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η-UMBILICAL HYPERSURFACES IN $P_2\mathbb{C}$ AND $H_2\mathbb{C}$

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Abstract. We characterize totally η-umbilical hypersurfaces in $P_2\mathbb{C}$ or $H_2\mathbb{C}$ by using the structure Jacobi operator or the Ricci operator.

1. Introduction

Let $(\widetilde{M}, J, \widetilde{g})$ be an n-dimensional Kähler manifold with a Kähler structure $(J, \widetilde{g})$ and let M be an orientable real hypersurface in $\widetilde{M}$ with a unit normal vector N on M. Then the *Reeb vector field* $\xi = -JN$ plays a fundamental role in real hypersurfaces in a Kähler manifold. In particular for a complex projective space $P_n\mathbb{C}$, Cecil and Ryan [1] proved that Hopf hypersurfaces (with ξ a principal curvature vector field) are realized as tubes over certain submanifolds in $P_n\mathbb{C}$, provided the rank of their focal maps is constant. In the geometry of hypersurfaces, the structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ (along the Reeb flow) has many interesting implications (cf. [2], [3], [4]). Recently, Ivey and Ryan [6] showed that there are no real hypersurfaces whose structure Jacobi operator vanishes in $P_2\mathbb{C}$ or $H_2\mathbb{C}$. In higher dimensions, it was proved by the present authors [5].

From Codazzi equation, we can show that there are no totally umbilical real hypersurfaces in a non-flat complex space form. In this context, some authors studies the so called *totally η-umbilical structure* in a real hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$, that is, its shape operator A is represented by

$$A = \lambda I + \mu\eta \otimes \xi$$

for $\lambda, \mu \in \mathbb{R}$. Indeed, totally η-umbilical real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ are classified in [1], [14] or [10]. They are realized as a geodesic hypersphere in $P_n\mathbb{C}$ and a horosphere, a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$ in $H_n\mathbb{C}$. By Gauss equation we find that R_ξ is proportional to I (identity transformation) on the orthogonal complement space $\xi^\perp$ of ξ for such spaces. In the present note, we characterize totally η-umbilical hypersurfaces in $P_2\mathbb{C}$ or $H_2\mathbb{C}$ by using the structure Jacobi operator or the Ricci operator.

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2. Preliminaries

All manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented. At first, we review the fundamental facts on a real hypersurface of a n-dimensional complex space form $\widetilde{M}_n(c)$ with constant holomorphic sectional curvature $4c$. Let M be an orientable real hypersurface of $\widetilde{M}_n(c)$ and let N be a unit normal vector on M. We denote by $\widetilde{g}$ and J a Kähler metric tensor and its Hermitian structure tensor, respectively. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \tag{2.1}$$

where ϕ is a (1,1)-type tensor field, η is a 1-form and ξ is a unit vector field on M, which is called *Reeb vector field*. The induced Riemannian metric on M is denoted by g. Then by properties of $(\widetilde{g}, J)$ we see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, from (2.1) it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \end{aligned} \tag{2.2}$$

for any vector fields X and Y tangent to M. From (2.2), we have

$$\phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(X) = g(X, \xi)$$

The Gauss and Weingarten formula for M are given as

$$\begin{aligned} &\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \\ &\widetilde{\nabla}_X N = -AX \end{aligned}$$

for any tangent vector fields X, Y, where $\widetilde{\nabla}$ and ∇ denote the Levi-Civita connections of $(\widetilde{M}_n(c), \widetilde{g})$ and (M, g), respectively, A is the shape operator. From (2.1) and $\widetilde{\nabla} J = 0$, we then obtain

$$\begin{aligned} (\nabla_X \phi)Y &= \eta(Y)AX - g(AX, Y)\xi, \\ \nabla_X \xi &= \phi AX. \end{aligned} \tag{2.3}$$

Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY. \end{aligned} \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \tag{2.5}$$

From (2.4) together with (2.2) the Ricci operator S is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + HAX - A^2 X, \tag{2.6}$$

where $H =$ trace of A. Also, from (2.4) the structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$, which is a self-adjoint operator, is given by

$$R_\xi(X) = c(X - \eta(X)\xi) + g(A\xi, \xi)AX - \eta(AX)A\xi. \tag{2.7}$$

Now we consider the vector field $U = \nabla_\xi \xi$ and denote $\alpha_m = \eta(A^m\xi)$. Then from (2.2) and (2.3) we easily observe that

$$g(U,\xi) = 0,\ \ g(U, A\xi) = 0,$$
$$\|U\|^2 = g(U,U) = \alpha_2 - \alpha_1^2.$$

Then we see at once that ξ is a principal curvature vector field if and only if $\alpha_2 - \alpha_1^2 = 0$. Moreover, at that time, α_1 is constant (cf. [8], [12]).

3. Real hypersurfaces satisfying $S\phi = \phi S$ and $R_\xi\phi = \phi R_\xi$

In [2] the first author studies a real hypersurface M in a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$, which satisfies $S\phi = \phi S$ and at the same time $R_\xi\phi = \phi R_\xi$. Unfortunately, there contain some incorrect arguments. So, in this section we add the correction of it.

From the condition $S\phi = \phi S$, we have

$$H(A\phi - \phi A)X - (A^2\phi - \phi A^2)X = 0. \tag{3.1}$$

Put $X = \xi$ in (3.1) to get $\phi A^2\xi = HU$. Applying ϕ, then we have

$$A^2\xi = HA\xi + (\alpha_2 - \alpha_1 H)\xi. \tag{3.2}$$

Using (2.7) the commutativity $R_\xi\phi = \phi R_\xi$ implies that

$$\alpha_1(A\phi - \phi A)X = -g(U,X)A\xi - \eta(AX)U. \tag{3.3}$$

Put $X = A\xi$ in (3.3) to get $\alpha_1 AU = \alpha_1\phi A^2\xi - \alpha_2 U$. Using (3.2) we get

$$\alpha_1 AU = (\alpha_1 H - \alpha_2)U. \tag{3.4}$$

Now, we shall prove that M is a Hopf hypersurface. Put $\Omega = \{p \in M : (\alpha_2 - \alpha_1^2)(p) \neq 0\}$. Suppose that Ω is non-empty and proceed our arguments in Ω. Then from (3.4) we can see that $\alpha_1 \neq 0$ in Ω. Use the relation:

$$(A^2\phi - \phi A^2)X = A(A\phi - \phi A)X + (A\phi - \phi A)AX.$$

Then, from (3.1) and (3.3) we have

$$\alpha_1 H(A\phi - \phi A)X = -\Big(g(U,X)A^2\xi + \eta(AX)AU + g(U,AX)A\xi + \eta(A^2X)U\Big). \tag{3.5}$$

Using (3.2), (3.3) and (3.4) in (3.5), then we obtain

$$(\alpha_2 - \alpha_1 H)\Big(\alpha_1 g(U,X)\xi - \eta(AX)U - g(U,X)A\xi + \alpha_1\eta(X)U\Big) = 0. \tag{3.6}$$

Put $X = U$ in (3.6) to get

$$(\alpha_2 - \alpha_1 H)(\alpha_2 - \alpha_1^2)A\xi = (\alpha_2 - \alpha_1 H)(\alpha_2 - \alpha_1^2)\alpha_1\xi, \tag{3.7}$$

which yields that $\alpha_2 - \alpha_1 H = 0$ in Ω. Hence, from (3.2) and (3.4) we obtain

Lemma 3.1. *In* Ω,

$$AU = 0 \tag{3.8}$$

and

$$A^2\xi = HA\xi. \tag{3.9}$$

Differentiating $\alpha_1 = g(A\xi, \xi)$ covariantly, using 2nd equation of (2.3) and (3.8) we easily get

$$g((\nabla_X A)\xi, \xi) = d\alpha_1(X), \tag{3.10}$$

where d denotes the exterior differentiation. Since $U = \phi A\xi$, by using 1st equation of (2.3), (2.5) and (3.8), we have

$$\nabla_\xi U = \alpha_1 A\xi - \alpha_2\xi + \phi \operatorname{grad}(\alpha_1), \tag{3.11}$$

where $\operatorname{grad}(\alpha_1)$ denotes the gradient vector field of α_1.

Differentiating (3.8) covariantly along Ω, then by using (2.5) and (3.11) we have

$$(\nabla_U A)\xi = -c\phi U - \alpha_1 A^2\xi + \alpha_2 A\xi - A\phi \operatorname{grad}(\alpha_1). \tag{3.12}$$

Also, if we differentiate (3.9) covariantly along Ω, then together with (2.3) we get

$$\begin{aligned} &g(A\xi, (\nabla_X A)Y) + g((\nabla_X A)\xi, AY) + g(\phi AX, A^2Y) \\ &= dH(X)g(A\xi, Y) + Hg((\nabla_X A)\xi, Y) + Hg(\phi AX, AY). \end{aligned} \tag{3.13}$$

From (3.13), using Codazzi equation (2.5), then it follows that

(3.14)
$$\begin{aligned} &c\big(\eta(X)g(A\xi, \phi Y) - \eta(Y)g(A\xi, \phi X) - 2\alpha_1 g(\phi X, Y)\big) \\ &\quad + g((\nabla_X A)\xi, AY) - g((\nabla_Y A)\xi, AX) + g(\phi AX, A^2Y) - g(\phi AY, A^2X) \\ &= dH(X)g(A\xi, Y) - dH(Y)g(A\xi, X) + Hg((\nabla_X A)\xi, Y) - Hg((\nabla_Y A)\xi, X) \\ &\quad + 2Hg(\phi AX, AY) \end{aligned}$$

for any vector fields X and Y tangent to Ω. Putting $X = U$ and making use of (2.5) and (3.8), then we have

$$g((\nabla_U A)\xi, AY) = c\big(2(\alpha_1 - H)g(\phi U, Y) - \eta(Y)g(U, U)\big) + dH(U)g(A\xi, Y). \tag{3.15}$$

Hence, from (3.12) and (3.15), we have

$$\begin{aligned} &- cg(\phi U, AY) + d\alpha_1(\phi A^2Y) \\ &= c\big(2(\alpha_1 - H)g(\phi U, Y) - \eta(Y)g(U, U)\big) + dH(U)g(A\xi, Y), \end{aligned} \tag{3.16}$$

where we have used $(\alpha_2 - \alpha_1 H) = 0$. If we put $Y = \xi$ in (3.16), then use (3.9) to obtain

$$\alpha_1 dH(U) - Hd\alpha_1(U) = 2c(\alpha_2 - {\alpha_1}^2).$$

Putting $Y = A\xi$ in (3.16) and using (3.9) again, then we obtain

$$H(\alpha_1 dH(U) - Hd\alpha_1(U)) = c(3\alpha_1 - H)(\alpha_2 - {\alpha_1}^2).$$

From the above two equations, we have $(H - \alpha_1)(\alpha_2 - \alpha_1^2) = 0$. But, since $\alpha_2 = \alpha_1 H$, we have $\alpha_2 - \alpha_1^2 = 0$. Eventually, we have shown that M is a Hopf hypersurface.

Moreover, from (3.3) we have $\alpha_1(A\phi - \phi A) = 0$. Therefore, due to Okumura [13] and Motiel-Romero [11] we have (cf. [2])

Theorem 3.2. *Let M be a real hypersurface of $P_n\mathbb{C}$ and $H_n\mathbb{C}$. If M satisfies $\phi S = S\phi$ and $\phi R_\xi = R_\xi\phi$ at the same time, then $A\xi = 0$ or M is locally congruent to one of the so-called real hypersurfaces of type (A).*

4. 3-dimensional real hypersurfaces

Let M be a real hypersurface in $P_2\mathbb{C}$ and $H_2\mathbb{C}$. Then, since the Weyl curvature tensor vanishes in dimension 3, we have

$$R(X,Y)Z = \rho(Y,Z)X - \rho(X,Z)Y + g(Y,Z)SX - g(X,Z)SY - r/2\big(g(Y,Z)X - g(X,Z)Y\big) \tag{4.1}$$

for any smooth vector fields X, Y, Z on M, where $\rho(X,Y) = g(SX,Y)$ and r denotes the scalar curvature. From (4.1) we get

$$R_\xi(X) = \rho(\xi,\xi)X - \rho(X,\xi)\xi + SX - \eta(X)S\xi - r/2(X - \eta(X)\xi). \tag{4.2}$$

It follows from (4.2) that

$$(R_\xi\phi - \phi R_\xi)(X) = (S\phi - \phi S)(X) - \rho(\phi X,\xi)\xi + \eta(X)\phi S\xi.$$

Then we can easily show the following result.

Proposition 4.1. *For a 3-dimensional real hypersurface M of $P_2\mathbb{C}$ and $H_2\mathbb{C}$, the following four conditions are equivalent:*
- *$S\phi = \phi S$;*
- *M is pseudo-Einstein (or η-Einstein), which means $S = aI + b\eta \otimes \xi$ for smooth functions a and b;*
- *$R_\xi\phi = \phi R_\xi$ and $S\xi = \sigma\xi$;*
- *$R_\xi = f(I - \eta \otimes \xi)$ and $S\xi = \sigma\xi$, where f, σ are smooth functions.*

Then, using Theorem 2 we have

Theorem 4.2. *Let M be a real hypersurface in $P_2\mathbb{C}$ and $H_2\mathbb{C}$ which satisfies one of four in Proposition 4.1. Then M is locally congruent to a geodesic hypersphere in $P_2\mathbb{C}$ and a horosphere, a geodesic hypersphere, a tube over a complex hyperbolic line $H_1\mathbb{C}$ in $H_2\mathbb{C}$, or a Hopf hypersurface with $A\xi = 0$ in $P_2\mathbb{C}$ and $H_2\mathbb{C}$.*

The *Reeb section* is defined by the plane spanned by $\{\xi, X\}$ for a unit vector X orthogonal to ξ and the *Reeb sectional curvature* is defined by $K(X,\xi) = g(R(X,\xi)\xi, X)$. Then, we have

Corollary 4.3. *Let M be a real hypersurface of $P_2\mathbb{C}$ or $H_2\mathbb{C}$ whose Ricci operator S satisfies $S\xi = \sigma\xi$ for a function σ. If the Reeb sectional curvature is pointwise constant, then $A\xi = 0$, or otherwise M is locally congruent to a geodesic hypersphere in $P_2\mathbb{C}$ and a horosphere, a geodesic hypersphere or a tube over a complex hyperbolic line $H_1\mathbb{C}$ in $H_2\mathbb{C}$.*

We close this paper by the following remark.

Remark 4.4. Using Cecil-Ryan's fundamental idea of tube construction in $P_n\mathbb{C}$ ([1]), Kimura and Maeda [9] found real hypersurfaces in $P_n\mathbb{C}$ with $A\xi = 0$, provided the rank of their focal maps is constant. Indeed, they are realized as tubes of certain complex submanifolds in $P_n\mathbb{C}$ of radius $\pi/4$. Very recently, Ivey and Ryan [7] constructed such a real hypersurface in $H_2\mathbb{C}$ with $A\xi = 0$ by a pair of Legendre curves in the unit 3-sphere.

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