Mem. Fac. Sci. Shimane Univ.,
27, pp. 11–15 Dec. 27, 1993

Note on Special Amalgamation for Regular Bands

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(Received September 6, 1993)

Let $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ be a special amalgam of regular bands. In [2], the author showed that the amalgam $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ is strongly embedded in a regular band whose structure semilattice is the free product of Γ_1 and Γ_2 amalgamating Δ in the class of semilattices. In this paper, we shall show that for any bundled semilattice Γ of a special amalgam $(\Gamma_1, \Gamma_2; \Delta)$ of semilattices, the amalgam $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ can be embedded in a regular band whose structure semilattice is Ω.

Let $\mathscr{A}$ be a class of algebras. For a family of algebras $\{A_i: i \in I\}$ from $\mathscr{A}$, each having $U \in \mathscr{A}$ as subalgebra, the list $(\{A_i: i \in I\}; U)$ is called an *amalgam* from $\mathscr{A}$. We say that an amalgam $(\{A_i: i \in I\}; U)$ is *strongly embedded* in an algebra B if there exist an algebra B in $\mathscr{A}$ and monomorphisms $\phi_i: A_i \to B$, $i \in I$, such that

(i) $\phi_i|_U = \phi_j|_U$ for all $i, j \in I$,

(ii) $A_i\phi_i \cap A_j\phi_j = U\phi_i$ for all $i, j \in I$ with $i \neq j$,

where $\phi_i|_U$ denotes the restriction of ϕ_i to U. We say that a class $\mathscr{A}$ of algebras has *the strong amalgamation property* if every amalgam from $\mathscr{A}$ is strongly embedded in an algebra from $\mathscr{A}$. If $A_i \cong A_j$ for all $i, j \in I$, an amalgam $(\{A_i: i \in I\}; U)$ is called a *special amalgam* from $\mathscr{A}$. We say that $\mathscr{A}$ has *the special amalgamation property* if each special amalgam from $\mathscr{A}$ is strongly embedded in an algebra from $\mathscr{A}$. It is well-known (see [4]) that in a class of algebras closed under isomorphisms and the formation of the union of any asccending chain of algebras, each amalgamation property follows from the case in which $|I| = 2$. If $|I| = 2$, we write an amalgam $(\{A_1, A_2\}; U)$ simply by $(A_1, A_2; U)$.

For an amalgam $(E_1, E_2; U)$ of semilattices, a semilattice E is called a *bundled semilattice* of the amalgam if $(E_1, E_2; U)$ is strongly embedded in E by monomorphisms $\phi_i: E_i \to E$, $i = 1, 2$, say, such that for $e_i \in E_i$ and $e_j \in E_j$ with $i \neq j$, if $e_i\phi_i \leq e_j\phi_j$ (in E) then there exists $u \in U$ satisfying $e_i \leqq u$ (in E_i) and $u \leqq e_j$ (in E_j). A band B is called a [*left*, *right*] *regular band* if it satisfies the identity $axya = axaya$ [$ax = axa$, $xa = axa$].

It is well-known (see [2]) that the class of regular bands does not have the strong amalgamation property but it has the special amalgamation property. In [3], the author introduced the concept of bundled semilattices and showed that an amalgam $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ of normal bands can be embedded in a normal band whose structure semilattice is a bundled semilattice of the amalgam $(\Gamma_1, \Gamma_2; \Delta)$ of semilattices and further that an amalgem $(S(\Gamma), T(\Lambda); U(\Delta))$ of generalized inverse semigroups can be embedded in a generalized inverse semigroup if the amalgam $(\Gamma, \Lambda; \Delta)$ of inverse semigroups can be strongly embedded in an inverse semigroup A such that $E(A)$ is a bundled semilattice of the amalgam $(E(\Gamma), E(\Lambda); E(\Delta))$ of semilattices, where $S(\Gamma)$ means that Γ is the structure inverse semigroup of S and $E(\Gamma)$ denotes the set of all idempotents of Γ. In his paper [1], T. E. Hall proved that the class of generalized inverse semigroups has the strong amalgamation property by showing that any amalgam $(S, T; U)$ of inverse semigroups is strongly embedded in an inverse semigroup A such that $E(A)$ is a bundled semilattice of the amalgam $(E(S), E(T); E(U))$ of semilattices.

Hall's result raises the question of whether the class of quasi-inverse semigroups (that is, regular semigroups whose idempotents form regular bands) has the special amalgamation property or not. Unfortunately, we can not answer the question yet. To solve the problem, firstly we have to know whether a special amalgam of regular bands is strongly embedded in a regular band whose structure semilattice is a bundled semilattice of a special amalgam of their structure semilattices. In this paper, we shall show that the latter problem is affirmative. The notation and terminology are those of [1] and [6], unless otherwise stated.

Let E be a left regular band and U its subband. By [5], there exist semilattices Γ and Δ and left zero semigroups E_γ, $\gamma\in\Gamma$, and U_δ, $\delta\in\Delta$, such that $\Gamma\supset\Delta$, $E_\delta\supset U_\delta$ for all $\delta\in\Delta$ and the structure decompositions of E and U are $E\sim\Sigma\{E_\alpha\colon \alpha\in\Gamma\}$ and $U\sim\Sigma\{U_\alpha\colon\alpha\in\Delta\}$, respectively. Let $\phi_i\colon E\to E_i$, $i=1, 2$, be isomorphisms such that $E_1\cap E_2=U$. Hereafter, we identify an element u in U to $u\phi_i$ in E_i, $i=1, 2$. Then there exist semilattices Γ_1, Γ_2 and left zero semigroups E_i^α, $\alpha\in\Gamma_i$, $i=1, 2$, such that $E_i\sim\Sigma\{E_i^\alpha\colon\alpha\in\Gamma_i\}$, $i=1, 2$, $\Gamma_1\cong\Gamma_2\cong\Gamma$, $\Gamma_1\cap\Gamma_2=\Delta$ and $E_1^\alpha\cap E_2^\alpha=U_\alpha$ for all $\alpha\in\Delta$. Let Ω be a bundled semilattice of the special amalgam $(\Gamma_1, \Gamma_2; \Delta)$, and let us consider that Γ_1 and Γ_2 are subsemilattices of Ω satisfying $\Gamma_1\cap\Gamma_2=\Delta$. Let Λ be the subsemilattice of Ω generated by $\Gamma_1\cup\Gamma_2$. It is obvious that Λ is also a bundled semilattice of $(\Gamma_1, \Gamma_2; \Delta)$.

Let F be the set of all finite non-empty words $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ in the alphabet $E_1\cup E_2$, where $a_{\alpha_k}\in E_i^{\alpha_k}$, $i=1, 2$. If $a_\alpha\in E_i^\alpha$, $\overline{\alpha}$ means i. The multiplication of two words in F is defined by juxtaposition. It is obvious that F is a semigroup. Let $\sim$ be the congruence on F generated by $\{(a_{\alpha_1}\cdots a_{\alpha_k}a_{\alpha_{k+1}}\cdots a_{\alpha_n}, a_{\alpha_1}\cdots(a_{\alpha_k}a_{\alpha_{k+1}})\cdots a_{\alpha_n})\in F\times F\colon \overline{\alpha_k}=\overline{\alpha_{k+1}}\}$. Denote $F/\sim$ by B. Let θ the congruence on B generated by $R=\{(a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}, a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m})\in B\times B\colon \alpha_1\alpha_2\cdots\alpha_n=\beta_1\beta_2\cdots\beta_m \text{ (in } \Lambda)\}$. It is clear that B/θ is a left regular band whose structure semilattice is Λ.

DEFINITION. Let $a\in E$. An element $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ in B is said to *have the property* $P_i(a)$ if there exist $u_{\sigma_1}, u_{\sigma_2},\dots,u_{\sigma_n}, v_{\tau_1}, v_{\tau_2},\dots,v_{\tau_n}\in U^1$, $\sigma_k, \tau_k\in\Delta^1$ such that

(i) $u_{\sigma_1} = 1$,

(ii) $\sigma_j\tau_j \geqq \alpha_1\alpha_2\cdots\alpha_n$ (in Λ) for all $1 \leqq j \leqq n$,

(iii) $u_{\sigma_j}(a_{\alpha_j}\phi_m^{-1})v_{\tau_j} \in U$ if $i \neq \overline{\alpha_j} = m$, say,

(iv) $a = \prod_{k=1}^{n} u_{\sigma_k}(a_{\alpha_k}\phi_{\overline{\alpha_k}}^{-1})v_{\tau_k}$.

LEMMA 1. *Let* $a \in E$. *If* $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ θ $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m}$ *and* $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ *has the property* $P_i(a)$, *then* $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m}$ *has the property* $P_i(a)$.

PROOF. Assume that $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ has the property $P_i(a)$. Then there exist $u_{\sigma_1}, u_{\sigma_2},\dots, u_{\sigma_n}, v_{\tau_1}, v_{\tau_2},\dots, v_{\tau_n} \in U^1$, $\sigma_k, \tau_k \in \Delta^1$ satifying (i), (ii), (iii) and (iv) in the definition above. For any $1 \leqq j \leqq n$, it follows from (iii) that $\sigma_j\alpha_j\tau_j \in \Gamma_i$ and that $(\sigma_1\alpha_1\tau_1)(\sigma_2\alpha_2\tau_2)\cdots(\sigma_n\alpha_n\tau_n) = \xi$, say, is an element of Γ_i.

In order to show the lemma, it is sufficient to prove that if $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ $(\boldsymbol{R} \cup \boldsymbol{R}^{-1})$ $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m}$ then $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m}$ has the property $P_i(a)$. Firstly, we consider in the case that $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m} = a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}c_{\gamma_1}c_{\gamma_2}\cdots c_{\gamma_t}$ (in B) and $\alpha_1\alpha_2\cdots\alpha_n = \gamma_1\gamma_2\cdots\gamma_t$ (in Λ). For any $1 \leqq j \leqq t$, let $\sigma_j' = \tau_j' = 1$ if $\gamma_j \in \Gamma_i$. If $\gamma_j \notin \Gamma_i$, there exits $\sigma' \in \Delta$ such that $\gamma_j \geq \sigma' \geq \xi$ (in Λ), since Λ is a bundled semilattice of $(\Gamma_1, \Gamma_2; \Delta)$. So, let $\sigma_j' = \tau_j' = \sigma'$, and pick up and fix an element u_j in every such $U_{\sigma_j'}$. For any $1 \leqq j \leqq t$, set

$$u_{\sigma_j'} = v_{\tau_j'} = \begin{cases} 1 & \text{if } \gamma_j \in \Gamma_i, \\ u_j & \text{if } \gamma_j \notin \Gamma_i. \end{cases}$$

If $\overline{\gamma_j} \neq i$, then $u_{\sigma_j'}c_{\gamma_j}v_{\tau_j'} = u_{\sigma_j'} \in U$, since $\sigma_j' = \gamma_j\tau_j' = \sigma'$ and $U_{\sigma'}$ is a left zero semigroup. Since $\prod_{k=1}^{n} \sigma_k\alpha_k\tau_k \prod_{k=1}^{t} \sigma_k'\gamma_k\tau_k' = \xi \in \Gamma_i$, we have

$$\prod_{k=1}^{n} u_{\sigma_k}(a_{\alpha_k}\phi_{\overline{\alpha_k}}^{-1})v_{\tau_k} \prod_{k=1}^{t} u_{\sigma_k'}(c_{\gamma_k}\phi_{\overline{\gamma_k}}^{-1})v_{\tau_k'} = a.$$

Thus $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m}$ has the property $P_i(a)$. Next, we consider in the case that $m < n$ and $b_{\beta_k} = a_{\alpha_k}$ for all $1 \leqq k \leqq m$. Since $\beta_1\beta_2\cdots\beta_m = (\alpha_1\cdots\alpha_m)(\alpha_{m+1}\cdots\alpha_n)$, we can easily verify that $b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m}$ has the property $P_i(a)$.

COROLLARY 2. *Let* a *be an element of* E_i, $i = 1, 2$. *If* a θ $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ (*in* B), *then* $a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}$ *has the property* $P_i(a\phi_i^{-1})$.

LEMMA 3. *We use the notation above. Let* $\psi: E_i \to B/\theta$, $i = 1, 2$, *be mappings defined by*

$$a\psi_i = a\theta \quad \text{for all } a \in E_i.$$

Then ψ_1 *and* ψ_2 *are monomorphisms such that* $\psi_1|_U=\psi_2|_U$ *and* $E_1\psi_1\cap E_2\psi_2=U\psi_1$. *Therefore, the special amalgam* $(E_1, E_2; U)$ *is strongly embedded in* B/θ.

PROOF. Let a_α and b_β be elements of E_i, $i=1, 2$, such that $a_\alpha\psi_i=b_\beta\psi_i$. Then $a_\alpha\ \theta\ b_\beta$. By the corollary above, we have $\alpha=\beta$ and $a_\alpha=b_\beta$. Thus ψ_i is a monomorphism. It is obvious that $\psi_1|_U=\psi_2|_U$. Let $a_\alpha\in E_1$ and $b_\beta\in E_2$ such that $a_\alpha\psi_1=b_\beta\psi_2$. Then $a_\alpha\ \theta\ b_\beta$. By the corollary above, there exist u_σ and v_τ in U such that $\sigma\geqq\alpha$, $\tau\geqq\beta$, $1(a_\alpha\psi_1^{-1})u_\sigma\in U$, $1(b_\beta\psi_2^{-1})v_\tau\in U$, $a_\alpha\psi_1^{-1}=1(b_\beta\psi_2^{-1})v_\tau$ and $b_\beta\psi_2^{-1}=1(a_\alpha\psi_1^{-1})u_\sigma$. Then $a_\alpha=b_\beta\in U$. Hence we have $E_1\psi_1\cap E_2\psi_2=U\psi_1$.

LEMMA 4. *Let* Λ *be a subsemilattice of a semilattice* Ω *and* B *a left regular band whose structure semilattice is* Λ. *Then* B *can be embedded in a left regular band whose structure semilattice is* Ω.

PROOF. Let $B\sim\Sigma\{E_\alpha: \alpha\in\Lambda\}$ be the structure decomposition of B. For each $\alpha\in\Omega$, take a symbol $e_\alpha\notin B$, and let $E=\{e_\alpha\in\Omega\}$. Define a multiplication on E by $e_\alpha e_\beta=e_{\alpha\beta}$. For any $\alpha\in\Omega$, let

$$A_\alpha=\begin{cases} E_\alpha\cup\{e_\alpha\} & \text{if } \alpha\in\Lambda, \\ \{e_\alpha\} & \text{if } \alpha\notin\Lambda. \end{cases}$$

Let $F=\{a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}: a_{\alpha_i}\in A_{\alpha_i}\}$ and the multiplication on F is defined by juxtaposition. Let θ be the congruence on B generated by $\{(a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n}, a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n} b_{\beta_1}b_{\beta_2}\cdots b_{\beta_m})\in F\times F: \alpha_1\alpha_2\cdots\alpha_n=\beta_1\beta_2\cdots\beta_m \text{ (in } \Omega)\}$. It is clear that F/θ is a left regular band whose structure semilattice is Ω and that $F/\theta\sim\Sigma\{B_\alpha: \alpha\in\Omega\}$, where $B_\alpha=\{(a_{\alpha_1}a_{\alpha_2}\cdots a_{\alpha_n})\theta: a_{\alpha_i}\in A_{\alpha_i}, \alpha_1\alpha_2\cdots\alpha_n=\alpha \text{ (in } \Omega)\}$.

Now, we have the following theorem.

THEOREM 5. *The special amalgam* $(E_1, E_2; U)$, *defined above, can be strongly embedded in a left regular band whose structure semilattice is any bundled semilattice of the special amalgam* $(\Gamma_1, \Gamma_2; \Delta)$ *of semilattices.*

Let $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ be a special amalgam of regular bands. By [6], there exist left regular bands $L_1(\Gamma_1), L_2(\Gamma_2), V(\Delta)$ and right regular bands $R_1(\Gamma_1), R_2(\Gamma_2), W(\Delta)$ such that $E_1=L_1\bowtie R_1(\Gamma_1)$, $E_2=L_2\bowtie R_2(\Gamma_2)$, $U=V\bowtie W(\Delta)$, $L_1\cap L_2=V$, $R_1\cap R_2=W$ and $\Gamma_1\cap\Gamma_2=\Delta$, where $L_1\bowtie R_1(\Gamma_1)$ denotes the spined product of L_1 and R_1 with respect to the common structure semilattice Γ_1. Let Ω be any bundled semilattice of a special amalgam $(\Gamma_1, \Gamma_2; \Delta)$ of semilattices. By the theorem above and its dual, there exist a left regular band $L(\Omega)$ and a right regular band $R(\Omega)$ such that the special amalgams $(L_1, L_2; V)$ and $(R_1, R_2; W)$ are strongly embedded in L and R, respectively. It is obvious that $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ is strongly embedded in a regular band $L\bowtie R(\Omega)$. Thus, we have the following main theorem.

THEOREM 6. *A special amalgam* $(E_1(\Gamma_1), E_2(\Gamma_2); U(\Delta))$ *of regular bands can be strongly embedded in a regular band whose structure semilattice is any bundled semilattice of the special amalgam* $(\Gamma_1, \Gamma_2; \Delta)$ *of semilattices.*

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