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On Normal Left Lie Subloops of Homogeneous Left Lie Loops

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We show that any homogeneous left loop is characterized by the homogeneous system associated with it. By applying this result to the theory of normal subsystems of analytic homogeneous systems developed in [5], we show the fundamental theorems concerning normal left Lie subloops of homogeneous left Lie loops, an exact generalization of the theory of Lie groups and Lie algebras.

§1. Homogeneous left Lie loops and analytic homogeneous systems

In the following, we consider homogeneous left Lie loops on a connected analytic manifold G (cf., e.g. [12]), where the underlying topology of G is assumed to be second countable.

DEFINITION 1.1. A *homogeneous left Lie loop* (G, μ) is an analytic binary operation $\mu: G \times G \to G$ on G with the two-sided identity element $e \in G$ satisfying;

(1.1.1) Each left translation

$$L_x: G \longrightarrow G;\ L_x y := \mu(x, y), \quad y \in G,$$

is an analytic diffeomorphism of G.

(1.1.2) The *inversion* $J: G \to G$; $J(x) = x^{-1}$ is an analytic diffeomorphism of G, where $x^{-1} := L_x^{-1} e$ for $x \in G$.

(1.1.3) The *left inverse property* is provided for μ, i.e.,

$$L_x^{-1} = L_{x^{-1}}$$

holds for each $x \in G$.

(1.1.4) Each *left inner mapping* $L_{x,y} := L_{\mu(x,y)}^{-1} \cdot L_x \cdot L_y$ is an automorphism of (G, μ).

By the same way as Lemma 1.8 in [2], we can show the following;

PROPOSITION 1.2. *The left inner mappings of a homogeneous left Lie loop satisfy the following equalities*:

(1.2.1) $$L_{x,y} = L_{y,\mu(y^{-1},x^{-1})}$$

(1.2.2) $$L_{x,y}^{-1} = L_{y^{-1},x^{-1}}.$$

We know that a ternary operation $\eta: G \times G \times G \to G$;

(1.3) $$\eta(x, y, z) := L_x \mu(L_x^{-1} y, L_x^{-1} z)$$

is associated with each μ (cf. [4]). In the following, we show that homogeneous left Lie loops are characterized by the associated ternary operations. In fact, this is a purely algebraic result:

THEOREM 1.3. *Let (G, μ) be an abstract homogeneous left loop on a set G with the identity element e. Then the associated ternary operation η on G given by* (1.3) *satisfies the relations*

(1.4.1) $$\eta(x, x, y) = y$$

(1.4.2) $$\eta(x, y, x) = y$$

(1.4.3) $$\eta(x, y, \eta(y, x, z)) = z$$

(1.4.4) $$\eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)).$$

In this case, the multiplication μ is expressed by η as follows;

(1.5) $$\mu(x, y) = \eta(e, x, y).$$

Conversely, assume that a given ternary operation η on a set G satisfies the relations (1.4.1)–(1.4.4). *Then, for any choice of the element $e \in G$, the multiplication μ defined by* (1.5) *gives a homogeneous left loop on G with the identity element e, and the associated ternary operation given by* (1.3) *coincides with η.*

PROOF. Assume that (G, μ) is a homogeneous left loop, that is, the multiplication μ on a set G satisfies the relations (1.1.1)–(1.1.4) in which the word 'analytic diffeomorphism' is replaced by 'bijection'. Then, by the definition (1.3) of η, it is evident that the relations (1.4.1) and (1.4.2) are satisfied. By using Proposition 1.2, we get

(1.6) $$\begin{aligned}\eta(x, y, z) &= L_y \cdot L_{x,\mu(x^{-1},y)} \cdot L_x^{-1} z \\ &= L_y \cdot L_{y^{-1},x} \cdot L_x^{-1} z\end{aligned}$$

and

$$\begin{aligned}\eta(x, y, \eta(y, x, z)) &= L_y \cdot L_{y^{-1},x} \cdot L_{x^{-1},y} \cdot L_y^{-1} z \\ &= z,\end{aligned}$$

which shows the relation (1.4.3). From (1.6) we obtain

$$L_y^{-1}\eta(x, y, v) = L_y \cdot L_{y^{-1},x} \cdot L_x^{-1}v$$

and analogous one for w instead of v. These equalities imply

$$\begin{aligned}\eta(y, \eta(x, y, v), \eta(x, y, w)) &= L_y \cdot L_{y^{-1},x}\mu(L_x^{-1}v, L_x^{-1}w)\\ &= \eta(x, y, \eta(x, v, w)).\end{aligned}$$

By using the last equality repeatedly, we can show the relation (1.4.4). In fact, we get

$$\begin{aligned}\eta(x, y, \eta(u, v, w)) &= \eta(x, y, \eta(x, u, \eta(u, x, \eta(u, v, w))))\\ &= \eta(y, \eta(x, y, u), \eta(x, y, \eta(x, \eta(u, x, v), \eta(u, x, w))))\\ &= \eta(y, \eta(x, y, u), \eta(y, \eta(x, y, \eta(u, x, v)), \eta(x, y, \eta(u, x, w))))\\ &= \eta(\eta(x, y, u), \eta(y, \eta(x, y, u), \eta(x, y, \eta(u, x, v))), \eta(y, \eta(x, y, u), \eta(x, y, \eta(u, x, w))))\\ &= \eta(\eta(x, y, u), \eta(x, y, \eta(x, u, \eta(u, x, v))), \eta(x, y, \eta(x, u, \eta(u, x, w))))\\ &= \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)).\end{aligned}$$

Conversely, let η be a ternary operation on G satisfying the relations (1.4.1)–(1.4.4). For any fixed element e of G, define the multiplication μ by (1.5). Then, it is easy to show that μ satisfies (1.1.1), (1.1.2) and (1.1.3) with the identity element e, where

(1.7) $$x^{-1} = \eta(x, e, e).$$

For any fixed u, v in G, denote by $\eta(u, v)$ the mapping from G onto itself given as follows:

(1.8) $$\eta(u, v)w := \eta(u, v, w) \qquad \text{for any } w \in G.$$

Then, any left inner mapping $L_{x,y}$ can be expressed by these maps as follows;

(1.9) $$L_{x,y} = \eta(\mu(x, y), e) \cdot \eta(x, \mu(x, y)) \cdot \eta(e, x)$$

The relation (1.4.4) assures that any map $\eta(x, y)$ is an automorphism of the ternary system (G, η). Therefore, the left inner mapping $L_{x,y}$ given by (1.9) is an automorphism of (G, η) leaving the element e fixed, that is, $L_{x,y}$ is an automorphism of the multiplicative system (G, μ). Thus we see that (G, μ) forms a homogeneous left loop. By (1.4.3) we see that

(1.10) $$\eta(x, y)^{-1} = \eta(y, x).$$

Hence we have

$$\begin{aligned}L_x\mu(L_x^{-1}y, L_x^{-1}z) &= \eta(e, x)\eta(e, \eta(x, e, y), \eta(x, e, z))\\ &= \eta(x, y, z),\end{aligned}$$

that is, the ternary operation associated with this homogeneous left loop (G, μ) is coincident with the given η. q.e.d.

DEFINITION 1.4. A ternary operation η on a set G satisfying the relations (1.4.1)–(1.4.4) is called a *homogeneous system* on G. If (G, μ) is a homogeneous left loop, the homogeneous system η given by (1.3) is said to be *associated with* (G, μ). For a homogeneous system (G, η), the mapping $\eta(u, v): G \to G$ given by (1.8) is called the *displacement* of G *from u to v*. Indeed, the relations (1.4.2) and (1.4.3) show that the displacement $\eta(u, v)$ is a bijection of G onto itself sending u to v, whose inverse map is $\eta(v, u)$.

Again we assume that G is a connected analytic manifold. It is easy to show that the multiplication μ and the inversion J of a homogeneous left loop (G, μ) are analytic if and only if the associated homogeneous system η is analytic. Therefore, Theorem 1.3 implies the following;

COROLLARY 1.5. *Any homogeneous left Lie loop (G, μ) on the manifold G is characterized by the homogeneous system (G, η) associated with it. More precisely, any analytic homogeneous system (G, η) on G with an arbitrarily fixed point $e \in G$ is associated with one and only one homogeneous left Lie loop (G, μ) whose identity element is e.*

COROLLARY 1.6. *Let (G, μ) be a homogeneous left Lie loop on G and η the associated homogeneous system. Then, for any fixed $u \in G$, there corresponds a homogeneous left Lie loop (G, μ_u) with its identity element u, where μ_u is given by*

$$\mu_u(x, y) = \eta(u, x, y). \tag{1.11}$$

Any two homogeneous left Lie loops μ_u and μ_v on G are isomorphic under the displacement $\eta(u, v)$ from u to v.

§2. Tangent Lie triple algebras of homogeneous left Lie loops

DEFINITION 2.1. (Cf. [2], [10], [11], [12]) For an analytic homogeneous system (G, η), the *canonical connection* ∇ is defined to be a linear connection on G given by;

$$(\nabla_X Y)_x := X_x Y - \eta(x, X_x, Y_x) \tag{2.1}$$

for any C^∞-class vector fields X and Y on G. Here, $X_x Y$ denotes the tangent vector at x given by

$$X_x Y := X_x^j \frac{\partial Y^i}{\partial u^j}\bigg|_x \frac{\partial}{\partial u^i}\bigg|_x \quad \text{for } X = X^i \frac{\partial}{\partial u^i},\ Y = Y^i \frac{\partial}{\partial u^i}$$

in any chart $(u^1, u^2, \ldots, u^n)$ around x, and

$$\eta(x, X_x, Y_x) = X_x^j Y_x^k \frac{\partial^2 \eta^i}{\partial v^j \partial w^k}\bigg|_{(x,x,x)} \frac{\partial}{\partial u^i}\bigg|_x$$

for

$$\eta(u, v, w) = (\eta^i(u^1, u^2, \ldots, u^n; v^1, v^2, \ldots, v^n; w^1, w^2, \ldots, w^n)), \qquad i = 1, \ldots, n.$$

The canonical connection of a homogneous left Lie loop (G, μ) is, by definition, the canonical connection of the associated homogeneous system η of (G, μ) (cf. [5-I], [10]). The homogeneous left Lie loop (G, μ) is said to be *geodesic* if the multiplication μ is coincident with the geodesic loop (cf. [1], Sabinin [15]) of ∇ in some neighborhood of the identity element e. Since each displacement of (G, μ) is an affine transformation of ∇, the geodesic loop at any point x in G is coincident with the multiplication μ_x in some neighborhood of x. In fact, (G, μ_x) is a homogeneous left Lie loop isomorphic to (G, μ) under the displacement $\eta(e, x) = L_x$, and the differential of a displacement $\eta(x, y)$ induces the parallel displacement of tangent vectors along the geodesic arc from x to y (if it exists). An analytic homogeneous system (G, η) is said to be *geodesic* if the induced homogeneous left Lie loop $\mu = \mu_e$, given by (1.5) (or (1.11)) for some element e of G (and hence for each element), is geodesic.

Let (G, μ) be a geodesic homogeneous left Lie loop with the canonical connection ∇. The torsion tensor field S and the curvature tensor field R of ∇ give rise to a concept of algebraic structure on the tangent space to G at the identity element e, called the *tangent Lie triple algebra* $\mathfrak{g} = (\mathfrak{g}_S, \mathfrak{g}_R)$, which is defined as follows:

(2.2.1) $$\mathfrak{g}_S\,;\, XY = S_e(X, Y),$$

(2.2.2) $$\mathfrak{g}_R\,;\, [X, Y, Z] = R_e(X, Y)Z,$$

for $X, Y, Z \in \mathfrak{g} = T_e(G)$.

In terms of homogeneous system, we know the following (cf. [5-I], [5-V]): Let $D(\eta)$ denote the group of displacements of an analytic homogeneous system (G, η), Λ_e the isotropy subgroup of $D(\eta)$ at the point e. They are subgroups of the affine transformation group Aff (∇) of the canonical connection ∇. By using (1.9), we can show that the group Λ_e is coincident with the left inner mapping group of (G, μ). Let $K = \overline{\Lambda_e}$ be the closure of Λ_e in Aff(∇). Then K is a closed Lie subgroup of Aff(∇) and the semi-direct product $A = G \times K$ forms a Lie group called the enveloping group of (G, η) by K (cf. [2], [9]). Then, G can be regarded as the reductive homogeneous space A/K with the canonical connection ∇ of 2nd kind (cf. [14]), and we may call $\mathfrak{g} = (\mathfrak{g}_S, \mathfrak{g}_R)$ above the *tangent Lie triple algebra* *of* (G, η) *at* $e \in G$.

In fact, the following theorem is well-known (cf., e.g. [2], [11], [12]):

THEOREM 2.2. *The tangent Lie triple algebra* $\mathfrak{g}=(\mathfrak{g}_S, \mathfrak{g}_R)=\{\mathfrak{g}; XY, [X, Y, Z]\}$ *of any analytic homogeneous system* (G, η) *at any point* e *satisfies the following axiom of Lie triple algebra* ([2]) (general Lie triple system of K. Yamaguti [16]):

$$XY=-YX \tag{2.3.1}$$

$$[X, Y, Z]=-[Y, X, Z] \tag{2.3.2}$$

$$\mathfrak{S}_{X,Y,Z}\{[X, Y, Z]+(XY)Z\}=0 \tag{2.3.3}$$

$$\mathfrak{S}_{X,Y,Z}\{[XY, Z, W]\}=0 \tag{2.3.4}$$

$$[X, Y, UV]=[X, Y, U]V+U[X, Y, V] \tag{2.3.5}$$

$$\begin{aligned}[X, Y, [U, V, W]]=[[X, Y, U], V, W]+[U, [X, Y, V], W]\\ +[U, V, [X, Y, W]]\end{aligned} \tag{2.3.6}$$

for any $X, Y, Z, U, V, W \in \mathfrak{g}$, *where* $\mathfrak{S}_{X,Y,Z}$ *denotes the cyclic sum with respect to* X, Y, Z.

THEOREM 2.3. *Let* (G, μ) *and* $(\tilde{G}, \tilde{\mu})$ *be geodesic homogeneous left Lie loops with the tangent Lie triple algebras* $\mathfrak{g}$ *and* $\tilde{\mathfrak{g}}$, *respectively. Let* $\phi: G \rightarrow \tilde{G}$ *be an analytic homomorphism of* (G, μ) *into* $(\tilde{G}, \tilde{\mu})$. *Then the differential map* $d\phi$ *of* ϕ *at the identity element* e *of* G *induces a homomorphism of the tangent Lie triple algebra* $\mathfrak{g}$ *into the tangent Lie triple algebra* $\tilde{\mathfrak{g}}$. *If* G *is simply connected, then every homomorphism of* $\mathfrak{g}$ *into* $\tilde{\mathfrak{g}}$ *is the differential map of one and only one analytic homomorphism of* (G, μ) *into* $(\tilde{G}, \tilde{\mu})$.

PROOF. Let η and $\tilde{\eta}$ be the associated homogeneous systems of (G, μ) and $(\tilde{G}, \tilde{\mu})$, respectively. Then the first half of the theorem follows from Theorem 2 in [5-III]. The remainning part is shown as follows: Let $\Phi: \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ be a homomorphism of the tangent Lie triple algebras. Since the torsions and curvatures of both of the canonical connections are parallel tensor fields, there exists an analytic local affine transformation $\phi: U \rightarrow \tilde{U}$ of the canonical connections ∇ and $\tilde{\nabla}$ such that $d\phi=\Phi$ (cf. e.g. Theorem 7.4, Ch. VI in [13]), where U (resp. $\tilde{U}$) is a neighborhood of the identity element e (resp. $\tilde{e}$). Since the homogeneous left Lie loops G and $\tilde{G}$ are assumed to be geodesic, ϕ induces a local homomorphism of the geodesic loops at the respective identity elements e and $\tilde{e}$, i.e.;

$$\phi\mu(x, y)=\tilde{\mu}(\phi x, \phi y). \tag{2.4}$$

The domain of the local affine transformation ϕ can be uniquely extended on G, for G is simply connected (cf. Theorem 6.1 Ch. VI in [13]). As the maps $\phi: G \rightarrow \tilde{G}$,

μ and $\tilde{\mu}$ are analytic and $G \times G$ is connected, (2.4) holds on whole of $G \times G$. Thus the proof is completed. q.e.d.

DEFINITION 2.4. An analytic homogeneous system (G, η) is said to be *regular* if it is geodesic and the Lie group $K = \overline{A_e}$ is coincident with the holonomy group Ψ_e of the canonical connection ∇ at the origin e. A homogeneous left Lie loop (G, μ) is *regular* if the associated homogeneous system is regular (cf. [5-1I).

In the following we consider regular (hence geodesic) homogeneous left Lie loops.

REMARK 2.5. Let $G = (G, \mu)$ be a connected Lie group. Then the canonical connection is reduced to the $(-)$-connection of E. Cartan, which is geodesic and regular. In this case, the tangent Lie triple algebra $\mathfrak{g}$ is reduced to the Lie algebra $\mathfrak{g} = \{\mathfrak{g};\ XY = [X, Y], [X, Y, Z] = 0\}$ of G.

§3. Normal left Lie subloops and quotient homogeneous left Lie loops

In this section, terminology and notations are refered to the series of papers [4] and [5-I]-[5-V]. In these articles, we have investigated various remarkable properties of analytic homogeneous systems, especially of their normal subsystems. By applying Theorem 1.3 to the results obtained there, we can present some basic results concerning normal left Lie subloops of homogeneous left Lie loops, which are exact generalization of the fundamental theory of Lie subgroups and corresponding Lie subalgebras. Related results have been obtained in [3] for homogeneous Lie loops.

Let G be a connected second countable analytic manifold and η an analytic homogeneous system on G.

DEFINITION 3.1. A *subsystem* $H = (H, \eta_H)$ of (G, η) is a submanifold H of G which is an algebraic bubsystem of (G, η), where η_H denotes the restriction of η to H. For any element x of G, denote by xH the subset $\eta(H, x, H) = \{\eta(u, x, v);\ u, v \in H\}$. A subsystem H is said to be *invariant* if;

(3.1) $$\eta(x, y)xH = yH$$

for any x, y in G. A subsystem H is said to be *normal* if it satisfies the following;

(3.2) $$\eta(xH, yH, zH) = \eta(x, y, z)H$$

for any x, y, z in G. Assume that (G, μ) is a homogeneous left Lie loop and η the associated homogeneous system. By Theorem 1.3, any subsystem $H = (H, \eta_H)$ containing the identity element e induces a homogeneous left Lie loop (H, μ_H) on

H, where μ_H is just equal to the restriction of μ to the submanifold H. We call (H, μ_H) a *left Lie subloop* of (G, μ). In particular, if H is a normal (resp. invariant) subsystem, then the induced left Lie subloop (H, μ_H) is called a *normal* (resp. *invariant*) left Lie subloop of (G, μ).

REMARK 3.2. By the equality (1.9) and (2.1), we can show that any left Lie subloop H of a homogeneous left Lie loop G is invariant if and only if the underlying submanifold H is invariant by the left inner mapping group Λ_e of G. In particular, any Lie subgroup of a Lie group is an invariant left Lie subloop since $\Lambda_e = \{\mathrm{id}\}$ for Lie groups.

DEFINITION 3.3. Let $\mathfrak{g} = \{\mathfrak{g}; XY, [X, Y, Z]\}$ be a Lie triple algebra over a field of characteristic zero. A *Lie triple subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ closed under the bilinear product and the trilinear product of $\mathfrak{g}$, i.e., $\mathfrak{h}\mathfrak{h} \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A Lie triple subalgebra $\mathfrak{h}$ is said to be *invariant* if it satisfies;

$$[\mathfrak{g}, \mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}, \tag{3.3}$$

and $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$ if;

$$\mathfrak{g}\mathfrak{h} \subset \mathfrak{h} \quad \text{and} \quad [\mathfrak{g}, \mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}. \tag{3.4}$$

If $\mathfrak{h}$ is a normal Lie triple subalgebra of $\mathfrak{g}$, the *quotient Lie triple algebra* $\mathfrak{g}/\mathfrak{h}$ is well defined in a natural way. Cf. [2], [3], [5], [6], [7], [8] and Yamaguti [16], [17].

THEOREM 3.4. *Let $G = (G, \mu)$ be a regular homogeneous left Lie loop with the tangent Lie triple algebra $\mathfrak{g}$. If H is an invariant left Lie subloop of G, then the tangent Lie triple algebra $\mathfrak{h}$ of H is an invariant Lie triple subalgebra of $\mathfrak{g}$. Every invariant Lie triple subalgebra of $\mathfrak{g}$ is the tangent Lie triple algebra of one and only one invariant left Lie subloop of (G, μ).*

PROOF. Let (G, η) be the associated analytic homogeneous system of (G, μ). If H is an invariant left Lie subloop of G, then the associated homogeneous system η_H is an invariant subsystem of η. By applying Theorem 5 in [5-I], we get the conclusion of the theorem. q.e.d.

THEOREM 3.5. *Let (G, μ) be a geodesic homogeneous left Lie loop, (H, μ_H) a closed normal left Lie subloop and η the associated homogeneous system. Let $\tilde{G}$ be the collection of all $xH = \eta(H, x, H)$, $x \in G$, and define $\tilde{\mu}: \tilde{G} \times \tilde{G} \to \tilde{G}$ by*

$$\tilde{\mu}(xH, yH) := \mu(x, y)H. \tag{3.5}$$

Then $(\tilde{G}, \tilde{\mu})$ forms a homogeneous left Lie loop and the natural projection $\pi: G \to \tilde{G}$ sending $x \in G$ to xH induces an analytic homomorphism of (G, μ) onto $(\tilde{G}, \tilde{\mu})$ with the kernel (H, μ_H).

PROOF. By Theorem 1 in [5-III] applied for the associated homogeneous system η, we see that $\tilde{G}$ has an analytic structure and that $(\tilde{G}, \tilde{\eta})$ forms an analytic homogeneous system, where $\tilde{\eta}$ is given by:

$$\tilde{\eta}(xH, yH, zH) := \eta(x, y, z)H \qquad \text{for } x, y, z \in G. \tag{3.6}$$

Moreover, the projection π is an analytic homomorphism of homogeneous systems η and $\tilde{\eta}$ with $H = \pi^{-1}(eH)$. Since $\tilde{\mu}$ defined by (3.5) is the homogeneous left Lie loop induced from $\tilde{\eta}$, $(\tilde{G}, \tilde{\mu})$ is a homogeneous left Lie loop on the analytic manifold $\tilde{G}$ with the identity element eH, and the theorem is proved. q.e.d.

DEFINITION 3.6. The homogeneous left Lie loop $(\tilde{G}, \tilde{\mu})$ above is called the *quotient homogeneous left Lie loop* of (G, μ) modulo (H, μ_H) and denoted by $\tilde{G} = G/H$.

Theorem 3 in [5-III] asserts the following;

THEOREM 3.7. *Let (G, μ) be a geodesic homogeneous left Lie loop, H a closed normal left Lie subloop. Then the tangent Lie triple algebra $\mathfrak{h}$ of H is an ideal of the tangent Lie triple algebra $\mathfrak{g}$ of (G, μ), and the tangent Lie triple algebra $\tilde{\mathfrak{g}}$ of the quotient homogeneous left Lie loop $\tilde{G} = G/H$ is isomorphic to the quotient Lie triple algebra $\mathfrak{g}/\mathfrak{h}$ under the differential map $d\pi$ of the natural projection π, at the identity element e.*

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