

## SMITH-TYPE CRITERION FOR THE ASYMPTOTIC STABILITY OF A PENDULUM WITH TIME-DEPENDENT DAMPING

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ABSTRACT. A necessary and sufficient condition is given for the asymptotic stability of origin of a pendulum with time-varying friction described by the equation

$$x'' + h(t)x' + \sin x = 0,$$

where  $h(t)$  is continuous and nonnegative for  $t \geq 0$ . This condition is expressed as a double integral on the friction  $h(t)$ . The method that is used to obtain the result is Lyapunov's stability theory and phase plane analysis of the positive orbits of an equivalent planar system to the above-mentioned equation.

### 1. INTRODUCTION

We consider the damped pendulum equation

$$(P) \quad x'' + h(t)x' + \sin x = 0,$$

where the prime denotes  $d/dt$  and the damping coefficient  $h(t)$  is continuous and nonnegative for  $t \geq 0$ . The origin  $(x, x') = (0, 0)$  is an equilibrium of  $(P)$ .

Let  $\mathbf{x}(t) = (x(t), x'(t))$  and  $\mathbf{x}_0 \in \mathbb{R}^2$ , and let  $\|\cdot\|$  be any suitable norm. We denote the solution of  $(P)$  through  $(t_0, \mathbf{x}_0)$  by  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ . The uniqueness of solutions of  $(P)$  is guaranteed for the initial value problem.

The origin is said to be *stable* if, for any  $\varepsilon > 0$  and any  $t_0 \geq 0$ , there exists a  $\delta(\varepsilon, t_0) > 0$  such that  $\|\mathbf{x}_0\| < \delta$  implies  $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$  for all  $t \geq t_0$ . The origin is *uniformly stable* if it is stable and  $\delta$  can be chosen independent of  $t_0$ . The origin is said to be *attractive* if, for any  $t_0 \geq 0$ , there exists a  $\delta_0(t_0) > 0$  such that  $\|\mathbf{x}_0\| < \delta_0$  implies  $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| \rightarrow 0$  as  $t \rightarrow \infty$ . The origin is *uniformly attractive* if  $\delta_0$  in the definition of attractivity can be chosen independent of  $t_0$ , and for every  $\eta > 0$  there is a  $T(\eta) > 0$  such that  $t_0 \geq 0$  and  $\|\mathbf{x}_0\| < \delta_0$  imply  $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \eta$  for  $t \geq t_0 + T(\eta)$ . The origin is *asymptotically stable* if it is stable and attractive. The origin is *uniformly asymptotically stable* if it is uniformly stable and is uniformly attractive. With respect to the various definitions of stability, the reader may refer to the books [1, 6, 7, 12, 16, 25] for example.

The purpose of this paper is to establish a criterion for judging whether the origin of  $(P)$  is asymptotically stable or not. Our main theorem is as follows:

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THEOREM 1. Suppose that there exists a  $\gamma_0$  with  $0 < \gamma_0 < \pi$  such that

$$(1) \quad \liminf_{t \rightarrow \infty} \int_t^{t+\gamma_0} h(s) ds > 0.$$

Then the origin of (P) is asymptotically stable if and only if

$$(2) \quad \int_0^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}} dt = \infty,$$

where

$$H(t) = \int_0^t h(s) ds.$$

The criterion (2) is the so-called growth condition on  $h(t)$ . This condition has first been given by Smith [17, Theorems 1 and 2]. He has considered the damped linear oscillator

$$(L) \quad x'' + h(t)x' + x = 0$$

under the assumption that there exists an  $\underline{h} > 0$  such that  $h(t) \geq \underline{h}$  for  $t \geq 0$  and showed that condition (2) is necessary and sufficient for the origin of (L) to be asymptotically stable. It is known that if  $h(t)$  is bounded from above or  $h(t) = t$ , then condition (2) holds; if  $h(t) = t^2$ , then condition (2) fails to hold (for details, see [10]).

The case in which  $\underline{h} \leq h(t) < \infty$  for  $t \geq 0$  is often called *large damping*. Later, many attempts were carried out to taken off the lower bound  $\underline{h}$  from the assumption of  $h(t)$  (for example, see [2, 8, 9, 11, 14, 15, 18–22]). Among them, we should mention specially about Hatvani and Totik's result [11, Theorem 3.1]. They showed that the growth condition (2) is necessary and sufficient for the origin of (L) to be asymptotically stable provided that condition (1) is satisfied. It is clear that condition (1) holds in the case of large damping. Hence, Hatvani and Totik's result is a generalization of the result by Smith. Even if intervals where  $h(t)$  becomes zero are infinitely many, condition (1) may be satisfied if the lengths of intervals are less than  $\pi$ . This is a good point of condition (1).

A lot of researches were carried out to solve the problem of the preservation of uniform asymptotic stability from the  $n$ -dimensional linear system

$$(S) \quad \mathbf{x}' = A(t)\mathbf{x}$$

to the quasi-linear system

$$(Q) \quad \mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t, \mathbf{x}),$$

where  $\mathbf{f}(t, \mathbf{x})$  is continuous in  $(t, \mathbf{x}) \in R \stackrel{\text{def}}{=} \{(t, \mathbf{x}) : t \geq 0 \text{ and } \|\mathbf{x}\| < a\}$  for some  $a > 0$  and has continuous first-order partial derivatives with respect to  $\mathbf{x}$  in  $R$ , and  $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}$ . As is well known, under the assumption that  $\mathbf{f}$  has the property

$$(3) \quad \lim_{\|\mathbf{x}\| \rightarrow 0} \frac{\|\mathbf{f}(t, \mathbf{x})\|}{\|\mathbf{x}\|} = 0$$

uniformly in  $t$ , if the origin of (L) is uniformly asymptotically stable, then the origin of (Q) is also uniformly asymptotically stable. However, if we drop the terms "uniformly", then the above statement is not true. Perron [13] has shown that the

asymptotic stability of the origin of  $(L)$  do not always imply the asymptotic stability of the origin of  $(Q)$ . As to Perron's example, see the books [3, pp. 42–43], [4, pp. 169–170], [5, p. 71], [23, pp. 92–93], [24, pp. 315–317], etc.

Equations  $(L)$  and  $(P)$  are rewritten as systems  $(S)$  and  $(Q)$ , where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -h(t) \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} 0 \\ x - \sin x \end{pmatrix},$$

respectively. It is easy to verify that  $\mathbf{f}$  possess the property (3). However, Theorem 1 is not an immediately consequence of Hatvani and Totik's result, because their result is a criterion for the asymptotic stability of the origin of  $(L)$ , but it is not a criterion for the uniform asymptotic stability.

As can be seen from Theorem 1 and the result of Hatvani and Totik, a necessary and sufficient condition for the origin of the damped pendulum equation  $(P)$  to be asymptotically stable is the same as that of the linear approximation  $(L)$ . Judging from this fact, there might be room for a further research on the problem of the preservation from system  $(S)$  to system  $(Q)$ .

## 2. PHASE PLANE ANALYSIS

To examine the asymptotic behavior of solutions of  $\mathbf{x}(t; t_0, \mathbf{x}_0)$ , it is very useful to change equation  $(P)$  into planar equivalent systems. By putting  $y = x'$  as a new variable, equation  $(P)$  becomes the system

$$(4) \quad \begin{aligned} x' &= y, \\ y' &= -\sin x - h(t)y. \end{aligned}$$

The whole  $x$ - $y$  plane is often called the *phase plane* of (4) and the phase plane is divided into four quadrants. We denote by  $Q_i$  the  $i$ -th quadrant ( $i = 1, 2, 3, 4$ ). We call the projection of a positive semitrajectory of (4) onto the phase plane a *positive orbit* and we denote by  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  the positive orbit of (4) starting from a point  $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$  at a time  $t_0 \geq 0$ . Since system (4) is nonautonomous, the forms of positive orbits starting from the same point  $\mathbf{x}_0$  are different according to the initial time  $t_0$ .

As a suitable Lyapunov function, we adopt

$$V(x, y) = 1 - \cos x + \frac{1}{2}y^2,$$

which is regarded as a total energy for system (4). Then, we obtain

$$\dot{V}_{(4)}(t, x, y) = (\sin x)x' + yy' = -h(t)y^2 \leq 0$$

on  $[0, \infty) \times \mathbb{R}^2$ . Hence, we see that the domain

$$D = \{(x, y) \in \mathbb{R}^2 : |x| \leq \pi \text{ and } V(x, y) \leq 2\}$$

is a positive invariant set of (4), namely, for any  $\mathbf{x}_0 \in D$  and  $t_0 \geq 0$ , the positive orbit  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  stays in  $D$  for all future time. Since  $V(x, y)$  is positive definite and decrescent in a neighborhood of the origin  $(0, 0)$ , we conclude that

**Lemma 1.** *The origin of  $(P)$  is uniformly stable.*

Let

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

Then, we can transform system (4) into the polar coordinates system

$$(5) \quad \begin{aligned} r' &= r \sin \theta \cos \theta - \sin \theta \sin(r \cos \theta) - h(t) r \sin^2 \theta, \\ \theta' &= -\frac{1}{r} \sin(r \cos \theta) \cos \theta - \sin^2 \theta - h(t) \sin \theta \cos \theta. \end{aligned}$$

Hence,  $r^2 \theta' = -x \sin x - y^2 - h(t)xy \leq 0$  if  $(x, y) \in (Q_1 \cup Q_3) \cap D$ . This means that every positive orbit turns clockwise around the origin  $(0, 0)$  as long as it moves through  $Q_1 \cap D$  or  $Q_3 \cap D$ . Then, does this orbit enter  $Q_4$  or  $Q_2$  by passing through the  $x$ -axis? The answer is yes.

**Lemma 2.** *There is no positive orbit which continues staying in  $(Q_1 \cup Q_3) \cap D$ .*

*Proof.* Suppose that there exists a point  $\mathbf{x}_0 \in Q_1 \cap D$  (resp.,  $Q_3 \cap D$ ) and a time  $t_0 \geq 0$  such that the positive orbit  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  stays in  $Q_1 \cap D$  (resp.,  $Q_3 \cap D$ ). Let  $(r(t), \theta(t))$  be the solution of (5) corresponding to this positive orbit. Then,

$$\theta'(t) \leq -\sin^2 \theta(t) \quad \text{for } t \geq t_0$$

and there exists a  $\theta_0 \in (0, \pi/2)$  (resp.,  $(\pi, 3\pi/2)$ ) such that

$$\theta(t) \searrow \theta_0 \quad \text{as } t \rightarrow \infty.$$

Hence, we have

$$\theta'(t) < -\sin^2 \theta_0 \quad \text{for } t \geq t_0.$$

Integrating this inequality from  $t_0$  to  $t$ , we obtain

$$\theta(t) < \theta(t_0) - (\sin^2 \theta_0)(t - t_0) \rightarrow -\infty \quad \text{as } t \rightarrow \infty.$$

This is a contradiction. Thus, such a positive orbit does not exist.  $\square$

From the vector field of (4), we see that the positive orbit moves to the left in  $Q_4$ , and moves to the right in  $Q_2$ . However, it does not always rotate around the origin  $(0, 0)$  in a clockwise direction, and may go up and down in  $Q_4$  and  $Q_2$ .

Let  $(x(t), y(t))$  be any solution of (4) with the initial time  $t_0 \geq 0$  and define

$$v(t) = V(x(t), y(t))$$

for  $t \geq t_0$ . Since  $v'(t) = -h(t)y^2(t) \leq 0$  for  $t \geq t_0$ ,  $v(t)$  is nonincreasing, and therefore, it has the limiting value  $v_0 \geq 0$ . If  $v_0$  is zero for all solutions of (4) staying in  $D$ , then the origin of  $(P)$  is attractive. If  $v_0$  is positive for a solution of (4) staying in  $D$ , then the positive orbit corresponding to this solution remains in the annulus

$$A = \{(x, y) \in \mathbb{R}^2 : v_0 < V(x, y) \leq 2\} \subset D.$$

Since the closed curve given by  $V(x, y) = v_0 > 0$  is a symmetric oval, this curve intersects with the  $x$ -axis only at two points  $(\alpha, 0)$  and  $(-\alpha, 0)$ , where  $0 < \alpha = \arccos(1 - v_0) < \pi$ . In the next section, we will show that the case of  $v_0 > 0$  does not occur provided conditions (1) and (2) hold.

## 3. PROOF OF THEOREM 1

We first prove that if the origin of  $(P)$  is attractive, then the growth condition (2) holds. We then prove the converse.

*Necessity.* Suppose that (2) does not hold. Then, we can choose a  $T \geq 0$  so large that

$$(6) \quad \int_T^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}} dt < \frac{1}{2}.$$

Consider the positive orbit  $\Gamma_{(4)}^+(T, (1, 0))$ . From the vector field of (4), it follows that  $\Gamma_{(4)}^+(T, (1, 0))$  goes into  $Q_4 \cap D$  afterwards. Let  $(x(t), y(t))$  be the solution of (4) corresponding to  $\Gamma_{(4)}^+(T, (1, 0))$ . Note that  $x(T) = 1$  and  $x'(T) = y(T) = 0$ . We will show that  $x(t) > 1/2$  for  $t \geq T$ .

By way of contradiction, we suppose that there exists a  $T_1 > T$  such that  $x(T_1) = 1/2$  and  $x(t) > 1/2$  for  $T \leq t < T_1$ . Since

$$x''(t) + h(t)x'(t) = -\sin x(t) \geq -1,$$

we have

$$(x'(t)e^{H(t)})' \geq -e^{H(t)} \quad \text{for } t \geq T.$$

Integrating both sides of this inequality from  $T$  to  $t$ , we get

$$x'(t)e^{H(t)} \geq x'(T)e^{H(T)} - \int_T^t e^{H(s)} ds = - \int_T^t e^{H(s)} ds,$$

and therefore,

$$x'(t) \geq - \frac{\int_T^t e^{H(s)} ds}{e^{H(t)}} \quad \text{for } t \geq T.$$

Integrate both sides of this inequality from  $T$  to  $T_1$  to obtain

$$\begin{aligned} x(T_1) &\geq x(T) - \int_T^{T_1} \frac{\int_T^t e^{H(s)} ds}{e^{H(t)}} dt \\ &\geq 1 - \int_T^\infty \frac{\int_T^t e^{H(s)} ds}{e^{H(t)}} dt \geq 1 - \int_T^\infty \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}} dt. \end{aligned}$$

From this estimation and (6) it follows that  $x(T_1) > 1/2$ , which contradicts the assumption that  $x(T_1) = 1/2$ . Hence, we see that  $x(t) > 1/2$  for  $t \geq T$ .

We therefore conclude that  $\Gamma_{(4)}^+(T, (1, 0))$  stays in the region

$$\{(x, y) \in \mathbb{R}^2 : 1/2 < x \leq 1, y \leq 0 \text{ and } V(x, y) \leq 2\}$$

for all future time and the origin of  $(P)$  is not attractive.

*Sufficiency.* As mentioned in the preceding section,  $v(t)$  has the limiting value  $v_0 \geq 0$  for any solution  $(x(t), y(t))$  of (4). We prove that if conditions (1) and (2) are satisfied, then  $v_0$  have to be zero for any solution of (4). This means that the origin of  $(P)$  is attractive. Condition (1) implies that  $H(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

The proof is by contradiction. Suppose that  $v_0$  is positive. Then, there exist a point  $\mathbf{x}_0 \in A$  and a time  $t_0 \geq 0$  such that the positive orbit  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  remains in the annulus  $A$  for all future time. Will  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  keep rotating around the origin  $(0, 0)$ ? From Lemma 2, we see that  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  cannot stay in  $(Q_1 \cup Q_3) \cap A$ , and it inevitably enters  $(Q_4 \cup Q_2) \cap A$ .

If  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  does not rotate, then we can find out a point  $\mathbf{x}_1 \in (Q_4 \cup Q_2) \cap A$  and a time  $T \geq t_0$  so that  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  passes through  $\mathbf{x}_1$  at  $T$  and remains in  $(Q_4 \cup Q_2) \cap A$  afterwards. From the uniqueness of solutions of (4) to initial value problems, we see that the positive orbit  $\Gamma_{(4)}^+(T, \mathbf{x}_1)$  is contained in  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$ . There are two cases that should be considered: (i)  $\Gamma_{(4)}^+(T, \mathbf{x}_1)$  remains in  $Q_4 \cap A$  for all the future and (ii)  $\Gamma_{(4)}^+(T, \mathbf{x}_1)$  remains in  $Q_2 \cap A$  for all the future. We consider only the former, because the latter is carried out in the same way.

Let  $(x(t), y(t))$  be the solution of (4) corresponding to  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$ . Since  $x'(t) \leq 0$  for  $t \geq T$ , there exists a  $c \in \mathbb{R}$  with  $0 \leq c < \pi$  such that  $x(t) \searrow c$  as  $t \rightarrow \infty$ . Recall that  $v(t) = 1 - \cos x(t) + y^2(t)/2 \searrow v_0$  as  $t \rightarrow \infty$ . Hence, it follows that

$$\frac{1}{2}y^2(t) \rightarrow v_0 - 1 + \cos c \quad \text{as } t \rightarrow \infty.$$

Note that  $1 - \cos c \leq v_0$ . If  $1 - \cos c < v_0$ , then we can choose a  $T_2 \geq T$  so that

$$y^2(t) > v_0 - 1 + \cos c > 0 \quad \text{for } t \geq T_2.$$

Hence, we have

$$v'(t) = -h(t)y^2(t) \leq -(v_0 - 1 + \cos c)h(t)$$

for  $t \geq T_2$ . Integrating this inequality from  $T_2$  to  $t$ , we obtain

$$v_0 - v(T_2) < v(t) - v(T_2) \leq -(v_0 - 1 + \cos c) \int_{T_2}^t h(s) ds,$$

which tends to  $-\infty$  as  $t \rightarrow \infty$ . This is a contradiction. Thus, we see that  $1 - \cos c = v_0$ , namely,  $c = \alpha = \arccos(1 - v_0)$ . We therefore conclude that  $(x(t), y(t)) \rightarrow (\alpha, 0)$  as  $t \rightarrow \infty$ . In other words,  $\Gamma_{(4)}^+(T, \mathbf{x}_1)$  approaches  $(\alpha, 0)$ , which is the intersection of the symmetric oval  $V(x, y) = v_0$  and the positive  $x$ -axis.

Let  $\beta = \min\{\sin \alpha, \sin x(T)\}$ . Since  $0 < \alpha < x(t) \leq x(T) < \pi$  for  $t \geq T$ , we have

$$(x'(t)e^{H(t)})' = -\sin x(t)e^{H(t)} \leq -\beta e^{H(t)}$$

for  $t \geq T$ . Hence, we obtain

$$\begin{aligned} x'(t)e^{H(t)} &\leq x'(t)e^{H(t)} - y(T)e^{H(T)} \\ &= x'(t)e^{H(t)} - x'(T)e^{H(T)} \leq -\beta \int_T^t e^{H(s)} ds \end{aligned}$$

for  $t \geq T$ . Since  $h(t) \geq 0$  for  $t \geq 0$ , it follows that  $\int_0^t e^{H(s)} ds \rightarrow \infty$  as  $t \rightarrow \infty$ . Hence, there exists a  $T_3 > T$  such that

$$\int_T^t e^{H(s)} ds = \int_0^t e^{H(s)} ds - \int_0^T e^{H(s)} ds > \frac{1}{2} \int_0^t e^{H(s)} ds \quad \text{for } t \geq T_3.$$

Using this estimation, we obtain

$$x'(t)e^{H(t)} < -\frac{\beta}{2} \int_0^t e^{H(s)} ds \quad \text{for } t \geq T_3,$$

namely,

$$x'(t) < -\frac{\beta}{2} \frac{\int_0^t e^{H(s)} ds}{e^{H(t)}} \quad \text{for } t \geq T_3.$$

Integrating this inequality from  $T_3$  to  $t$ , we get

$$-\pi < \alpha - x(T_3) < x(t) - x(T_3) < -\frac{\beta}{2} \int_{T_3}^t \frac{\int_0^u e^{H(s)} ds}{e^{H(u)}} du$$

for  $t \geq T_3$ . This contradicts condition (2). Thus,  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  have to keep rotating around the origin  $(0, 0)$ .

Since  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  turns around the origin while remaining in the annulus  $A$ , it crosses with the  $y$ -axis, the straight lines  $y = (\tan \varepsilon)x$  and  $y = (\tan(\pi - \varepsilon))x$  infinitely many times, where  $\varepsilon$  is any number satisfying

$$(7) \quad 0 < \varepsilon < \frac{\pi - \gamma_0}{2}$$

( $\gamma_0$  is the number given in condition (1)). Let  $(r(t), \theta(t))$  be the solution of (5) corresponding to  $(x(t), y(t))$ . Then, there exist four divergent sequences  $\{\tau_n\}$ ,  $\{t_n\}$ ,  $\{\sigma_n\}$  and  $\{s_n\}$  with  $t_0 \leq \tau_n < t_n < \sigma_n < s_n$  such that  $\theta(\tau_n) = 3\pi/2$ ,  $\theta(t_n) = \pi - \varepsilon$ ,  $\theta(\sigma_n) = \pi/2$  and  $\theta(s_n) = \varepsilon$ . Recall that

$$r^2(t)\theta'(t) = -x(t) \sin x(t) - y^2(t) - h(t)x(t)y(t) \leq 0$$

as long as  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  is in  $(Q_1 \cup Q_3) \cap A$ . This means that  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  moves clockwise in  $(Q_1 \cup Q_3) \cap A$ . However, since  $\theta'(t)$  is not necessarily negative in  $(Q_2 \cup Q_4) \cap A$ , the form of  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  may be not so simple. The point in the set  $\{t \in (\tau_n, \sigma_n) : \theta(t) = \pi - \varepsilon\}$  might not be only one. In such a case, we should select the supremum of all  $t \in (\tau_n, \sigma_n)$  for which  $\theta(t) \geq \pi - \varepsilon$  as the point  $t_n$ . Then, we have

$$\varepsilon < \theta(t) < \pi - \varepsilon \quad \text{for } t_n < t < s_n.$$

Since the closed curve  $V(x, y) = v_0$  is a symmetric oval, it intersects with the half-line  $\theta = \varepsilon$  at only one point. Let  $\delta(\varepsilon)$  be the  $y$ -component of the intersection. Since  $\Gamma_{(4)}^+(t_0, \mathbf{x}_0)$  does not go out of the annulus  $A$ , we see that  $y(t) > \delta$  for  $t_n \leq t \leq s_n$ . Hence, we obtain

$$v'(t) = -h(t)y^2(t) < -h(t)\delta^2 \quad \text{for } t_n \leq t \leq s_n.$$

Needless to say,  $v'(t) \leq 0$  otherwise.

Suppose that there exists an  $N \in \mathbb{N}$  such that  $s_n - t_n \geq \gamma_0$  for  $n \geq N$ . Then, we can estimate that

$$v(s_n) - v(t_n) < -\delta^2 \int_{t_n}^{s_n} h(t) dt \leq -\delta^2 \int_{t_n}^{t_n + \gamma_0} h(t) dt$$

for  $n \geq N$  and  $v(t_{n+1}) - v(s_n) \leq 0$  for  $n \in \mathbb{N}$ . Adding these two evaluations, we obtain

$$v(t_{n+1}) - v(t_n) < -\delta^2 \int_{t_n}^{t_n + \gamma_0} h(t) dt \quad \text{for } n \geq N.$$

This inequality yields that

$$v_0 - v(t_N) \leq v(t_{n+1}) - v(t_N) < -\delta^2 \sum_{i=N}^n \int_{t_i}^{t_i + \gamma_0} h(t) dt.$$

This contradicts condition (1), because  $t_n$  tends to  $\infty$  as  $n \rightarrow \infty$ . Thus, there exists a sequence  $\{n_k\}$  with  $n_k \in \mathbb{N}$  and  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$  such that

$$(8) \quad s_{n_k} - t_{n_k} < \gamma_0.$$

Since

$$\begin{aligned}\theta'(t) &\geq -\frac{1}{r(t)}|\sin(r(t)\cos\theta(t))||\cos\theta(t) - \sin^2\theta(t) - h(t)|\sin\theta(t)||\cos\theta(t)| \\ &\geq -\cos^2\theta(t) - \sin^2\theta(t) - h(t) \geq -1 - h(t)\end{aligned}$$

for  $t \geq t_0$ , it follows from (8) that

$$\begin{aligned}\varepsilon - (\pi - \varepsilon) &= \theta(s_{n_k}) - \theta(t_{n_k}) \\ &\geq -(s_{n_k} - t_{n_k}) - \int_{t_{n_k}}^{s_{n_k}} h(t)dt > -\gamma_0 - \int_{t_{n_k}}^{s_{n_k}} h(t)dt\end{aligned}$$

for each  $k \in \mathbb{N}$ . Hence, from (7) it turns out that

$$\int_{t_{n_k}}^{s_{n_k}} h(t)dt > \pi - \gamma_0 - 2\varepsilon > 0 \quad \text{for } k \in \mathbb{N}.$$

Repeating the above-mentioned argument with this estimation, we obtain

$$v(s_{n_k}) - v(t_{n_k}) < -\delta^2 \int_{t_{n_k}}^{s_{n_k}} h(t)dt < -\delta^2(\pi - \gamma_0 - 2\varepsilon)$$

and  $v(t_{n_{k+1}}) - v(s_{n_k}) \leq 0$  for  $k \in \mathbb{N}$ . Hence, we have

$$v(t_{n_{k+1}}) - v(t_{n_k}) < -\delta^2(\pi - \gamma_0 - 2\varepsilon) \quad \text{for } k \in \mathbb{N},$$

and therefore,

$$v_0 - v(t_0) \leq \sum_{k=1}^{\infty} (v(t_{n_{k+1}}) - v(t_{n_k})) = -\infty.$$

This contradicts the assumption that  $v_0$  is positive.

The proof of Theorem 1 is thus complete.  $\square$

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