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Global asymptotic stability for oscillators with superlinear damping

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Abstract A necessary and sufficient condition is established for the equilibrium of the damped superlinear oscillator

$$x'' + a(t)\phi_q(x') + \omega^2 x = 0$$

to be globally asymptotically stable. The obtained criterion is judged by whether the integral of a particular solution of the first-order nonlinear differential equation

$$u' + \omega^{q-2}a(t)\phi_q(u) + 1 = 0$$

is divergent or convergent. Since this nonlinear differential equation cannot be solved in general, it can be said that the presented result is expressed by an implicit condition. Explicit sufficient conditions and explicit necessary conditions are also given for the equilibrium of the damped superlinear oscillator to be globally attractive. Moreover, it is proved that a certain growth condition of $a(t)$ guarantees the global asymptotic stability for the equilibrium of the damped superlinear oscillator.

Keywords Damped oscillator · Superlinear differential equations · Global asymptotic stability · Newtonian damping · Growth condition

Mathematics Subject Classification (2000) 34D05 · 34D23 · 34D45 · 37B25

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1 Introduction

The damped linear oscillator

$$x'' + a(t)x' + \omega^2 x = 0 \tag{L}$$

is one of the most famous models which describe a number of physical phenomena. Here, the prime denotes d/dt, the spring constant ω is positive, the damping coefficient $a(t)$ is continuous and nonnegative for $t \geq 0$. This model has been researched from many angles in a wide range of fields which covers pure science, applied science, and technology. Needless to say, in this model, the damping force is assumed to be proportional to the velocity in this model. However, in specific types of phenomena, this assumption is not necessarily suitable. For example, in fluid mechanics, it is well known that the air resistance is approximately proportional to the square of the velocity. A model of viscosity in which the damping force is proportional to the square of the velocity is called *Newtonian damping*.

When a small fishing vessel is on still water, the extinction of free rolling motion is caused by wave and vortex that occur because of the rolling of the vessel. The damping forces are called wave resistance and eddy-making resistance, respectively. Besides, it is thought that resistance of the friction works in the rolling motion of the vessel. The wave resistance is said to be proportional to the angular velocity. On the other hand, the eddy-making resistance and the frictional resistance are said to be proportional to the square of the angular velocity. Hence, the damping term is regarded as a function of the angular velocity. In the latter half of the 19th century, the expressions of such a function were first given by William Froude who was an English engineer and by Louis-Émile Bertin who was a French naval engineer. Afterwards, by experiments, a lot of engineers examined causes that influence the extinction of free rolling motion (for example, see [4, 5, 7, 10, 12, 19, 24, 30]). Because analysis is difficult, in most cases, damping coefficients of the function is assumed to be constants.

Since Eq. (L) is very simple, it may be difficult to apply it to a specific model such as Newtonian damping. We intend to establish an attenuation criterion which is applicable even to physical models with Newtonian damping. For this purpose, we consider the second-order differential equation

$$x'' + a(t)\phi_q(x') + \omega^2 x = 0, \tag{E}$$

and present a necessary and sufficient condition for the equilibrium of (E) to be globally asymptotically stable. In Eq. (E), the damping coefficient $a(t)$ is continuous and nonnegative for $t \geq 0$ and the function $\phi_q(z)$ is defined by

$$\phi_q(z) = |z|^{q-2}z, \qquad z \in \mathbb{R}$$

with $q \geq 2$. It is clear that the only equilibrium of (E) is the origin $(x, x') = (0, 0)$. Eq. (E) naturally contains Eq. (L) as the special case in which $q = 2$. Since $q \geq 2$, we call Eq. (E) a *damped superlinear oscillator*.

Let $\mathbf{x}(t) = (x(t), x'(t))$ and $\mathbf{x}_0 \in \mathbb{R}^2$, and let $\|\cdot\|$ be any suitable norm. We denote the solution of (E) through $(t_0, \mathbf{x}_0)$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. The global existence and uniqueness of solutions of (E) are guaranteed for the initial value problem.

The equilibrium is said to be *stable* if, for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geq t_0$. The equilibrium is said to be *attractive* if, for any $t_0 \geq 0$, there exists a $\delta_0(t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta_0$ implies $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \to 0$ as $t \to \infty$. The equilibrium is said to be *globally attractive* if, for any $t_0 \geq 0$, any $\eta > 0$, and any $\mathbf{x}_0 \in \mathbb{R}^2$, there is a $T(t_0, \eta, \mathbf{x}_0) > 0$ such that $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$ for all

$t \geq t_0 + T(t_0, \eta, \mathbf{x}_0)$. The equilibrium is *asymptotically stable* if it is stable and attractive. The equilibrium is *globally asymptotically stable* if it is stable and globally attractive. With respect to the various definitions of stability, the reader may refer to the books [2, 6, 8, 9, 13, 14, 23, 27, 31] for example.

Stability and attractivity are local properties of the equilibrium. In general, the equilibrium is not always globally attractive (resp., globally asymptotically stable) even if it is locally attractive (resp., locally asymptotically stable). However, it is correct in the linear differential equations such as Eq. (L). The research of the (global) asymptotic stability is one of the qualitative theoretical main themes of the differential equation. A large number of papers has been devoted to find sufficient conditions and/or necessary conditions for the asymptotic stability of (L) and more general equations (for example, see [1, 3, 11, 15–18, 20, 22, 25, 26, 28, 29]). The historical development of this research is concisely summarized in Sugie [29, Section 1]. Among them, we should mention specially the following result given by Hatvani and Totik [18, Theorem 3.1].

Theorem A *Suppose that there exists a* γ_0 *with* $0 < \gamma_0 < \pi/\omega$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+\gamma_0} a(s)ds > 0. \tag{1.1}$$

Then the equilibrium of (L) *is asymptotically stable if and only if*

$$\int_0^\infty \frac{\int_0^t e^{A(s)}ds}{e^{A(t)}}\, dt = \infty, \tag{1.2}$$

where

$$A(t) = \int_0^t a(s)ds.$$

The criterion (1.2) is the so-called growth condition on $a(t)$. This condition was first presented by Smith [28, Theorems 1 and 2] under the assumption that the lower bound of $a(t)$ was positive. Clearly, this assumption is stronger than condition (1.1). Even if intervals where $a(t)$ becomes zero are infinitely many, condition (1.1) may be satisfied if the lengths of intervals are less than π/ω. Hence, Theorem A is a natural generalization of Smith's result.

Let us look at condition (1.2) from another viewpoint. We consider the scalar differential equation

$$u' + a(t)u + 1 = 0. \tag{1.3}$$

Then, the solution $u(t)$ of (1.3) satisfying the initial condition $u(0) = 0$ is given by

$$u(t) = -\frac{\int_0^t e^{A(s)}ds}{e^{A(t)}}.$$

Hence, condition (1.2) coincides with

$$\int_0^\infty u(t)dt = -\infty.$$

In other word, whether the integral of $u(t)$ is divergent or convergent determines the asymptotic stability for Eq. (L). Since Eq. (1.3) bears a close relation with the damped linear oscillator (L), we call it a *characteristic equation*. We will extend Theorem A from the viewpoint of characteristic equations.

Our main theorem is as follows:

Theorem 1.1 *Under the assumption* (1.1), *the equilibrium of* (E) *is globally asymptotically stable if and only if*

$$\int_0^\infty u(t)dt = -\infty,$$

where $u(t)$ *is the solution of*

$$u' + \omega^{q-2}a(t)\phi_q(u) + 1 = 0$$

satisfying $u(0) = 0$.

2 Characteristic equation

Consider the scalar characteristic equation

$$u' + \omega^{q-2}a(t)\phi_q(u) + 1 = 0, \tag{2.1}$$

where $\omega > 0$ and $q \geq 2$, and $a(t)$ is continuous and nonnegative for $t \geq 0$. As well as Eq. (E), the global existence and uniqueness of solutions of (2.1) are guaranteed for the initial value problem. Let T be a nonnegative number. We denote the solution of (2.1) through $(T,0)$ by $u(t;T)$. Then,

$$u(t;T) < 0 \quad \text{for } t > T.$$

In fact, since $u(T;T) = 0$ and $u'(T;T) = -1$, we see that $u(t;T) < 0$ in a right-hand neighborhood of T. Suppose that there exists a $t_1 > T$ such that $u(t_1;T) = 0$ and

$$u(t;T) < 0 \quad \text{for } T < t < t_1.$$

Then, $u'(t_1;T) = -1$. Hence, there exists a small $\delta > 0$ such that $u'(t;T) < 0$ for $t \in [t_1 - \delta, t_1]$. From this inequality it follows that $u(t_1 - \delta;T) > u(t_1;T) = 0$, which contradicts the definition of t_1.

In the special case in which $q = 2$, Eq. (2.1) coincides with Eq. (1.3). The solution $u(t;T)$ of (1.3) satisfying the initial condition $u(T;T) = 0$ is given by

$$u(t;T) = -\int_T^t e^{-\int_s^t a(u)du}ds$$

for $t \geq T \geq 0$. Let us compare solutions $u(t;0)$ and $u(t;T)$ of (1.3). For the sake of convenience, let

$$\psi(t,s) = e^{-\int_s^t a(u)du} > 0.$$

Then, $\psi(t,t) \equiv 1$ and $\psi(t,0) = e^{-\int_0^t a(s)ds}$ is decreasing for $t \geq 0$ and tends to zero as $t \to \infty$. It is clear that

$$\frac{\partial}{\partial s}\psi(t,s) = a(s)\psi(t,s) \geq 0$$

for $0 \leq s \leq t$, and therefore,

$$\int_0^T \psi(t,s)ds \leq \int_T^{2T} \psi(t,s)ds.$$

Hence, we obtain

$$\begin{aligned}\int_T^\infty\int_T^t \psi(t,s)ds\,dt &\le \int_0^\infty\int_0^t \psi(t,s)ds\,dt\\ &= \int_0^{2T}\int_0^t \psi(t,s)ds\,dt + \int_{2T}^\infty\int_0^T \psi(t,s)ds\,dt\\ &\quad + \int_{2T}^\infty\int_T^{2T} \psi(t,s)ds\,dt + \int_{2T}^\infty\int_{2T}^t \psi(t,s)ds\,dt\\ &< \int_0^{2T}\int_0^t \psi(t,s)ds\,dt + 2\int_{2T}^\infty\int_T^{2T} \psi(t,s)ds\,dt + 2\int_{2T}^\infty\int_{2T}^t \psi(t,s)ds\,dt\\ &= \int_0^{2T}\int_0^t \psi(t,s)ds\,dt - 2\int_T^{2T}\int_T^t \psi(t,s)ds\,dt + 2\int_T^\infty\int_T^t \psi(t,s)ds\,dt.\end{aligned}$$

Since

$$\int_0^{2T}\int_0^t \psi(t,s)ds\,dt - 2\int_T^{2T}\int_T^t \psi(t,s)ds\,dt$$

is bounded for each $T \ge 0$, we conclude that

$$\int_0^\infty u(t;0)dt = -\infty$$

if and only if

$$\int_T^\infty u(t;T)dt = -\infty.$$

If $q > 2$, then we cannot know a concrete expression of $u(t;T)$ any longer. In general, however, the integral from 0 to ∞ of $u(t;0)$ has the following equivalence relation, which plays a key role in this paper.

Lemma 2.1 *For any $T \ge 0$,*

$$\int_T^\infty u(t;T)dt = -\infty$$

if and only if

$$\int_0^\infty u(t;0)dt = -\infty.$$

Proof Let us fix T arbitrarily and compare two solutions $u(t;0)$ and $u(t;T)$ of (2.1). Since $u(T;0) < 0 = u(T;T)$, it follows that $u(t;0) < u(t;T) < 0$ in a right-hand neighborhood of T. Hence,

$$\begin{aligned}u'(t;0) &= -1 - \omega^{q-2}a(t)\phi_q(u(t;0))\\ &> -1 - \omega^{q-2}a(t)\phi_q(u(t;T)) = u'(t;T)\end{aligned}$$

as long as $u(t;0) < u(t;T) < 0$.

If $u(t^*;0) = u(t^*;T)$ for some $t^* > T$, then

$$\begin{aligned}u'(t^*;0) &= -1 - \omega^{q-2}a(t^*)\phi_q(u(t^*;0))\\ &= -1 - \omega^{q-2}a(t^*)\phi_q(u(t^*;T)) = u'(t^*;T).\end{aligned}$$

Hence, from the uniqueness of solutions of (2.1) for the initial value problem, it turns out that

$$u(t;0) = u(t;T) \quad \text{for } t \ge t^*,$$

and therefore,

$$\begin{aligned}\int_0^\infty u(t;0)dt &= \int_0^{t^*} u(t;0)dt + \int_{t^*}^\infty u(t;0)dt \\ &= \int_0^{t^*} u(t;0)dt + \int_{t^*}^\infty u(t;T)dt \\ &= \int_0^{t^*} u(t;0)dt - \int_T^{t^*} u(t;T)dt + \int_T^\infty u(t;T)dt,\end{aligned}$$

as required.

If such a t^* does not exist, then

$$u(t;0) < u(t;T) < 0 \quad \text{for } t > T.$$

Let $t_1 > T$ be given. We choose a ρ so that

$$0 < \rho < \frac{u(t_1;T)}{u(t_1;0)}.$$

Then, $\rho < 1$. Define

$$\eta(t) = \rho\, u(t;0)$$

for $t \geq 0$. Then,

$$\begin{aligned}\eta'(t) = \rho\, u'(t;0) &= -\rho - \rho\, \omega^{q-2} a(t)\phi_q(u(t;0)) \\ &= -\rho - \frac{\rho}{\phi_q(\rho)}\omega^{q-2} a(t)\phi_q(\eta(t))\end{aligned}$$

for $t \geq 0$. Since $0 < \rho < 1$ and $q \geq 2$, we see that

$$\frac{\rho}{\phi_q(\rho)} = \left(\frac{1}{\rho}\right)^{q-2} \geq 1.$$

Noticing that $\eta(t) \leq 0$ for $t \geq 0$, we obtain

$$\eta'(t) > -1 - \omega^{q-2} a(t)\phi_q(\eta(t))$$

for $t \geq 0$. From the definition of ρ it follows that

$$0 > \eta(t_1) = \rho\, u(t_1;0) > u(t_1;T).$$

Suppose that there exists a $t_2 > t_1$ such that $\eta(t_2) = u(t_2;T)$ and $\eta(t) > u(t;T)$ for $t_1 \leq t < t_2$. Then,

$$\begin{aligned}\eta'(t_2) &> -1 - \omega^{q-2} a(t_2)\phi_q(\eta(t_2)) \\ &= -1 - \omega^{q-2} a(t_2)\phi_q(u(t_2;T)) = u'(t_2;T).\end{aligned}$$

Hence, $\eta'(t) > u'(t;T)$ in a left-hand neighborhood of t_2; namely, there exists a small $\delta > 0$ such that

$$\eta'(t) > u'(t;T) \quad \text{for } t_2 - \delta \leq t \leq t_2.$$

Integrating both sides of this inequality from $t_2 - \delta$ to t_2 and using that $\eta(t_2) = u(t_2;T)$, we obtain

$$\eta(t_2 - \delta) < u(t_2 - \delta;T),$$

which is a contradiction. Thus, we see that

$$0 > \eta(t) > u(t;T) \quad \text{for } t \geq t_1.$$

From this estimation, we obtain

$$\begin{aligned}
\int_T^\infty u(t;T)dt &= \int_T^{t_1} u(t;T)dt + \int_{t_1}^\infty u(t;T)dt \\
&\leq \int_T^{t_1} u(t;T)dt + \int_{t_1}^\infty \eta(t)dt \\
&= \int_T^{t_1} u(t;T)dt + \rho \int_{t_1}^\infty u(t;0)dt \\
&= \int_T^{t_1} u(t;T)dt - \rho \int_0^{t_1} u(t;0)dt + \rho \int_0^\infty u(t;0)dt.
\end{aligned}$$

On the other hand, since $u(t;0) < u(t;T) < 0$ for $t > T$, it follows that

$$\int_0^\infty u(t;0)dt < \int_T^\infty u(t;T)dt < 0.$$

We therefore conclude that convergence and divergence of the integrals of $u(t;0)$ and $u(t;T)$ happen simultaneously. □

We next consider a more general scalar differential equation

$$u' = f(t,u), \tag{2.2}$$

where $f(t,u)$ is continuous on $[0,\infty) \times \mathbb{R}$ and satisfies locally a Lipschitz condition with respect to u. For Eq. (2.2), the following results are well known (for example, see [31, p. 5]).

Lemma 2.2 *Let $u(t)$ be a solution of (2.2) on an interval $[a,b]$. Suppose that $\eta(t)$ is continuous on $[a,b]$ and satisfies the inequality*

$$\eta'(t) \geq f(t,\eta(t)) \quad \textit{for } a < t < b.$$

If $\eta(a) \geq u(a)$, then $\eta(t) \geq u(t)$ for $a \leq t \leq b$.

Lemma 2.3 *Let $u(t)$ be a solution of (2.2) on an interval $[a,b]$. Suppose that $\eta(t)$ is continuous on $[a,b]$ and satisfies the inequality*

$$\eta'(t) \leq f(t,\eta(t)) \quad \textit{for } a < t < b.$$

If $\eta(a) \leq u(a)$, then $\eta(t) \leq u(t)$ for $a \leq t \leq b$.

3 Necessary and sufficient conditions for global asymptotic stability

By putting $y = x'/\omega$ as a new variable, Eq. (E) becomes the planar system

$$\begin{aligned} x' &= \omega y, \\ y' &= -\,\omega x - \omega^{q-2} a(t)\phi_q(y). \end{aligned} \tag{3.1}$$

The whole x-y plane is divided into four quadrants. As is customary,

$$Q_1 = \big\{(x,y) : x > 0 \ \text{ and } \ y \geq 0\big\},$$

$$Q_2 = \big\{(x,y) : x \leq 0 \ \text{ and } \ y > 0\big\},$$

$$Q_3 = \big\{(x,y) : x < 0 \ \text{ and } \ y \leq 0\big\},$$

$$Q_4 = \big\{(x,y) : x \geq 0 \ \text{ and } \ y < 0\big\}.$$

We call the projection of a positive semitrajectory of (3.1) onto the x-y plane a *positive orbit* and we denote by $\Gamma^+(t_0, \mathbf{x}_0)$ the positive orbit of (3.1) starting from a point $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ at a time $t_0 \geq 0$.

The total energy

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$$

is the most suitable as a Lyapunov function for system (3.1). Differentiate $V(x,y)$ along any solution of (3.1) to obtain

$$\dot{V}_{(3.1)}(t,x,y) = xx' + yy' = -\,\omega^{q-2} a(t)|y|^q \leq 0$$

on $[0,\infty) \times \mathbb{R}^2$. Since $V(x,y)$ is positive definite and $\dot{V}_{(3.1)}(t,x,y)$ is nonpositive, by a basic Lyapunov's direct method, we obtain the following result.

Proposition 3.1 *The equilibrium of (E) is stable.*

Now, let us move on to the next subject; namely, the global attractivity. To begin with, we present necessary conditions for the equilibrium of (E) to be attractive.

Theorem 3.2 *If the equilibrium of (E) is attractive, then*

$$\int_0^\infty u(t)dt = -\infty, \tag{3.2}$$

where $u(t)$ is the solution of (2.1) satisfying $u(0) = 0$.

Proof. Let $L = \max\{1, \omega\}$. By way of contradiction, suppose that (3.2) does not holds. Then, we can choose a $T > 0$ so large that

$$\int_T^\infty u(t)dt > -\frac{1}{2\omega L}.$$

As shown in the proof of Lemma 2.1, we see that

$$u(t) = u(t;0) \leq u(t;T) < 0$$

for $t > T$. Hence, we have

$$\int_T^\infty u(t;T)dt > -\frac{1}{2\omega L}. \tag{3.3}$$

Consider the positive orbit $\Gamma^+(T,(1,0))$. From the vector field of (3.1), it turns out that $\Gamma^+(T,(1,0))$ goes into Q_4 afterwards. Let $(x(t),y(t))$ be the solution of (3.1) corresponding to $\Gamma^+(T,(1,0))$. Then, $x(T) = 1$ and $y(T) = 0$. If

$$x(t) > \frac{1}{2} \quad \text{for } t \geq T, \tag{3.4}$$

then the solution $(x(t),y(t))$ does not approach the origin; namely, the equilibrium of (E) is not attractive. This completes the proof. Hereafter, we will show that (3.4) holds. Suppose that there exists a $T_1 > T$ such that $x(T_1) = 1/2$ and $1/2 < x(t) \leq 1$ for $T \leq t < T_1$. Then,

$$y(t) < 0 \quad \text{for } T < t \leq T_1.$$

Let $\eta(t) = y(t)/L < 0$. Then, from the second equation of (3.1) it follows that

$$\begin{aligned}
\eta'(t) &= -\frac{\omega}{L}x(t) - \frac{\omega^{q-2}}{L}a(t)\phi_q(y(t)) \\
&\geq -1 - \omega^{q-2}\frac{\phi_q(L)}{L}a(t)\phi_q(\eta(t)) \geq -1 - \omega^{q-2}a(t)\phi_q(\eta(t))
\end{aligned}$$

for $T \leq t \leq T_1$. Let $f(t,u) = -1 - \omega^{q-2}a(t)\phi_q(u)$. Then, $\eta'(t) \geq f(t,\eta(t))$ for $T \leq t \leq T_1$. We compare $\eta(t)$ with the solution $u(t;T)$ of (2.1) satisfying $u(T;T) = 0$. Since $\eta(T) = y(T)/L = 0$, by Lemma 2.2, we see that

$$Lu(t;T) \leq L\eta(t) = y(t) \leq 0$$

for $T \leq t \leq T_1$. Hence, we have

$$x'(t) \geq \omega L u(t;T) \quad \text{for } T \leq t \leq T_1.$$

Integrating both sides of this inequality from T to T_1, we obtain

$$x(T_1) \geq x(T) + \omega L\int_T^{T_1} u(t;T)dt > 1 + \omega L\int_T^\infty u(t;T)dt > \frac{1}{2}$$

by (3.3). This contradicts the assumption that $x(T_1) = 1/2$.

We have thus proved the theorem. □

We next transform system (3.1) to polar coordinates by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

to find

$$\begin{aligned}
r' &= -\omega^{q-2}a(t)\phi_q(r)|\sin\theta|^q, \\
\theta' &= -\omega - \omega^{q-2}a(t)r^{q-2}\phi_q(\sin\theta)\cos\theta.
\end{aligned} \tag{3.5}$$

Consider the positive orbit $\Gamma^+(t_0,\mathbf{x}_0)$ starting from a point $\mathbf{x}_0 \in Q_1 \cup Q_3$ at a time $t_0 \geq 0$. Let $(r(t),\theta(t))$ be the solution of (3.5) corresponding to this positive orbit. Since

$$r^2\theta' = -\omega(x^2+y^2) - \omega^{q-2}a(t)xy|y|^{q-2} \leq 0$$

if $(x,y) \in Q_1 \cup Q_3$, we see that $\Gamma^+(t_0,\mathbf{x}_0)$ moves clockwise around the origin as long as it is in $Q_1 \cup Q_3$. Then, since

$$\sin\theta(t)\cos\theta(t) \geq 0 \quad \text{for } t \geq t_0,$$

it follows that

$$\begin{aligned}\theta'(t) &= -\omega - \omega^{q-2}a(t)(r(t))^{q-2}\phi_q(\sin\theta(t))\cos\theta(t) \\ &= -\omega - a(t)(\omega r(t)|\sin\theta(t)|)^{q-2}\sin\theta(t)\cos\theta(t) \leq -\omega\end{aligned}$$

for $t \geq t_0$. Hence, we obtain

$$\theta(t) \leq \theta(t_0) - \omega(t-t_0),$$

which tends to $-\infty$ as $t \to \infty$. This is a contradiction. Thus, we have the following result.

Lemma 3.3 *There is no positive orbit of* (3.1) *which continues staying in $Q_1 \cup Q_3$ ultimately.*

Judging from Lemma 3.3, system (3.1) has three types of positive orbits. Positive orbits of the first type keep rotating around the origin. Those of the second type remain in Q_4 (resp., Q_2) and approach the origin through Q_4 (resp., Q_2). Those of the third type stay in Q_4 (resp., Q_2) and tend to an interior point in Q_4 (resp., Q_2).

We are now ready to prove 'if'-part of the main theorem; namely, Theorem 1.1.

Theorem 3.4 *Assume* (1.1) *and* (3.2). *Then the equilibrium of* (E) *is globally attractive.*

Proof. Recall that Eq. (E) is equivalent to system (3.1). Let $x(t)$ be any solution of (E) with the initial time $t_0 \geq 0$ and let $(x(t),y(t))$ be the solution of (3.1) corresponding to $x(t)$. Define

$$v(t) = V(x(t),y(t))$$

for $t \geq t_0$. To prove the theorem, it is enough to show that

$$v(t) \to 0 \quad \text{as } t \to \infty.$$

Since $v'(t) = \dot{V}_{(3.1)}(t,x(t),y(t)) = -\omega^{q-2}a(t)|y(t)|^q \leq 0$ for $t \geq t_0$, $v(t)$ has the limiting value $v_0 \geq 0$. If $v_0 = 0$, then the proof is complete. We will show that the case of $v_0 > 0$ does not happen provided (1.1) and (3.2) hold.

Suppose that v_0 is positive. Then the closed curve given by $V(x,y) = v_0$ is the circumference of a circle whose center is at the origin and whose radius is $\sqrt{2v_0}$. Hence, this curve crosses with the x-axis only at two points $(\sqrt{2v_0},0)$ and $(-\sqrt{2v_0},0)$. Let $\mathbf{x}_0 = (x(t_0),y(t_0))$ and consider the positive orbit $\Gamma^+(t_0,\mathbf{x}_0)$.

As already mentioned, if $\Gamma^+(t_0,\mathbf{x}_0)$ does not rotate around the origin, then it remains in Q_2 or Q_4 ultimately; that is, there exist a point $\mathbf{x}_1 \in Q_4$ (resp., Q_2) and a time $T \geq t_0$ so that $\Gamma^+(t_0,\mathbf{x}_0)$ passes through $\mathbf{x}_1$ at T and remains in Q_4 (resp., Q_2) afterwards. We consider only the case in which $\Gamma^+(t_0,\mathbf{x}_0)$ remains in Q_4 ultimately, because the other case is carried out in the same way.

Since $(x(t),y(t)) \in Q_4$ for $t \geq T$, we see that $x'(t) = \omega y(t) < 0$ for $t \geq T$. Hence, there exists an $\alpha \geq 0$ such that $x(t) \to \alpha$ as $t \to \infty$, and therefore, it follows that

$$\frac{1}{2}y^2(t) \to v_0 - \frac{1}{2}\alpha^2 \quad \text{as } t \to \infty.$$

Of course, $v_0 \geq \alpha^2/2$. If $v_0 > \alpha^2/2$, then we can choose a $T_1 \geq T$ so large that

$$y^2(t) > v_0 - \frac{1}{2}\alpha^2 > 0 \quad \text{for } t \geq T_1.$$

Hence, we have

$$v'(t) = -\,\omega^{q-2}a(t)|y(t)|^q \leq -\,\omega^{q-2}\big(v_0 - \alpha^2/2\big)^{q/2}a(t)$$

for $t \geq T_1$. Integrating this inequality from T_1 to t, we obtain

$$v_0 - v(T_1) < v(t) - v(T_1) \leq -\,\omega^{q-2}\big(v_0 - \alpha^2/2\big)^{q/2}\int_{T_1}^t a(s)ds,$$

which tends to $-\infty$. This is a contradiction. Thus, we see that $\alpha = \sqrt{2v_0}$. We therefore conclude that $\Gamma^+(t_0,\mathbf{x}_0)$ approaches the point $(\sqrt{2v_0},0)$ which is an intersection of the closed curve $V(x,y) = v_0$ and the x-axis.

Let $\varepsilon_0 = \min\{1,\, \omega\sqrt{2v_0}\}$. Then, taking into account that $\phi_q(\varepsilon_0) \leq \varepsilon_0$, and

$$\sqrt{2v_0} < x(t) \leq x(T) \quad \text{and} \quad y(t) < 0$$

for $t \geq T$, we can estimate that

$$\begin{aligned}\left(\frac{y(t)}{\varepsilon_0}\right)' &= -\,\frac{\omega x(t)}{\varepsilon_0} - \frac{\omega^{q-2}a(t)\phi_q(y(t))}{\varepsilon_0}\\ &< -\,\frac{\omega\sqrt{2v_0}}{\varepsilon_0} - \frac{\omega^{q-2}a(t)\phi_q(y(t))}{\phi_q(\varepsilon_0)} \leq -1 - \omega^{q-2}a(t)\phi_q\!\left(\frac{y(t)}{\varepsilon_0}\right)\end{aligned}$$

for $t \geq T$. Let $\eta(t) = y(t)/\varepsilon_0$ for $t \geq t_0$ and let $f(t,u) = -1 - \omega^{q-2}a(t)\phi_q(u)$. Then, $\eta'(t) \leq f(t,\eta(t))$ for $t \geq T$. We compare $\eta(t)$ with the solution $u(t;T)$ of (2.1) satisfying $u(T;T) = 0$. Since $\eta(T) = y(T)/\varepsilon_0 < 0$, by Lemma 2.3, we see that

$$\frac{y(t)}{\varepsilon_0} = \eta(t) \leq u(t;T) \leq 0$$

for $t \geq T$. Hence, we have

$$x'(t) \leq \omega\,\varepsilon_0\,u(t;T) \quad \text{for } t \geq T.$$

Integrate both sides of this inequality from T to t to obtain

$$\sqrt{2v_0} - x(T) < x(t) - x(T) \leq \omega\,\varepsilon_0\int_T^t u(s;T)ds.$$

By (3.2) and Lemma 2.1, however,

$$\int_T^t u(s;T)ds \to -\infty \quad \text{as } t \to \infty.$$

This is a contradiction. Thus, $\Gamma^+(t_0,\mathbf{x}_0)$ have to keep rotating around the origin.

Let ε be so small that

$$0 < \varepsilon < \frac{\pi - \omega\gamma_0}{2}, \tag{3.6}$$

where γ_0 is the number given in (1.1). Consider the straight lines $y = (\tan\varepsilon)x$ and $y = (\tan(\pi - \varepsilon))x$. Since $\Gamma^+(t_0,\mathbf{x}_0)$ continues going around the origin, it naturally crosses the lines and the y-axis infinitely many times. Let $(r(t),\theta(t))$ be the solution of (3.5) corresponding to $\Gamma^+(t_0,\mathbf{x}_0)$. Then, we can find four divergent sequences $\{\tau_n\}$, $\{t_n\}$, $\{\sigma_n\}$ and $\{s_n\}$ with $t_0 \le \tau_n < t_n < \sigma_n < s_n$ such that $\theta(\tau_n) = 3\pi/2$, $\theta(t_n) = \pi-\varepsilon$, $\theta(\sigma_n) = \pi/2$ and $\theta(s_n) = \varepsilon$. Although $\Gamma^+(t_0,\mathbf{x}_0)$ moves clockwise around the origin when it passes through $(Q_1 \cup Q_3)$, the behavior of $\Gamma^+(t_0,\mathbf{x}_0)$ is not so simple when it is in $(Q_2 \cup Q_4)$. Since

$$\theta'(t) = -\omega - a(t)(\omega r(t)|\sin\theta(t)|)^{q-2}\sin\theta(t)\cos\theta(t),$$

$\Gamma^+(t_0,\mathbf{x}_0)$ does not always move clockwise in $(Q_2 \cup Q_4)$; namely, it might advance temporarily anti-clockwise. In such a case, we should select the supremum of all $t \in (\tau_n,\sigma_n)$ for which $\theta(t) \ge \pi-\varepsilon$ as the point t_n. Then, we have

$$\varepsilon < \theta(t) < \pi-\varepsilon \quad \text{for } t_n < t < s_n.$$

Recall that the closed curve $V(x,y) = v_0$ is the circumference of a circle with radius $\sqrt{2v_0}$, and $\Gamma^+(t_0,\mathbf{x}_0)$ does not enter in the circle. The curve intersects with the half-line $\theta = \varepsilon$ at only one point. Let $h(\varepsilon)$ be the y-coordinate of the intersection. Then, it turns out that $y(t) = r(t)\sin\theta(t) > h$ for $t_n \le t \le s_n$. Hence,

$$v'(t) = -\omega^{q-2}a(t)|y(t)|^q < -\omega^{q-2}h^q a(t) \tag{3.7}$$

for $t_n \le t \le s_n$. Needless to say, $v'(t) \le 0$ otherwise.

Suppose that there exists an $N \in \mathbb{N}$ such that $s_n - t_n \ge \gamma_0$ for $n \ge N$. Then, it follows from (3.7) that

$$v(s_n) - v(t_n) < -\omega^{q-2}h^q\int_{t_n}^{s_n} a(t)dt \le -\omega^{q-2}h^q\int_{t_n}^{t_n+\gamma_0} a(t)dt$$

for $n \ge N$. Since $v(t_{n+1}) - v(s_n) \le 0$ for $n \in \mathbb{N}$, we obtain

$$v(t_{n+1}) - v(t_n) < -\omega^{q-2}h^q\int_{t_n}^{t_n+\gamma_0} a(t)dt \quad \text{for } n \ge N,$$

and therfore,

$$v_0 - v(t_N) \le v(t_{n+1}) - v(t_N) < -\omega^{q-2}h^q\sum_{i=N}^{n}\int_{t_i}^{t_i+\gamma_0} a(t)dt.$$

However, from (1.1) it turns out that

$$\sum_{n=N}^{\infty}\int_{t_n}^{t_n+\gamma_0} a(t)dt = \infty.$$

This is a contradiction. Thus, there exists a sequence $\{n_k\}$ with $n_k \in \mathbb{N}$ and $n_k \to \infty$ as $k \to \infty$ such that

$$s_{n_k} - t_{n_k} < \gamma_0. \tag{3.8}$$

Since $r'(t) = -\omega^{q-2}a(t)\phi_q(r(t))|\sin\theta(t)|^q \le 0$ for $t \ge t_0$, we see that $r(t) \le r(t_0)$ for $t \ge t_0$. Hence,

$$\begin{aligned}\theta'(t) &\ge -\omega - a(t)(\omega r(t)|\sin\theta(t)|)^{q-2}|\sin\theta(t)||\cos\theta(t)| \\ &\ge -\omega - (\omega r(t_0))^{q-2}a(t)\end{aligned}$$

for $t \geq t_0$. From (3.8) it follows that

$$\begin{aligned}\varepsilon - (\pi - \varepsilon) &= \theta(s_{n_k}) - \theta(t_{n_k}) \\ &\geq -\omega(s_{n_k} - t_{n_k}) - (\omega r(t_0))^{q-2}\int_{t_{n_k}}^{s_{n_k}} a(t)dt \\ &> -\omega\gamma_0 - (\omega r(t_0))^{q-2}\int_{t_{n_k}}^{s_{n_k}} a(t)dt\end{aligned}$$

for each $k \in \mathbb{N}$; namely,

$$(\omega r(t_0))^{q-2}\int_{t_{n_k}}^{s_{n_k}} a(t)dt > \pi - \omega\gamma_0 - 2\varepsilon \quad \text{for } k \in \mathbb{N}.$$

Using this estimation and (3.7), we obtain

$$v(s_{n_k}) - v(t_{n_k}) < -\omega^{q-2}h^q\int_{t_{n_k}}^{s_{n_k}} a(t)dt < -\frac{h^q}{r(t_0)^{q-2}}(\pi - \omega\gamma_0 - 2\varepsilon)$$

for $k \in \mathbb{N}$. Since $v(t_{n_{k+1}}) - v(s_{n_k}) \leq 0$ for $k \in \mathbb{N}$, we see that

$$v(t_{n_{k+1}}) - v(t_{n_k}) < -\frac{h^q}{r(t_0)^{q-2}}(\pi - \omega\gamma_0 - 2\varepsilon) \quad \text{for } k \in \mathbb{N}.$$

Taking (3.6) into consideration, we can conclude that

$$v_0 - v(t_0) \leq \sum_{k=1}^{\infty}\big(v(t_{n_{k+1}}) - v(t_{n_k})\big) = -\infty,$$

which is a contradiction.

The proof of the theorem is now complete. □

Combining Theorems 3.2 and 3.4 with Proposition 3.1, we can conclude that Theorem 1.1 holds.

4 Explicit conditions

As shown in Section 1, in the special case in which $q = 2$, we can seek the solution $u(t)$ of (2.1) satisfying $u(0) = 0$ concretely. In general, however, it is difficult to confirm whether condition (3.2) is satisfied or not. For this reason, it is safe to say that Theorem 1.1 will give an implicit necessary and sufficient condition for global asymptotic stability. Hereafter, we will give some explicit sufficient conditions for the equilibrium of (E) to be globally attractive.

To state our results, we define the inverse function of ϕ_q as follows. Let q^* be the conjugate number of q; namely,

$$\frac{1}{q} + \frac{1}{q^*} = 1,$$

then q^* is also greater than 1. Let

$$w = \phi_q(z) = \begin{cases} z^{q-1} & \text{if } z \geq 0 \\ -(-z)^{q-1} & \text{if } z < 0. \end{cases}$$

Then, $z \ge 0$ if and only if $w \ge 0$, and $z = \phi_{q^*}(w)$. In fact, since

$$z = \begin{cases} w^{1/(q-1)} & \text{if } w \ge 0 \\ -(-w)^{1/(q-1)} & \text{if } w < 0, \end{cases}$$

it follows from $(q-1)(q^*-1) = 1$ that $w^{1/(q-1)} = w^{q^*-1} = |w|^{q^*-2}w = \phi_{q^*}(w)$ if $w \ge 0$ and $-(-w)^{1/(q-1)} = -(-w)^{q^*-1} = (-w)^{q^*-2}w = |w|^{q^*-2}w = \phi_{q^*}(w)$ if $w < 0$.

Corollary 4.1 *Suppose that assumption* (1.1) *holds. Suppose also that there exist a differentiable function $b(t)$ and a $T > 0$ such that*

$$b(t) > 0 \quad \textit{and} \quad a(t) \le b(t)$$

for $t \ge T$. If, in addition, $b(t)$ is nondecreasing for $t \ge T$ and

$$\int_T^\infty \phi_{q^*}\left(\frac{1}{b(t)}\right) dt = \infty,$$

then the equilibrium of (E) *is globally attractive.*

Proof. Define

$$g(t) = -\,\phi_{q^*}\left(\frac{1}{\omega^{q-2}b(t)}\right)$$

for $t \ge T$. Then, it is clear that $g(t) < 0$ and

$$\omega^{q-2}b(t)\phi_q(g(t)) = -1 \quad \text{for } t \ge T.$$

From the assumption of $b(t)$ it follows that $g(t)$ is negative, differentiable and nondecreasing for $t \ge T$.

Consider the solution $u(t;T)$ of (2.1) satisfying $u(T;T) = 0$. Since $u'(T;T) = -1$, we can find a $\delta > 0$ such that

$$u(t;T) < 0 \quad \text{for } T < t < T + \delta.$$

Taking into account that $g(T) < 0 = u(T;T)$, we see that

$$g(t^*) \le u(t^*;T) < 0$$

for some $t^* \in (T, T+\delta)$.

Let us compare $u(t;T)$ with $\eta(t) = \lambda\, g(t)$, where

$$\lambda = \frac{u(t^*;T)}{g(t^*)}.$$

Note that $\eta(t) < 0$ for $t \ge T$ and

$$\eta(t^*) = \lambda\, g(t^*) = u(t^*;T).$$

Since $0 < \lambda \le 1$, we have

$$\omega^{q-2}b(t)\phi_q(\eta(t)) = \phi_q(\lambda)\omega^{q-2}b(t)\phi_q(g(t)) = -\,\phi_q(\lambda) \ge -1$$

for $t \geq T$. Let $f(t,u) = -1 - \omega^{q-2}a(t)\phi_q(u)$. Then,

$$\begin{aligned}\eta'(t) = \lambda g'(t) \geq 0 &\geq -1 - \omega^{q-2}b(t)\phi_q(\eta(t))\\ &\geq -1 - \omega^{q-2}a(t)\phi_q(\eta(t)) = f(t,\eta(t))\end{aligned}$$

for $t \geq T$. Hence, by Lemma 2.2,

$$\eta(t) \geq u(t;T) \quad \text{for } t \geq t^*.$$

Integrating both sides of this inequality from t^* to t, we obtain

$$\int_{t^*}^{t} \eta(s)ds \geq \int_{t^*}^{t} u(s;T)ds \quad \text{for } t \geq t^*.$$

Hence, it follows that

$$\begin{aligned}\int_T^\infty u(t;T)dt &= \int_T^{t^*} u(t;T)dt + \int_{t^*}^\infty u(t;T)dt\\ &\leq \int_T^{t^*} u(t;T)dt + \int_{t^*}^\infty \eta(t)dt\\ &= \int_T^{t^*} u(t;T)dt - \int_T^{t^*} \eta(t)dt + \int_T^\infty \eta(t)dt\\ &= \int_T^{t^*} \big(u(t;T) - \eta(t)\big)dt + \lambda \int_T^\infty g(t)dt\\ &= \int_T^{t^*} \big(u(t;T) - \eta(t)\big)dt - \frac{\lambda}{\omega^{2-q^*}} \int_T^\infty \phi_{q^*}\left(\frac{1}{b(t)}\right)dt = -\infty,\end{aligned}$$

and therefore, by Theorem 3.4 and Lemma 2.1, we conclude that the equilibrium of (E) is globally attractive. □

In Corollary 4.1, we assumed the existence of an upper nondecreasing function $b(t)$ for the damping coefficient $a(t)$. The nondecreaseness of $b(t)$ is not always necessary for the equilibrium of (E) to be globally attractive. The following result shows that another condition on $b(t)$ can substitute for the nondecreaseness.

Corollary 4.2 *Suppose that assumption* (1.1) *holds. Suppose also that there exist a differentiable function $b(t)$ and positive numbers β and T such that*

$$b(t) \geq \beta \quad \textit{and} \quad a(t) \leq b(t)$$

for $t \geq T$. If, in addition,

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)} = 0 \quad \textit{and} \quad \int_T^\infty \phi_{q^*}\left(\frac{1}{b(t)}\right)dt = \infty,$$

then the equilibrium of (E) *is globally attractive.*

Proof. As in the proof of Corollary 4.1, we define

$$g(t) = -\phi_{q^*}\left(\frac{1}{\omega^{q-2}b(t)}\right)$$

for $t \geq T$. Then, it is easy to verify that

$$-\phi_{q^*}\left(\frac{1}{\omega^{q-2}\beta}\right) \leq g(t) < 0, \qquad \omega^{q-2}b(t)\phi_q(g(t)) = -1$$

and

$$g'(t) = (q^*-1)\left(\frac{1}{\omega^{q-2}b(t)}\right)^{q^*-2}\frac{b'(t)}{\omega^{q-2}b^2(t)} = -(q^*-1)g(t)\frac{b'(t)}{b(t)}$$

for $t \geq T$. Since $g(t)$ is bounded and $b'(t)/b(t)$ tends to 0 as $t \to \infty$, we see that

$$|g'(t)| \to 0 \quad \text{as } t \to \infty.$$

Hence, we can choose $T_1 \geq T$ so that

$$g'(t) > -1 \quad \text{for } t \geq T_1.$$

Consider the solution $u(t;T_1)$ of (2.1) satisfying $u(T_1;T_1) = 0$. Since $u'(T_1;T_1) = -1$, we can find a $\delta > 0$ such that

$$u(t;T_1) < 0 \quad \text{for } T_1 < t < T_1 + \delta.$$

From the inequality $g(T_1) < 0 = 2u(T_1;T_1)$ it follows that

$$g(t^*) \leq 2u(t^*;T_1) < 0$$

for some $t^* \in (T_1, T_1 + \delta)$.

Let

$$\mu = \frac{u(t^*;T_1)}{g(t^*)} \quad \text{and} \quad \eta(t) = \mu g(t).$$

Then, $0 < \phi_q(\mu) < \mu \leq 1/2$, $\eta(t) < 0$ and

$$\omega^{q-2}b(t)\phi_q(\eta(t)) = \phi_q(\mu)\omega^{q-2}b(t)\phi_q(g(t)) = -\phi_q(\mu) \geq -\frac{1}{2}$$

for $t \geq T$. Hence, we obtain

$$\begin{aligned}\eta'(t) = \mu g'(t) > -\mu \geq -\frac{1}{2} &\geq -1 - \omega^{q-2}b(t)\phi_q(\eta(t)) \\ &\geq -1 - \omega^{q-2}a(t)\phi_q(\eta(t)) = f(t, \eta(t))\end{aligned}$$

for $t \geq T_1$, where $f(t,u) = -1 - \omega^{q-2}a(t)\phi_q(u)$. Since

$$\eta(t^*) = \mu g(t^*) = u(t^*;T_1),$$

it follows from Lemma 2.2 that

$$\eta(t) \geq u(t;T_1) \quad \text{for } t \geq t^*.$$

By means of the same argument as in the proof of Corollary 4.1, we can estimate that

$$\int_{T_1}^{\infty} u(t;T_1)dt = -\infty.$$

Hence, by Theorem 3.4 and Lemma 2.1, we see that the equilibrium of (E) is globally attractive. $\square$

Karsai and Graef [21, Corollary 2.4] have given a sufficient condition for the equilibrium of the damped nonlinear oscillator

$$x'' + a(t)\phi_q(x') + f(x) = 0 \tag{4.1}$$

to be globally attractive. Here, $f(x)$ is continuous and satisfied the signum condition that

$$xf(x) > 0 \quad \text{if } x \neq 0. \tag{4.2}$$

Their result is as follows.

Theorem B *Suppose that $f(x)$ is nondecreasing and*

$$0 < a_0 < \underline{a}(t) \leq a(t) < \overline{a}(t) \tag{4.3}$$

for $t \geq 0$. Suppose also that

$$\lim_{t\to\infty} \frac{\underline{a}'(t)}{\underline{a}(t)} = 0, \tag{4.4}$$

and either $\overline{a}(t)/(\underline{a}(t))^{(q-2)/(q-1)}$ is nondecreasing or

$$\lim_{t\to\infty} \frac{\left(\overline{a}(t)/(\underline{a}(t))^{(q-2)/(q-1)}\right)'}{\overline{a}(t)/(\underline{a}(t))^{(q-2)/(q-1)}} = 0. \tag{4.5}$$

If

$$\int_0^\infty \frac{(\underline{a}(t))^{(q-2)/(q-1)}}{\overline{a}(t)}dt = \infty, \tag{4.6}$$

then the equilibrium of (4.1) *is globally attractive.*

Let us compare our results with Theorem B. The biggest difference between our results and Theorem B is whether the lower bound is allowed to be zero. Theorem B can be applied to only the case in which $a(t)$ is not less than a positive constant for $t \geq 0$. Such a case is often called *large damping*. On the other hand, our results can be applied to not only the case of large damping but also the case in which the set $\{t \geq 0 : a(t) = 0\}$ is permitted to be the union of infinitely many disjoint intervals whose length are less than π (see, condition (1.1)).

In the case of large damping, it is easy to extend our results to be able to apply Eq. (4.1), because strong assumptions, such as (4.2) and nondecreasing, are imposed on $f(x)$.

Actually, condition (4.4) is unnecessary in Theorem B. To confirm this fact, let

$$b(t) = \phi_q\left(\frac{\overline{a}(t)}{(\underline{a}(t))^{(q-2)/(q-1)}}\right)$$

for $t \geq 0$. Then, (4.3) implies that

$$\begin{aligned}\phi_{q^*}(b(t)) = \frac{\overline{a}(t)}{(\underline{a}(t))^{(q-2)/(q-1)}} &> \frac{\overline{a}(t)}{(\overline{a}(t))^{(q-2)/(q-1)}} \\ &= (\overline{a}(t))^{q^*-1} = \phi_{q^*}(\overline{a}(t));\end{aligned}$$

namely, $\overline{a}(t) < b(t)$ for $t \geq 0$. Hence,

$$b(t) > a_0 \quad \text{and} \quad a(t) < b(t)$$

for $t \geq 0$. Since

$$\int_0^\infty \phi_{q^*}\left(\frac{1}{b(t)}\right) dt = \int_0^\infty \frac{(\underline{a}(t))^{(q-2)/(q-1)}}{\overline{a}(t)}\, dt,$$

(4.6) coincides with

$$\int_0^\infty \phi_{q^*}\left(\frac{1}{b(t)}\right) dt = \infty.$$

If $\overline{a}(t)/(\underline{a}(t))^{(q-2)/(q-1)}$ is nondecreasing, then $b(t)$ is also nondecreasing. Since

$$\frac{(\phi_{q^*}(b(t)))'}{\phi_{q^*}(b(t))} = (q^* - 1)\frac{b'(t)}{b(t)},$$

if (4.5) holds, then

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)} = 0.$$

Thus, all the conditions of Corollaries 4.1 and 4.2 are satisfied, and therefore, Theorem B follows from Corollaries 4.1 and 4.2 without assuming (4.4).

Corollary 4.1 yields the following simple result.

Corollary 4.3 *Assume* (1.1) *holds. Suppose that there exist positive numbers* σ *and* T *such that*

$$0 \leq a(t) \leq t^\sigma \quad \textit{for } t \geq T.$$

If $\sigma \leq q-1$, *then the equilibrium of* (E) *is globally attractive.*

Proof. Let $b(t) = t^\sigma$. Then, it is clear that $b(t)$ is positive and nondecreasing for $t \geq T$. Let $T_1 = \max\{1, T\}$. If $\sigma \leq q-1$, then

$$\phi_{q^*}\left(\frac{1}{b(t)}\right) = \left(\frac{1}{b(t)}\right)^{q^*-1} = \left(\frac{1}{t}\right)^{\sigma/(q-1)} \geq \frac{1}{t}$$

for $t \geq T_1$. Hence,

$$\int_T^\infty \phi_{q^*}\left(\frac{1}{b(t)}\right) dt \geq \int_T^{T_1} \phi_{q^*}\left(\frac{1}{b(t)}\right) dt + \int_{T_1}^\infty \frac{1}{t}\, dt = \infty.$$

Thus, from Corollary 4.1, it turns out that the equilibrium of (E) is globally attractive. □

Remark 4.1 Let $b(t) = t^\sigma$. Then,

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)} = 0.$$

Hence, we can also lead Corollary 4.3 from Corollary 4.2.

Applying Corollary 2.5 of Karsai and Graef [21] to Eq. (E), we see that if $t^\gamma \leq a(t) \leq t^\sigma$ with

$$0 < \gamma \leq \sigma \quad \text{and} \quad \sigma - 1 \leq \gamma\frac{q-2}{q-1},$$

then the equilibrium is globally attractive. Since $a(t) \geq t^\gamma$ for $t \geq 0$, our assumption (1.1) is naturally satisfied. Also, since $0 < \gamma \leq \sigma$, we obtain

$$\sigma - 1 \leq \gamma\frac{q-2}{q-1} \leq \sigma\frac{q-2}{q-1};$$

namely, $\sigma \le q-1$. Thus, Corollary 4.3 essentially includes their result.

Next, we give some explicit necessary conditions for the equilibrium of (E) to be attractive. We judge that the equilibrium of (E) is not attractive by using a lower function instead of the damping coefficient $a(t)$.

Corollary 4.4 *Suppose that there exist a differentiable function $c(t)$ and positive numbers β and T such that*

$$\beta \le c(t) \le a(t)$$

for $t \ge T$. If

$$\lim_{t\to\infty} \frac{c'(t)}{c(t)} = 0 \quad \textit{and} \quad \int_T^\infty \phi_{q^*}\left(\frac{1}{c(t)}\right) dt < \infty,$$

then the equilibrium of (E) is not attractive.

Proof. Let

$$g(t) = -\phi_{q^*}\left(\frac{1}{\omega^{q-2}c(t)}\right)$$

for $t \ge T$. Then, we can easily verify that $g(t)$ is negative and bounded for $t \ge T$, and it satisfies

$$\omega^{q-2}c(t)\phi_q(g(t)) = -1 \quad \text{and} \quad g'(t) = -(q^*-1)g(t)\frac{c'(t)}{c(t)}$$

for $t \ge T$. Since $c'(t)/c(t)$ tends to 0 as $t \to \infty$, we see that

$$|g'(t)| \to 0 \quad \text{as } t \to \infty,$$

and therefore, there exists a $T_1 \ge T$ such that

$$g'(t) < \frac{1}{2} \quad \text{for } t \ge T_1.$$

Consider the solution $u(t;T_1)$ of (2.1) satisfying $u(T_1;T_1) = 0$. As in the proof of Corollary 4.2, taking into account that $u'(T_1;T_1) = -1$, we can choose a $t^* > T_1$ such that

$$g(t^*) \le \frac{1}{2}u(t^*;T_1) < 0.$$

Let $\eta(t) = 2g(t)$. Then, $\eta(t) < 0$ and

$$\omega^{q-2}c(t)\phi_q(\eta(t)) = \phi_q(2)\omega^{q-2}c(t)\phi_q(g(t)) = -\phi_q(2) \le -2$$

for $t \ge T$. Hence, we obtain

$$\begin{aligned}\eta'(t) = 2g'(t) < 1 &\le -1 - \omega^{q-2}c(t)\phi_q(\eta(t)) \\ &\le -1 - \omega^{q-2}a(t)\phi_q(\eta(t))\end{aligned}$$

for $t \ge T_1$. Let $f(t,u) = -1 - \omega^{q-2}a(t)\phi_q(u)$. Then, $\eta'(t) < f(t,\eta(t))$ for $t \ge T_1$. Since

$$\eta(t^*) = 2g(t^*) \le u(t^*;T_1),$$

it follows from Lemma 2.3 that

$$\eta(t) \le u(t;T_1) \quad \text{for } t \ge t^*.$$

Integrate both sides of this inequality from t^* to t to obtain

$$\int_{t^*}^{t}\eta(s)ds \le \int_{t^*}^{t}u(s;T_1)ds \quad \text{for } t \ge t^*.$$

From this inequality, we can estimate that

$$\begin{aligned}
\int_{T_1}^{\infty}u(t;T_1)dt &= \int_{T_1}^{t^*}u(t;T_1)dt + \int_{t^*}^{\infty}u(t;T_1)dt \\
&\ge \int_{T_1}^{t^*}u(t;T_1)dt + \int_{t^*}^{\infty}\eta(t)dt \\
&= \int_{T_1}^{t^*}\big(u(t;T_1)-\eta(t)\big)dt + \int_{T_1}^{\infty}\eta(t)dt \\
&= \int_{T_1}^{t^*}\big(u(t;T_1)-\eta(t)\big)dt + 2\int_{T_1}^{\infty}g(t)dt \\
&= \int_{T_1}^{t^*}\big(u(t;T_1)-\eta(t)\big)dt - 2\int_{T}^{T_1}g(t)dt + 2\int_{T}^{\infty}g(t)dt \\
&= \int_{T_1}^{t^*}u(t;T_1)dt - \int_{T}^{t^*}\eta(t)dt - \frac{2}{\omega^{2-q^*}}\int_{T}^{\infty}\phi_{q^*}\left(\frac{1}{c(t)}\right)dt > -\infty.
\end{aligned}$$

Hence, by Theorem 3.2 and Lemma 2.1, we conclude that the equilibrium of (E) is not attractive. □

The following result is a direct consequence of Corollary 4.4.

Corollary 4.5 *Suppose that there exist positive numbers γ and T such that*

$$t^{\gamma} \le a(t) \quad \textit{for } t \ge T.$$

If $\gamma > q-1$, then the equilibrium of (E) is not attractive.

Proof. We may assume without loss of generality that $T > 1$. Let $c(t) = t^{\gamma}$ and $\beta = T^{\gamma}$. Then, it is clear that $\beta \le c(t) \le a(t)$ for $t \ge T$ and

$$\lim_{t\to\infty}\frac{c'(t)}{c(t)} = 0.$$

Since $\gamma > q-1$, we can choose an $\varepsilon_0 > 0$ so that

$$1+\varepsilon_0 \le \frac{\gamma}{q-1} = \gamma(q^*-1).$$

Hence, we obtain

$$\int_{T}^{\infty}\phi_{q^*}\left(\frac{1}{c(t)}\right)dt = \int_{T}^{\infty}\left(\frac{1}{t}\right)^{\gamma(q^*-1)}dt \le \int_{T}^{\infty}\left(\frac{1}{t}\right)^{1+\varepsilon_0}dt < \infty,$$

and therefore, by Corollary 4.4, we see that the equilibrium of (E) is not attractive. □

5 Growth condition on $a(t)$

Hatvani, Krisztin and Totik [17] have considered the damping linear oscillator (L) and proved that under the assumption that $A(t)$ tends to ∞ as $t \to \infty$, the growth condition (1.2) on $a(t)$ is equivalent to

$$\sum_{n=1}^{\infty} \left(A^{-1}(nc) - A^{-1}((n-1)c)\right)^2 = \infty \tag{5.1}$$

for any $c > 0$, where

$$A^{-1}(s) = \min\left\{t \geq 0 : A(t) \geq s\right\}.$$

It is clear that if $a(t) > 0$ for $t \geq 0$, then $A(t)$ is increasing for $t \geq 0$, and therefore, $A^{-1}(s)$ is the inverse function of $s = A(t)$. Using their ingenious idea and method, we see that the discrete condition (5.1) is also equivalent to

$$\int_0^{\infty} \frac{\int_0^t e^{kA(s)} ds}{e^{kA(t)}}\, dt = \infty \tag{5.2}$$

for any $k > 0$. Consequently, we have the following result.

Lemma 5.1 *Suppose that $A(t)$ tends to ∞ as $t \to \infty$. Then conditions* (1.2) *and* (5.2) *are equivalent.*

Combining Theorem 3.4 with Lemmas 2.1, 2.2, and 5.1, we obtain the following result.

Corollary 5.2 *Assume* (1.2) *and suppose that there exist positive numbers β and T such that*

$$a(t) \geq \beta \quad \text{for } t \geq T. \tag{5.3}$$

Then the equilibrium of (E) is globally attractive.

Proof. Let $\gamma_0 > 0$. From (5.3) it follows that

$$A(t + \gamma_0) - A(t) \geq \beta\gamma_0 > 0.$$

Hence, condition (1.1) holds.

Define

$$g(t) = -\frac{\omega^{q-2}\beta \int_0^t e^{\omega^{q-2}A(s)} ds}{2\, e^{\omega^{q-2}A(t)}}$$

for $t \geq 0$. Then,

$$g'(t) = -\frac{\omega^{q-2}\beta}{2} - \omega^{q-2}a(t)g(t)$$

for $t \geq 0$. By (5.3), we have

$$A(t) - A(s) \geq \beta(t - s) \quad \text{for } T \leq s \leq t.$$

Hence,

$$
\begin{aligned}
0 > g(t) &= -\frac{\omega^{q-2}\beta\int_0^T e^{\omega^{q-2}A(s)}ds}{2\,e^{\omega^{q-2}A(t)}} - \frac{\omega^{q-2}\beta}{2}\int_T^t e^{-\omega^{q-2}(A(t)-A(s))}ds \\
&\geq -\frac{\omega^{q-2}\beta\int_0^T e^{\omega^{q-2}A(s)}ds}{2\,e^{\omega^{q-2}A(t)}} - \frac{\omega^{q-2}\beta}{2}\int_T^t e^{-\omega^{q-2}\beta(t-s)}ds \\
&= -\frac{\omega^{q-2}\beta\int_0^T e^{\omega^{q-2}A(s)}ds}{2\,e^{\omega^{q-2}A(t)}} - \frac{1}{2}\left(1-\frac{e^{\omega^{q-2}\beta T}}{e^{\omega^{q-2}\beta t}}\right) \\
&\geq -\frac{\omega^{q-2}\beta\int_0^T e^{\omega^{q-2}A(s)}ds}{2\,e^{\omega^{q-2}A(t)}} - \frac{1}{2}
\end{aligned}
$$

for $t \geq T$. Since $A(t)$ diverges to ∞ as t tends to ∞, we can find a $T_1 > T$ so that

$$\frac{\omega^{q-2}\beta\int_0^T e^{\omega^{q-2}A(s)}ds}{2\,e^{\omega^{q-2}A(t)}} < \frac{1}{2} \quad \text{for } t \geq T_1.$$

We therefore conclude that

$$-1 < g(t) < 0 \quad \text{for } t \geq T_1.$$

Consider the solution $u(t;T_1)$ of (2.1) satisfying $u(T_1;T_1) = 0$. Since $u'(T_1;T_1) = -1$, we can choose a $\delta > 0$ such that

$$u(t;T_1) < 0 \quad \text{for } T_1 < t < T_1 + \delta.$$

Let $\nu = \min\{1, 2/(\omega^{q-2}\beta)\}$. From the inequality $g(T_1) < 0 = u(T_1;T_1)/\nu$, it turns out that

$$\nu g(t^*) \leq u(t^*;T_1) < 0$$

for some $t^* \in (T_1, T_1 + \delta)$.

Let

$$\mu = \frac{u(t^*;T_1)}{g(t^*)} \quad \text{and} \quad \eta(t) = \mu g(t).$$

Then, $0 < \mu \leq \nu$ and

$$-1 < -\nu < \eta(t) < 0 \quad \text{for } t \geq T_1,$$

and therefore,

$$
\begin{aligned}
\eta'(t) = \mu g'(t) &= -\frac{\omega^{q-2}\beta\mu}{2} - \omega^{q-2}a(t)\mu g(t) \\
&\geq -1 - \omega^{q-2}a(t)\eta(t) \geq -1 - \omega^{q-2}a(t)\phi_q(\eta(t)) = f(t,\eta(t)),
\end{aligned}
$$

where $f(t,u) = -1 - \omega^{q-2}a(t)\phi_q(u)$. Since

$$\eta(t^*) = \mu g(t^*) = u(t^*;T_1),$$

it follows from Lemma 2.2 that

$$\eta(t) \geq u(t;T_1) \quad \text{for } t \geq t^*.$$

Using Lemma 5.1, we see that (1.2) implies that

$$\int_0^\infty g(t)dt = -\infty.$$

Hence,

$$\begin{aligned}
\int_{T_1}^\infty u(t;T_1)dt &= \int_{T_1}^{t^*} u(t;T_1)dt + \int_{t^*}^\infty u(t;T_1)dt \\
&\le \int_{T_1}^{t^*} u(t;T_1)dt + \int_{t^*}^\infty \eta(t)dt \\
&= \int_{T_1}^{t^*} u(t;T_1)dt + \mu \int_{t^*}^\infty g(t)dt \\
&= \int_{T_1}^{t^*} u(t;T_1)dt - \mu \int_0^{t^*} g(t)dt + \mu \int_0^\infty g(t)dt = -\infty.
\end{aligned}$$

Thus, by means of Theorem 3.4 and Lemma 2.1, we conclude that the equilibrium of (E) is globally attractive. □

Recall that the proof of Theorem 3.4 was divided into three steps as follows:

(i) For any solution $(x(t),y(t))$ of (3.1), the function $v(t) \stackrel{\text{def}}{=} V(x(t),y(t))$ is nonincreasing for $t \ge 0$. Hence, $v(t)$ has the limiting value $v_0 \ge 0$. If v_0 is zero, then the proof is complete. In the second and third steps, it is shown that the case of v_0 does not occur. Afterwards, we assume that v_0 is positive.

(ii) If the positive orbit of (3.1) corresponding to $(x(t),y(t))$ does not rotate around the origin, then it has to converge to a point on the x-axis. However, comparing $y(t)$ with a certain solution of (2.1) and using condition (3.2), we can conclude that the positive orbit does not approach the point. This is a contradiction.

(iii) The positive orbit keeps rotating around the origin, Since v_0 is positive, the orbit does not enter in the circle of radius $\sqrt{2v_0}$. However, by using condition (1.1), we can show that the orbit approaches the origin by a constant distance each time it passes through a sector whose central angle is almost π. Hence, the orbit arrives at the origin. This is a contradiction.

Condition (3.2) was used only in the second step of the proof of Theorem 3.4. Making use of the growth condition (1.2) instead of condition (3.2), we obtain the following result.

Corollary 5.3 *Assume* (1.1) *and* (1.2). *Then the equilibrium of* (E) *is globally attractive*.

Proof. As mentioned above, the proof is completed in the three steps. The first and third steps are the same as those of Theorem 3.4. We will confirm only the second step by using (1.2). Let $x(t)$ be any solution of (E) with the initial time $t_0 \ge 0$ and let $(x(t),y(t))$ be the solution of (3.1) corresponding to $x(t)$. Suppose that $(x(t),y(t))$ stays in Q_2 or Q_4 ultimately. We consider only the case in which $(x(t),y(t))$ is in Q_4 ultimatey, because the other case is carried out in the same manner.

As in the proof of Theorem 3.4, we can show that

$$(x(t),y(t)) \to (\sqrt{2v_0},0) \quad \text{as } t \to \infty,$$

where $v_0 > 0$ is the limiting value of $v(t) = (x^2(t) + y^2(t))/2$. Hence, there exists a $T > 0$ such that

$$\sqrt{2v_0} < x(t) \le x(T) \quad \text{and} \quad -1 < y(t) < 0$$

for $t \ge T$. Since $q \ge 2$, we see that

$$0 > \phi_q(y(t)) = (-y(t))^{q-2} y(t) > y(t),$$

and therefore,

$$y'(t) = -\omega x(t) - \omega^{q-2} a(t) \phi_q(y(t)) < -\omega x(t) - \omega^{q-2} a(t) y(t)$$

for $t \ge T$. Hence, we get

$$\left(e^{\omega^{q-2}A(t)} y(t)\right)' < -\omega\sqrt{2v_0}\, e^{\omega^{q-2}A(t)} \quad \text{for } t \ge T.$$

Integrating both sides of this inequality from T to t, we obtain

$$e^{\omega^{q-2}A(t)} y(t) < e^{\omega^{q-2}A(T)} y(T) - \omega\sqrt{2v_0} \int_T^t e^{\omega^{q-2}A(s)} ds < -\omega\sqrt{2v_0} \int_T^t e^{\omega^{q-2}A(s)} ds;$$

namely,

$$x'(t) = \omega y(t) < -\omega^2 \sqrt{2v_0} \int_T^t \frac{e^{\omega^{q-2}A(s)}}{e^{\omega^{q-2}A(t)}} ds$$

for $t \ge T$. From Lemma 5.1 and (1.2) it follows that $x(t)$ tends to $-\infty$ as $t \to \infty$. This is a contradiction.

Thus, the theorem is proved. □

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References

1. Artstein, Z., Infante, E.F.: On the asymptotic stability of oscillators with unbounded damping. Quart. Appl. Math. **34**, 195–199 (1976/77)
2. Bacciotti, A., Rosier, L.: Liapunov functions and stability in control theory. Springer, Berlin (2005)
3. Ballieu, R.J., Peiffer, K.: Attractivity of the origin for the equation $\ddot{x} + f(t, \dot{x}, \ddot{x})|\dot{x}|^{\alpha}\dot{x} + g(x) = 0$. J. Math. Anal. Appl. **65**, 321–332 (1978)
4. Bass, D.W., Haddara, M.R.: Nonlinear models of ship roll damping. International Shipbuilding Progress **35**, 5–24 (1988)
5. Bass, D.W., Haddara, M.R.: Roll and sway-roll damping for three small fishing vessels. International Shipbuilding Progress **38**, 51–71 (1991)
6. Brauer, F., Nohel, J.: The qualitative theory of ordinary differential equations. W.A. Benjamin, New York (1969); (revised) Dover, New York (1989)
7. Cardo, A., Francescutto, A., Nabergoj, R.: On damping models in free and forced rolling motion. Ocean Engineering **9**, 171–179 (1982)
8. Cesari, L.: Asymptotic behavior and stability problems in ordinary differential equations. Springer, Berlin (1959); (2nd ed.) Springer, Berlin (1963)
9. Coppel, W.A.: Stability and asymptotic behavior of differential equations. Heath, Boston (1965)
10. Dalzell, J.F.: A note on the form of ship roll damping. J. Ship Res. **22**, 178–185 (1978)
11. Duc, L.H., Ilchmann, A., Siegmund, S., Taraba, P.: On stability of linear time-varying second-order differential equations. Quart. Appl. Math. **64**, 137–151 (2006)
12. Haddara, M.R., Bass, D.W.: On the form of roll damping moment for small fishing vessels. Ocean Engineering **17**, 525–539 (1990)
13. Halanay, A.: Differential equations: stability, oscillations, time lags. Academic Press, New York (1966)

14. Hale, J.K.: Ordinary differential equations. Wiley, New York (1969); (revised) Krieger, Malabar (1980)
15. Hatvani, L.: On the asymptotic stability for a two-dimensional linear nonautonomous differential system. Nonlinear Anal. **25**, 991-1002 (1995)
16. Hatvani, L.: Integral conditions on the asymptotic stability for the damped linear oscillator with small damping. Proc. Amer. Math. Soc. **124**, 415–422 (1996)
17. Hatvani, L., Krisztin, T., Totik, V.: A necessary and sufficient condition for the asymptotic stability of the damped oscillator. J. Differ. Equ. **119**, 209–223 (1995)
18. Hatvani, L., Totik, V.: Asymptotic stability of the equilibrium of the damped oscillator. Diff. Integral Eqns. **6**, 835–848 (1993)
19. Hinemo, Y.: Prediction of ship roll damping-state of art. Dept. of Naval Architecture and Marine Engineering, The University of Michigan, Report No. 239 (1981)
20. Ignatyev, A.O.: Stability of a linear oscillator with variable parameters. Electron. J. Differential Equations **1997**, No. 17, 1–6 (1997)
21. Karsai, J, Graef, J.R.: Attractivity properties of oscillator equations with superlinear damping. Discrete Contin. Dyn. Syst. **2005**, suppl., 497–504 (2005)
22. Levin, J.J., Nohel, J.A.: Global asymptotic stability for nonlinear systems of differential equations and application to reactor dynamics. Arch. Rational Mech. Anal. **5**, 194–211 (1960)
23. Michel, A. N., Hou, L., Liu, D.: Stability dynamical systems: continuous, discontinuous, and discrete systems. Birkhäuser, Boston (2008)
24. Neves, M.A.S., Pérez, N.A., Valerio, L.: Stability of small fishing vessels in longitudinal waves. Ocean Engineering **26**, 1389–1419 (1999)
25. Pucci, P., Serrin, J.: Precise damping conditions for global asymptotic stability for nonlinear second order systems. Acta Math. **170**, 275–307 (1993)
26. Pucci, P., Serrin, J.: Asymptotic stability for intermittently controlled nonlinear oscillators. SIAM J. Math. Anal. **25**, 815–835 (1994)
27. Rouche, N., Habets, P., Laloy, M.: Stability theory by liapunov's direct method. Applied mathematical sciences, vol. 22. Springer, New York (1977)
28. Smith, R.A: Asymptotic stability of $x'' + a(t)x' + x = 0$. Quart. J. Math. Oxford (2) **12**, 123–126 (1961)
29. Sugie, J.: Global asymptotic stability for damped half-linear oscillators. Nonlinear Anal. **74**, 7151–7167 (2011)
30. Taylan, M.: The effect of nonlinear damping and restoring in ship rolling. Ocean Engineering **27**, 921–932 (2000)
31. Yoshizawa, T.: Stability theory and the existence of periodic solutions and almost periodic solutions. Applied mathematical sciences, vol. 14. Springer, New York (1975)