{R}^n \mid -1 \le v(x) \le 1\}$ and $\text{int}S \neq \emptyset$.

$$h^*(s) = \begin{cases} -s & \text{if} \quad s \in [-1, 0) \\ \frac{1}{2}s^2 + s & \text{if} \quad s \in [0, +\infty) \\ +\infty & \text{if} \quad s \in (-\infty, -1), \end{cases}$$

$$g^*(y) = \begin{cases} -v(y) & \text{if} \quad y \in [-1,0)v \\ \frac{1}{2}(v(y))^2 + v(y) & \text{if} \quad y \in [0,+\infty)v \\ +\infty & \text{if} \quad \text{otherwise.} \end{cases}$$

So,

$$\text{coneepi}g^* = \{(\alpha v, r) \in \mathbb{R}^{n+1} \mid |\alpha| \leq r\} \setminus \{(\alpha v, r) \in \mathbb{R}^3 \mid 0 < \alpha < r\},$$

$$\text{clconeepi}g^* = \{(\alpha v, r) \in \mathbb{R}^{n+1} \mid \alpha \in \mathbb{R}, |\alpha| \leq 0\}.$$

For each $\bar{x} \in S$,

$$\text{coneepi}g^* \cap V(\bar{x}) = \begin{cases} \{(0,0)\} & \text{if} \quad \bar{x} \in \text{int}S \\ \{(0,0)\} & \text{if} \quad v(\bar{x}) = 1 \\ (-\infty,0]\{(v,1)\} & \text{if} \quad v(\bar{x}) = -1, \end{cases}$$

$$\text{clconeepi}g^* \cap V(\bar{x}) = \begin{cases} \{(0,0)\} & \text{if} \quad \bar{x} \in \text{int}S \\ [0,+\infty)\{(v,1)\} & \text{if} \quad v(\bar{x}) = 1 \\ (-\infty,0]\{(v,1)\} & \text{if} \quad v(\bar{x}) = -1. \end{cases}$$

Thus, σ does not satisfy BCQ at each $x \in \{x \in \mathbb{R}^n \mid v(x) = 1\}$, and σ satisfies BCQ at each $x \in \text{int}S \cup \{x \in \mathbb{R}^n \mid v(x) = -1\}$.

Example 3.8. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be two functions as follows:

$$g_1(x_1,x_2) = \begin{cases} -1 - x_1 & \text{if} \quad x_1 \in (-\infty,-1) \\ 0 & \text{if} \quad x_1 \in [-1,1) \\ \frac{1}{2}(x_1-1)^2 & \text{if} \quad x_1 \in [1,+\infty), \end{cases}$$

$$g_2(x_1,x_2) = \begin{cases} \frac{1}{2}(x_2+1)^2 & \text{if} \quad x_2 \in (-\infty,-1) \\ 0 & \text{if} \quad x_2 \in [-1,1) \\ -1 + x_2 & \text{if} \quad x_2 \in [1,+\infty). \end{cases}$$

Put $v_1 = (1,0)$, $v_2 = (0,1)$,

$$h_1(t) = \begin{cases} -1 - t & \text{if} \quad t \in (-\infty,-1) \\ 0 & \text{if} \quad t \in [-1,1) \\ \frac{1}{2}(t-1)^2 & \text{if} \quad t \in [1,+\infty), \end{cases}$$

and

$$h_2(t) = \begin{cases} \frac{1}{2}(t+1)^2 & \text{if} \quad t \in (-\infty,-1) \\ 0 & \text{if} \quad t \in [-1,1) \\ -1 + t & \text{if} \quad t \in [1,+\infty), \end{cases}$$

where $i = 1,2$. Then $g_i = h \circ v_i$, $i = 1,2$, $S = [-1,1]^2$ and $\text{int}S \neq \emptyset$.

$$h_1^*(s) = \begin{cases} -s & \text{if} \quad s \in [-1,0] \\ \frac{1}{2}s^2 + s & \text{if} \quad s \in [0,+\infty) \\ +\infty & \text{if} \quad s \in (-\infty,-1), \end{cases}$$

$$h_2^*(s) = \begin{cases} \frac{1}{2}s^2 - s & \text{if} \quad s \in (-\infty, 0] \\ s & \text{if} \quad s \in [0,1] \\ +\infty & \text{if} \quad s \in (1, +\infty), \end{cases}$$

$$g_1^*(y_1, y_2) = \begin{cases} -y_1 & \text{if} \quad (y_1, y_2) \in [-1, 0] \times \{0\} \\ \frac{1}{2}y_1^2 + y_1 & \text{if} \quad (y_1, y_2) \in [0, +\infty) \times \{0\} \\ +\infty & \text{if} \quad \text{otherwise}, \end{cases}$$

$$g_2^*(y_1, y_2) = \begin{cases} \frac{1}{2}y_2^2 - y_2 & \text{if} \quad (y_1, y_2) \in \{0\} \times (-\infty, 0] \\ y_2 & \text{if} \quad (y_1, y_2) \in \{0\} \times [0, 1] \\ +\infty & \text{if} \quad \text{otherwise}. \end{cases}$$

So,

$$\begin{aligned} \text{coneco}(\text{epi}g_1^* \cup \text{epi}g_2^*) = \; & \{(y_1, y_2, r) \in \mathbb{R}^3 \mid |y_1| + |y_2| < r\} \\ & \cup\{(y_1, y_2, r) \in \mathbb{R}^3 \mid y_1 \leq 0, y_2 \geq 0, -y_1 + y_2 = r\}, \\ \text{clconeco}(\text{epi}g_1^* \cup \text{epi}g_2^*) = \; & \{(y_1, y_2, r) \in \mathbb{R}^3 \mid |y_1| + |y_2| \leq r\}. \end{aligned}$$

σ does not satisfy FM. For each $(x_1, x_2) \in \{-1\} \times (-1, 1] \cup [-1, 1) \times \{1\}$, coneco$(\text{epi}g_1^* \cup \text{epi}g_2^*)$ is closed regarding the set $V(x_1, x_2)$. Thus, σ satisfies BCQ at each $(x_1, x_2) \in \text{int}S \cup \{-1\} \times (-1, 1] \cup [-1, 1) \times \{1\}$. For each $(x_1, x_2) \in \text{bd}S \setminus (\{-1\} \times (-1, 1] \cup [-1, 1) \times \{1\})$, coneco$(\text{epi}g_1^* \cup \text{epi}g_2^*)$ is not closed regarding the set $V(x_1, x_2)$. Thus, σ does not satisfy BCQ at each $(x_1, x_2) \in \text{bd}S \setminus (\{-1\} \times (-1, 1] \cup [-1, 1) \times \{1\})$.

Example 3.9. Let $I = [-\frac{1}{2}\pi, \frac{1}{2}\pi)$. We consider for each $\theta \in I$, $g_\theta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g_\theta(x) = \begin{cases} 1 - v_\theta(x) & \text{if} \quad v_\theta(x) \leq -1 \\ 0 & \text{if} \quad -1 < v_\theta(x) \leq 1 \\ \frac{1}{2}(v_\theta(x) - 1)^2 & \text{if} \quad 1 < v_\theta(x). \end{cases}$$

Put $v_\theta = (\cos\theta, \sin\theta)$ for each $\theta \in I$ and

$$h(t) = \begin{cases} -t - 1 & \text{if} \quad t \in (-\infty, -1) \\ 0 & \text{if} \quad t \in [-1, 1) \\ \frac{1}{2}(t-1)^2 & \text{if} \quad t \in [1, +\infty). \end{cases}$$

Then $g_\theta = h \circ v_\theta$ and $S = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$,

$$h^*(s) = \begin{cases} +\infty & \text{if} \quad s \in (-\infty, -1) \\ -s & \text{if} \quad s \in [-1, 0) \\ \frac{1}{2}s^2 + s & \text{if} \quad s \in [0, +\infty), \end{cases}$$

$$g_\theta^*(y) = \begin{cases} v_\theta(-y) & \text{if} \quad y \in [-1, 0)v_\theta \\ \frac{1}{2}(v_\theta(y))^2 + v_\theta(y) & \text{if} \quad y \in [0, +\infty)v_\theta \\ +\infty & \text{if} \quad y \in \mathbb{R}^2 \setminus [-1, +\infty)v_\theta. \end{cases}$$

So,

$$\begin{aligned}\mathrm{coneco}\bigcup_{\theta\in I}\mathrm{epi}g_\theta = & \ \{(-\alpha\cos\theta, -\alpha\sin\theta, r)\in\mathbb{R}^3 \mid 0\le\alpha=r, \theta\in[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)\} \\ & \cup\{(y_1,y_2,r)\in\mathbb{R}^3 \mid \|(y_1,y_2)\|<r\}, \\ \mathrm{clconeco}\bigcup_{\theta\in I}\mathrm{epi}g_\theta = & \ \{(y_1,y_2,r)\in\mathbb{R}^3 \mid \|(y_1,y_2)\|\le r\}.\end{aligned}$$

σ does not satisfy FM. For each $(x_1,x_2)\in\{(-\cos\theta,-\sin\theta)\mid\theta\in[-\frac{1}{2}\pi,\frac{1}{2}\pi)\}$, $\mathrm{coneco}\bigcup_{\theta\in I}\mathrm{epi}g_\theta$ is closed regarding the set $V(x_1,x_2)$. Thus σ satisfies BCQ at each $(x_1,x_2)\in\mathrm{int}S\cup\{(-\cos\theta,-\sin\theta)\mid\theta\in[-\frac{1}{2}\pi,\frac{1}{2}\pi)\}$. For each $(x_1,x_2)\in\mathrm{bd}S\setminus\{(-\cos\theta,-\sin\theta)\mid\theta\in[-\frac{1}{2}\pi,\frac{1}{2}\pi)\}$, $\mathrm{coneco}\bigcup_{\theta\in I}\mathrm{epi}g_\theta$ is not closed regarding the set $V(x_1,x_2)$. Thus σ does not satisfy BCQ at each $(x_1,x_2)\in\mathrm{bd}S\setminus\{(-\cos\theta,-\sin\theta)\mid\theta\in[-\frac{1}{2}\pi,\frac{1}{2}\pi)\}$.

Chapter 4

Constraint qualifications for locally Lipschitz inequality systems

Recently, the KKT optimality conditions for a convex optimization problem, whose constraint set S is described by the inequality constraints but every constraint functions are not necessarily convex, was studied. In 2013, a convex optimization problem, whose objective function is convex not necessarily differentiable and constraint functions are locally Lipschitz but not necessarily convex or differentiable, was discussed, and a constraint qualification for the optimality condition was given by Dutta and Lalitha, see [5]. In this chapter, we investigate several constraint qualifications, which are modifications of well-known constraint qualifications, for the KKT optimality in condition the convex optimization problem (P), which was discussed by Dutta and Lalitha in [5], and compare our results and previous ones. All results of this chapter is based on [26].

4.1 Definition of constraint qualifications for a locally Lipschitz systems

In this section, we consider the following convex optimization problem:

$$(P)\begin{cases} \min f(x) \\ \text{s.t. } x \in S, \end{cases}$$

where f is a real-valued convex function on $\mathbb{R}^n$ and S is a convex set. Throughout this section we assume that the feasible set S is given as

$$S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\},$$

where g_i, $i \in I = \{1, \ldots, m\}$, are real-valued locally Lipschitz functions on $\mathbb{R}^n$ and g_i is regular at every $x \in S$ and every $i \in I(x)$, where $I(x) = \{i \in I \mid g_i(x) = 0\}$.

The following theorem is shown by Dutta and Lalitha in [5].

Theorem 4.1. ([5]) Let $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I = \{1, \dots, m\}$, be locally Lipschitz functions, and let $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I\}$. Assume that S is a convex set, all g_i are regular at $\bar{x}$, the Slater condition holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for each $i \in I$, and $0 \notin \partial^\circ g_i(\bar{x})$ for each $i \in I(\bar{x})$. Then for each real-valued convex function f on $\mathbb{R}^n$, the following statements are equivalent:

(i) for each $x \in S$, $f(\bar{x}) \leq f(x)$,

(ii) there exists $\lambda \in \mathbb{R}^I_+$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x})$ and for each $i \in I$, $\lambda_i g_i(\bar{x}) = 0$.

Condition (ii) of this theorem is the KKT optimality condition of the problem (P).

In this section, we discuss the following conditions:

(A) $N_S(\bar{x}) = \text{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$,

(B) $T_S(\bar{x}) = \bigcap_{i \in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-)$ and $\text{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed,

(C) there exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi_i, y_0 \rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$,

(D) the Slater condition holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for each $i \in I$, and $0 \notin \partial^\circ g_i(\bar{x})$, for each $i \in I(\bar{x})$,

(E) $0 \notin \text{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$,

(F) $\text{int} S \neq \emptyset$ and $0 \notin \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$,

(G) for each $y_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, $\{y_i\}_{i \in I(\bar{x})}$ is linearly independent.

4.2 Observations of constraint qualifications

At first, we provide the following lemma, which is important to show our results:

Lemma 4.1. Let $\bar{x} \in S$. Then for each $i \in I(\bar{x})$, $\xi_i \in \partial^\circ g_i(\bar{x})$ and $x \in S$,

$$\langle \xi_i, x - \bar{x} \rangle \leq 0.$$

That is, $\partial^\circ g_i(\bar{x}) \subseteq N_S(\bar{x})$ for each $i \in I(\bar{x})$.

Proof. For each $i \in I(\bar{x})$, $\xi_i \in \partial^\circ g_i(\bar{x})$ and $x \in S$,

$$\langle \xi_i, x - \bar{x} \rangle \leq g_i^\circ(\bar{x}, x - \bar{x}).$$

From the regularity of g_i at $\bar{x}$,

$$\langle \xi_i, x - \bar{x} \rangle \leq g_i'(\bar{x}, x - \bar{x}) = \lim_{t \downarrow 0} \frac{g_i(\bar{x} + t(x - \bar{x})) - g_i(\bar{x})}{t}.$$

Since $\bar{x} + t(x - \bar{x}) \in S$ for each $t \in (0, 1)$ and $i \in I(\bar{x})$, we have $g_i'(\bar{x}, x - \bar{x}) \leq 0$, so $\langle \xi_i, x - \bar{x} \rangle \leq 0$. □

Now we show a result that conditions (A) and (B) are necessary and sufficient constraint qualifications for the optimality conditions in convex optimization problem (P).

Theorem 4.2. Let $\bar{x} \in S$. Then the following statements are equivalent:

(A) $N_S(\bar{x}) = \operatorname{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$,

(B) $T_S(\bar{x}) = \bigcap_{i \in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-)$ and $\operatorname{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed,

(O) for each real-valued convex function f on $\mathbb{R}^n$, the following statements are equivalent:

  (i) $f(x) \geq f(\bar{x})$ for each $x \in S$,

  (ii) there exists $\lambda \in \mathbb{R}^I_+$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x})$ and for each $i \in I$, $\lambda_i g_i(\bar{x}) = 0$.

Proof. First, we prove (A)⇔(B). It is clear that (A) holds if and only if $N_S(\bar{x}) = \operatorname{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ and $\operatorname{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed. From convexity of S, we have $N_S(\bar{x})^- = T_S(\bar{x})$. Therefore, it is enough to show that $(\bigcap_{i \in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-))^- = \operatorname{coneco} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$. This equality is given by the following property:

$$\bigcap_{i \in I} (A_i^-) = (\bigcup_{i \in I} A_i)^- \text{ for any } A_i \subseteq \mathbb{R}^n (i \in I).$$

Next, we prove (A)⇒(O). Let f be a real-valued convex function on $\mathbb{R}^n$. The proof that (ii) implies (i) is easy and omitted. Conversely, assume (i). For each $x \in S$, since $\bar{x} + \alpha(x - \bar{x}) \in S$ for each $\alpha \in (0, 1)$,

$$f(\bar{x}) \leq f(\bar{x} + \alpha(x - \bar{x})),$$

that is,

$$0 \leq f'(\bar{x}, x - \bar{x}) = \max_{\xi \in \partial f(\bar{x})} \langle \xi, x - \bar{x} \rangle .$$

Therefore $0 \le \inf_{x\in S} \max_{\xi\in\partial f(\bar{x})} \langle \xi, x-\bar{x}\rangle$. According to Sion's minimax theorem (see e.g. [18, 14]), we can invert the infimum and the maximum, and we get $0 \le \max_{\xi\in\partial f(\bar{x})} \inf_{x\in S} \langle \xi, x-\bar{x}\rangle$. Then there exists $\eta \in \partial f(\bar{x})$ such that

$$\langle -\eta, x-\bar{x}\rangle \le 0 \text{ for each } x \in S.$$

Thus, $-\eta \in N_S(\bar{x})$. From (A), $-\eta \in \operatorname{coneco}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. Then there exist $\mu_i \ge 0$ and $\xi_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, such that $-\eta = \sum_{i\in I(\bar{x})} \mu_i \xi_i$. Put

$$\lambda_i = \begin{cases} \mu_i & \text{if} \quad i \in I(\bar{x}), \\ 0 & \text{if} \quad i \in I \setminus I(\bar{x}), \end{cases}$$

for each $i \in I$. Then it is clear that $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$. Moreover,

$$-\eta = \sum_{i\in I(\bar{x})} \lambda_i \xi_i = \sum_{i\in I} \lambda_i \xi_i \in \sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x}).$$

Hence, $0 = \eta + (-\eta) \in \partial f(\bar{x}) + \sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x})$. Finally, we prove (O)⇒(A), $\operatorname{coneco}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x}) \subseteq N_S(\bar{x})$ is shown by using Lemma 4.1. Conversely, let $\eta \in N_S(\bar{x})$. Then

$$\langle -\eta, \bar{x}\rangle \le \langle -\eta, x\rangle \text{ for each } x \in S.$$

Put $f = \langle -\eta, \cdot\rangle$, then f is a convex function, and (i) of (O) holds. So, (ii) of (O) holds. Hence, there exists $\lambda \in \mathbb{R}^I_+$ such that

$$\begin{cases} 0 \in \partial f(\bar{x}) + \sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x}), \\ \lambda_i g_i(\bar{x}) = 0 \text{ for each } i \in I. \end{cases}$$

From $\partial f(\bar{x}) = \{-\eta\}$ and $0 \in \partial f(\bar{x}) + \sum_{i\in I} \partial^\circ g_i(\bar{x})$, $\eta \in \sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x})$. Since $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$, we have

$$\sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x}) = \sum_{i\in I(\bar{x})} \lambda_i \partial^\circ g_i(\bar{x}) \subseteq \operatorname{coneco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x}).$$

Thus, $\eta \in \operatorname{coneco}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. This completes the proof. □

Remark 4.1. (1) We remark that Theorem 4.2 holds even if the index set I is infinite. In this case, (ii) of (O) is as follows: there exist a finite subset $J \subseteq I(\bar{x})$ and $\lambda \in \mathbb{R}^J_+$ such that $0 \in \partial f(\bar{x}) + \sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x})$ and for each $i \in I, \lambda_i g_i(\bar{x}) = 0$.

(2) When all g_i are convex, then condition (A),

$$N_S(\bar{x}) = \operatorname{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}),$$

is called basic constraint qualification (BCQ).

(3) When all g_i are continuously differentiable at $\bar{x}$ and S is not necessarily convex, then condition (A),

$$N_S(\bar{x}) = \text{coneco} \bigcup_{i \in I(\bar{x})} \{\nabla g_i(\bar{x})\},$$

which is equivalent to

$$\text{clco}T_S(\bar{x}) = \{x \in \mathbb{R}^n \mid \langle \nabla g_i(\bar{x}), x \rangle \leq 0, \forall i \in I(\bar{x})\},$$

is called Guignard's constraint qualification, and condition (B),

$$T_S(\bar{x}) = \{x \in \mathbb{R}^n \mid \langle \nabla g_i(\bar{x}), x \rangle \leq 0, \forall i \in I(\bar{x})\},$$

is called Abadie's constraint qualification, see [21]. In this case, both Guignard's and Abadie's constraint qualifications are necessary and sufficient constraint qualifications for optimality condition of (P).

Next we show a result that condition (C) is a sufficient constraint qualification for the optimality conditions in convex optimization problem (P). When all g_i are continuously differentiable at $\bar{x}$, condition (C), that is,

$$\text{there exists } y_0 \in \mathbb{R}^n \text{ such that } \langle \nabla g_i(\bar{x}), y_0 \rangle < 0 \text{ for each } i \in I(\bar{x}),$$

is called Cottle's constraint qualification, see [21]. To show the result, we give the following lemma:

Lemma 4.2. Let Λ be an index set, and let $A_\lambda \subseteq \mathbb{R}^n$, $\lambda \in \Lambda$, be non-empty convex sets. If $\bigcap_{\lambda \in \Lambda} \text{int}A_\lambda \neq \emptyset$, then $\text{cl} \bigcap_{\lambda \in \Lambda} \text{int}A_\lambda = \bigcap_{\lambda \in \Lambda} \text{cl}A_\lambda$.

Proof. The equality $\text{cl} \bigcap_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} \text{cl}A_\lambda$ is shown straightforwardly and omitted. Since $\text{clint}A_\lambda = \text{cl}A_\lambda$ for each $\lambda \in \Lambda$, the equality of this lemma holds. □

Theorem 4.3. Let $\bar{x} \in S$. Then (C) implies (B).

Proof. Assume (C). There exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi_i, y_0 \rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$. That is, for each $i \in I(\bar{x})$,

$$g_i^\circ(\bar{x}, y_0) = \max_{\xi \in \partial^\circ g_i(\bar{x})} \langle \xi_i, y_0 \rangle < 0.$$

Since $g_i^\circ(\bar{x}, \cdot)$ is a real-valued convex function on $\mathbb{R}^n$ and $g_i^\circ(\bar{x}, y_0) < 0$, by using Theorem 1.6,

$$\text{int}\{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) \leq 0\} = \{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) < 0\}.$$

Also, it is clear that $\partial^\circ g_i(\bar{x})^- = \{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) \leq 0\}$. Thus,

$$\text{int}\partial^\circ g_i(\bar{x})^- = \{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) < 0\} \ni y_0. \tag{4.1}$$

Consequently, we have $\bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-) \neq \emptyset$. By using Lemma 4.2, we have

$$\operatorname{cl} \bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-) = \bigcap_{i\in I(\bar{x})} \operatorname{cl}(\partial^\circ g_i(\bar{x})^-) = \bigcap_{i\in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-). \tag{4.2}$$

Next, we show

$$\bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-) \subseteq T_S(\bar{x}). \tag{4.3}$$

Let $y \in \bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-)$. For each $i \in I(\bar{x})$, from (4.1) and the regularity of g_i at $\bar{x}$, we have $g_i'(\bar{x}, y) < 0$. Then, there exists $t_i > 0$ such that $g_i(\bar{x} + ty) < 0$ for each $t \in (0, t_i]$. Moreover, for each $i \in I \setminus I(\bar{x})$, from the continuity of g_i and $g_i(\bar{x}) < 0$, there exists $t_i > 0$ such that $g_i(\bar{x} + ty) < 0$ for each $t \in (0, t_i]$. Put $t_0 = \min\{t_i \mid i \in I\}$, for each $t \in (0, t_0)$

$$\text{for each } i \in I, g_i(\bar{x} + ty) < 0. \tag{4.4}$$

Then $\bar{x}+ty \in S$ for each $t \in (0, t_0]$. For each $k \in \mathbb{N}$, put $x_k = \bar{x} + \frac{t_0}{k} y$ and $\alpha_k = \frac{k}{t_0}$. Then $\{\alpha_k(x_k - \bar{x})\} \subseteq \operatorname{cone}(S - \bar{x})$ and $\alpha_k(x_k - \bar{x}) \to y$, that is, $y \in T_S(\bar{x})$. Thus (4.3) holds. By using (4.2) and (4.3), we have

$$\bigcap_{i\in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-) \subseteq T_S(\bar{x}).$$

The converse inclusion $T_S(\bar{x}) \subseteq \bigcap_{i\in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-)$ holds from Lemma 4.1.

Finally, we prove that $\operatorname{coneco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed, that is,

$$\operatorname{clconeco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x}) \subseteq \operatorname{coneco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x}).$$

We may assume that $I(\bar{x}) \neq \emptyset$. Let $y \in \operatorname{clconeco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. There exists $\{y_k\} \subseteq \operatorname{coneco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ such that $y_k \to y$. For each $k \in N$, there exist $\lambda^k = (\lambda_i^k)_{i\in I(\bar{x})} \in \mathbb{R}_+^{I(\bar{x})}$ and $x^k = (x_i^k)_{i\in I(\bar{x})} \in \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ such that $y_k = \sum_{i\in I(\bar{x})} \lambda_i^k x_i^k$. From (C), there exists $y_0 \in \mathbb{R}^n$ such that $g_i^\circ(\bar{x}, y_0) < 0$. Put $r = \max_{i\in I(\bar{x})} g_i^\circ(\bar{x}, y_0)$. For each $i \in I(\bar{x})$, $\langle x_i^k, y_0 \rangle \leq r < 0$. Thus, $\langle y_k, y_0 \rangle \leq r \sum_{i\in I(\bar{x})} \lambda_i^k$. Since $\langle y_k, y_0 \rangle \to \langle y, y_0 \rangle$,

$$\langle y, y_0 \rangle - 1 < \langle y_k, y_0 \rangle \leq r \sum_{i\in I(\bar{x})} \lambda_i^k$$

hold for sufficiently large k, that is,

$$\|\lambda^k\| \leq \sum_{i\in I(\bar{x})} \lambda_i^k \leq \frac{\langle y, y_0 \rangle - 1}{r} (=: K).$$

Therefore, $\{(\lambda^k, x^k)\} \subseteq \mathrm{cl}B(0,K) \times \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. From the compactness of $\mathrm{cl}B(0,K) \times \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$, there exist $(\lambda, x) = (\lambda_i, x_i)_{i\in I(\bar{x})} \in \mathrm{cl}B(0,K) \times \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ and a subsequence $\{(\lambda^{k_j}, x^{k_j})\}$ of $\{(\lambda^k, x^k)\}$ such that $(\lambda^{k_j}, x^{k_j}) \to (\lambda, x)$. Moreover, we have $\lambda_i \geq 0$, $x_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, and $y = \sum_{i\in I(\bar{x})} \lambda_i x_i$. Thus, $y \in \mathrm{coneco} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. This completes the proof. □

Remark 4.2. (1) The converse of Theorem 4.3 is not true in general, see Example 4.1.

(2) From (4.4), (C) implies the Slater condition. However, the converse is not true in general, see Example 4.2.

(3) In Example 4.2, the Slater condition does not imply (A). Therefore, the Slater condition is not a constraint qualification for the optimality conditions in convex optimization problem (P).

Example 4.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$g(x) = |x|.$$

Then $S = \{0\}$, $T_S(0) = \{0\}$ and $\partial^\circ g(0) = [-1, 1]$. So that, $\partial^\circ g(0)^- = \{0\}$ and $\partial^\circ g(0)$ is closed. Thus (B) holds. On the other hand, for each $y \in \mathbb{R}$, $\frac{y}{|y|+1} \in \partial^\circ g(0)$ and $\frac{y}{|y|+1} y \geq 0$, and then (C) does not hold.

Example 4.2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$g(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if} \quad x_1 \geq 0, x_2 \geq 0, \\ \|(x_1, x_2)\| + x_2 & \text{if} \quad x_1 \geq 0, x_2 < 0, \\ \|(x_1, x_2)\| + x_1 & \text{if} \quad x_1 < 0, x_2 \geq 0, \\ -x_1 x_2 & \text{if} \quad x_1 < 0, x_2 < 0. \end{cases}$$

Then $S = -\mathbb{R}^2_+$, S is convex, g is regular at $(0,0)$ and the Slater condition holds. On the other hand, $N_S(0,0) = \mathbb{R}^2_+$ and $\mathrm{cone}\partial^\circ g(0,0) = \{(0,0)\} \cup \mathrm{int}\mathbb{R}^2_+$. Hence, (A) does not hold. Thus (C) does not hold.

Next we consider the relationship of (C), (D), (E) and (F). From Theorem 4.1, condition (D), given by Dutta and Lalitha, is a sufficient constraint qualification for the optimality conditions in convex optimization problem (P). Conditions (E) and (F) are motivated by (C) and (D), respectively.

We show the relationship of (C), (D), (E) and (F) as follows:

Theorem 4.4. Let $\bar{x} \in S$. Then (C), (D), (E) and (F) are equivalent.

Proof. First, we prove (C) implies (D). Assume (C). There exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi_i, y_0 \rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$. It is clear that $0 \notin \partial^\circ g_i(\bar{x})$ for each $i \in I(\bar{x})$. In addition, Slater condition holds from (2) of Remark 4.2. Thus (D) holds.

Next, we prove (D) implies (F). Assume (D). Then $0 \notin \partial^\circ g_i(\bar{x})$ for each $i \in I(\bar{x})$, and it is easy to show that int$S$ is non-empty from Slater condition and the continuity of all g_i. Thus (F) holds.

Next, we prove (F) implies (E). Assume that (E) dos not hold. Then, there exist $\lambda_i \in \mathbb{R}_+$ and $\xi_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, such that

$$\begin{cases} \sum_{i\in I(\bar{x})} \lambda_i = 1, \\ \sum_{i\in I(\bar{x})} \lambda_i \xi_i = 0. \end{cases}$$

From (F), we have $\xi_i \neq 0$ for each $i \in I(\bar{x})$. Also from (F), there exists $x_0 \in \mathbb{R}^n$ and $r > 0$ such that $B(x_0, r) \subseteq S$. For each $i \in I(\bar{x})$, since $x_0 + \frac{r}{2\|\xi_i\|}\xi_i \in B(x_0, r) \subseteq S$, then for each $i \in I(\bar{x})$, $\partial^\circ g_i(\bar{x}) \subseteq N_S(\bar{x})$ from Lemma 4.1, that is, $\xi_i \in N_S(\bar{x})$. So for each $i \in I(\bar{x})$,

$$\langle \xi_i, x_0 - \bar{x}\rangle + \frac{r}{2}\|\xi_i\| = \left\langle \xi_i, x_0 + \frac{r}{2\|\xi_i\|}\xi_i - \bar{x} \right\rangle \leq 0.$$

Therefore,

$$\frac{r}{2}\sum_{i\in I(\bar{x})} \lambda_i\|\xi_i\| = \left\langle \sum_{i\in I(\bar{x})} \lambda_i\xi_i, x_0 - \bar{x} \right\rangle + \frac{r}{2}\sum_{i\in I(\bar{x})} \lambda_i\|\xi_i\| \leq 0.$$

From $\sum_{i\in I(\bar{x})} \lambda_i = 1$ and $\xi_i \neq 0$ for each $i \in I(\bar{x})$,

$$0 < \frac{r}{2}\sum_{i\in I(\bar{x})} \lambda_i\|\xi_i\|.$$

This is a contradiction.

Finally, we prove (E) implies (C). Assume (E). Since $\mathrm{co}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is a non-empty closed convex set and $0 \notin \mathrm{co}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ from (E), there exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi, y_0\rangle < 0$ for each $\xi \in \mathrm{co}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ from Theorem 1.1. Thus, $\langle \xi_i, y_0\rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$. Therefore (C) holds. This completes the proof. □

Finally, we consider the relationship of (E) and (G). When all g_i are continuously differentiable at $\bar{x}$, condition (G), that is

$$\{\nabla g_i(\bar{x})\}_{i\in I(\bar{x})} \text{ is linearly independent,}$$

is called the linearly independent constraint qualification, see [6, 21].

Theorem 4.5. Let $\bar{x} \in S$. Then (G) implies (E).

Proof. Assume that (E) does not hold. Then, there exist $\lambda_i \in \mathbb{R}_+$ and $x_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, such that

$$\begin{cases} \sum_{i\in I(\bar{x})} \lambda_i = 1, \\ \sum_{i\in I(\bar{x})} \lambda_i x_i = 0. \end{cases}$$

Thus (G) does not hold. □

The converse of Theorem 4.5 is not true in general. See the following example:

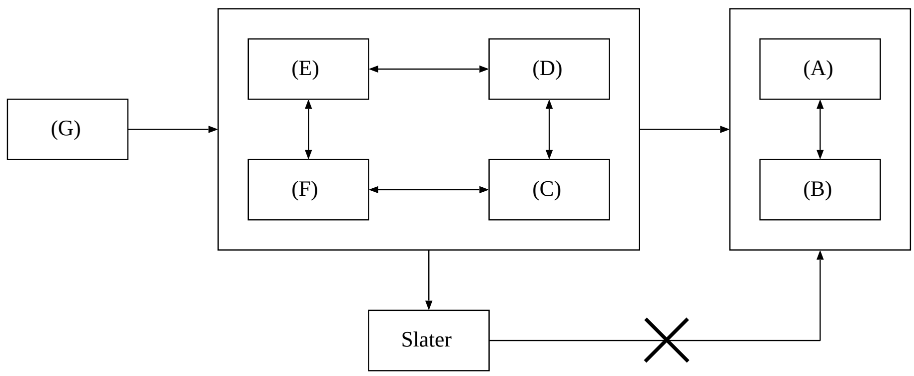
Example 4.3. Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be functions as follows:

$$g_1(x) = (x-1)(x+1), g_2(x) = \frac{1}{2}(x-1)(x+1).$$

Then $S = [-1, 1]$, $\mathrm{int}S \neq \emptyset$, $I(1) = \{1, 2\}$, $\partial^\circ g_1(1) = \{2\}$ and $\partial^\circ g_2(1) = \{1\}$. Thus (F) holds. On the other hand, it is clear that $\{2, 1\}$ is not linearly independent. Hence (G) does not hold.

4.3 Conclusion

In this chapter, we have presented constraint qualifications for KKT optimality condition in a convex optimization problem under locally Lipschitz constraints which was discuss by Dutta and Lalitha in [5], and compared our results to previous ones. First, we introduced two necessary and sufficient constraint qualifications for KKT optimality condition. Moreover we proposed constraint qualifications, and discussed the relationship of these constraint qualifications. On the other hand, it was shown that the Slater condition was not a constraint qualification in this optimization. The following figure shows the relationship of the constraint qualifications, which were introduced in this paper, for optimality conditions:



The figure is reprinted from [26].

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Bibliography

[1] R. I. Boţ, Conjugate Duality in Convex Optimization, Springer-Verlag, Berlin (2010).

[2] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, (1983).

[3] N. Dinh, M. A. Goberna, M. A. Lopez, *From linear to convex systems: Consistency, Farkas' lemma and applications*, Journal of Convex Analysis, 13 (2006) pp. 113–133.

[4] N. Dinh, M. A. Goberna, M. A. Lopez, T. Q. Son, *New Farkas-type constraint qualification in convex infinite programming*, ESAIM: COCV, 13 (2007) pp. 580–597.

[5] J. Dutta and C. S. Lalitha, *Optimality conditions in convex optimization revisited*, Optim. Lett. 7 (2013), pp. 221–229.

[6] R. G. Eustaquio, E. W. Karas, A. A. Ribeiro, *Constraint qualifications for nonlinear programming*, Federal University of Parana, Brazil, (2008).

[7] J. Farkas, *Theorie der einfachen Ungleichungen*, J. Reine Angew. Math. 124 (1902), pp. 1–27.

[8] M. A. Goberna, V. Jeyakumar, M. A. Lopez, *Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities*, Nonlinear Analysis 68 (2008), pp. 1184–1194.

[9] J. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms I*, Grundlehren der Mathematschen Wissenschaften 305 (1993), Springer-verlag, New York.

[10] V. Jeyakumar, *Constraint qualifications characterizing Lagrangian Duality in convex optimization*, J Optim Theory Appl 136 (2008) pp. 31–41.

[11] V. Jeyakumar, N. Dinh, G. M. Lee, *A new closed cone constraint qualification for convex optimization*, Applied Mathematics Report AMR 04/8, University of New South Wales (2004).

[12] V. Jeyakumar and G. Y. Li, *Farkas' lemma for separable sublinear inequalities without qualifications*, Optim. Lett. 3 (2009) pp. 537–545.

[13] V. Jeyakumar and G. Y. Li, *New strong duality results for convex programs with separable constraints*, European J. Oper. Res. 207 (2010) pp. 1203–1209.

[14] H. Komiya, *Elementary proof for Sions minimax theorem*, Kodai Math. J. 11 (1988) pp. 5–7.

[15] J. B. Lasserre, *On representations of the feasible set in convex optimization*, Optim. Lett. 4 (2010) pp. 1–5.

[16] C. Li, K. F. Ng, T. K. Pong, *Constraint qualifications for convex inequality system with applications in constrained optimization*, SIAM J. Optim. 19 (2008) pp. 163–187.

[17] R. T. Rockafellar, *Convex Analysis*, Priceton University Press, Princeton, (1970).

[18] M. Sion, *On general minimax theorems*, Pac. J. Math. 8 (1958) pp. 171–176.

[19] S. Suzuki and D. Kuroiwa, *Set containment characterization for quasiconvex programming*, J. Glob. Optim. 45 (2009) pp. 551–563.

[20] P. Tseng, *Some convex programs without a duality gap*, Math. Program. Ser. B, 116 (2009) pp. 553–578.

[21] Z. Wang, S.-C. Fang, W. X. Xing, *On constraint qualification, Design and inter-relations*, J. Ind. Manag. Optim. 9 (2013) pp. 983–1001.

[22] S. Yamamoto, S. Suzuki, D. Kuroiwa, *An observation of alternative theorem for separable convex functions*, The Seventh International Conference on Nonlinear Analysis and Convex Analysis proceedings (2011) pp. 297–304

[23] S. Yamamoto and D. Kuroiwa, *A method for checking the basic constraint qualification and its application to convex quadratic functions*, Proceedings of the Third Asian conference on nonlinear analysis and optimization (2012) pp. 353–359

[24] S. Yamamoto and D. Kuroiwa, *Observations of constraint qualifications for a class of convex composite functions*, Proceedings of the 8th International conference on nonlinear analysis and convex analysis, accepted.

[25] S. Yamamoto, S. Suzuki and D. Kuroiwa, *How to check the basic constraint qualification via the characteristic cone of the conical epigraph hull property*, submitted.

[26] S. Yamamoto and D. Kuroiwa, *Constraint qualifications for KKT optimality condition in convex optimization with locally Lipschitz inequality constraints*, Linear and Nonlinear Analysis, accepted.

[27] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge (2002).