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ON THE SERIES EXPANSIONS OF STEP FUNCTIONS

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ABSTRACT. We express $[x]$ as an infinite series and give a new formula for the partial sums of the divisor function.

Let $[x]$ be the greatest integer $\leq x$ and let $\lfloor x \rfloor$ be the least integer $\geq x$. Vinogradov [3, pp. 37–38] proved that

$$x - [x] = \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{n} \frac{\sin 2\pi k x}{k} + \frac{\theta}{\pi(n+1)\sin \pi x},$$

where $|\theta| \leq 1$. In this paper, using the following series, we express the step functions $[x]$ and $\lfloor x \rfloor$ as infinite series.

Lemma 1. *Let $f(x)$ be a real-valued function of a real variable that satisfies $|f(x)| < \infty$ and let*

$$C_f(x) = 1 - \sum_{k=1}^{\infty} \frac{f(x)^2}{(1+f(x)^2)^k}.$$

Then we have

$$C_f(x) = \begin{cases} 1 & \text{if } x \text{ is a zero point of } f(x), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If x is a zero point of $f(x)$, then the lemma holds trivially. If x is no zero point of $f(x)$, then

$$\begin{aligned} C_f(x) &= 1 + f(x)^2 - \sum_{k=0}^{\infty} \frac{f(x)^2}{(1+f(x)^2)^k} \\ &= 1 + f(x)^2 - \frac{f(x)^2}{1 - 1/(1+f(x)^2)} = 0. \end{aligned}$$

□

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In particular, taking $f(x) = \sin \pi x$, we have

$$C_f(x) = C_{\sin}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Z}. \end{cases}$$

Theorem 2. *For* $x \in \mathbb{R}$

$$[x] = x + \sum_{k=1}^{\infty} \left(\frac{\sin 2k\pi x}{k\pi} - \frac{\sin^2 \pi x}{2(1+\sin^2 \pi x)^k} \right), \tag{1}$$

$$\lfloor x \rfloor = x + \sum_{k=1}^{\infty} \left(\frac{\sin 2k\pi x}{k\pi} + \frac{\sin^2 \pi x}{2(1+\sin^2 \pi x)^k} \right). \tag{2}$$

Proof. (1) The function

$$g(x) = \begin{cases} x - 2k\pi & \text{if } (2k-1)\pi < x < (2k+1)\pi \text{ for } k \in \mathbb{Z}, \\ 0 & \text{if } x = (2k+1)\pi \text{ for } k \in \mathbb{Z}. \end{cases}$$

is expanded in the Fourier series:

$$g(x) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k}.$$

Hence

$$\begin{aligned} x - [x] &= \frac{g(2\pi x - \pi)}{2\pi} + \frac{1}{2} - \frac{C_{\sin}(x)}{2} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin(2k\pi x - k\pi)}{k\pi} + \sum_{k=1}^{\infty} \frac{\sin^2 \pi x}{2(1+\sin^2 \pi x)^k} \\ &= \sum_{k=1}^{\infty} \left(\frac{\sin^2 \pi x}{2(1+\sin^2 \pi x)^k} - \frac{\sin 2k\pi x}{k\pi} \right). \end{aligned}$$

(2) Since $\lfloor x \rfloor = [x] + 1 - C_{\sin}(x)$, (2) follows. □

Similarly, using $C_{\sin}(x)$, we can express a discontinuous periodic function $f(x)$ that satisfies

$$f(x_0) \neq \frac{f(x_0+0) + f(x_0-0)}{2}$$

at a point of discontinuity x_0 as infinite series.

Further, from (1) the following result is derived immediately.

Theorem 3. *Let $f(x)$ be a real-valued function on a real variable. Then for $m, n \in \mathbb{Z}$ we have*

$$\sum_{j=m}^{n} [f(j)] = \sum_{j=m}^{n} f(j) + \sum_{k=1}^{\infty} \sum_{j=m}^{n} \left(\frac{\sin(2k\pi f(j))}{k\pi} - \frac{\sin^2(\pi f(j))}{2\left(1+\sin^2(\pi f(j))\right)^k} \right).$$

Next we consider the partial sums of the diviser function. Let $d(n)$ be the number of positive divisors of the positive integer n. Voronoï [4] proved the explicit formula

$$\sum_{n\le x}{}' d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} - \frac{2}{\pi}x^{1/2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/2}}\left(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})\right)$$

for $x \ge 2$, where γ is Euler's constant, K_1 and Y_1 are Bessel functions, and $\sum'$ indicates that the last term is to be halved if x is an integer. The latest result on the remainder term defined by

$$\Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1) \quad \text{for } x \ge 2$$

is due to Huxley [2], who proved that

$$\Delta(x) = O\left(x^{131/416}(\log x)^{26947/8320}\right).$$

As a particular case of Theorem 3, we obtain a new formula for the partial sums of the divisor function.

Corollary 4.

$$\sum_{j=1}^{n} d(j) = n\sum_{j=1}^{n}\frac{1}{j} + \sum_{k=1}^{\infty}\sum_{j=1}^{n}\left(\frac{\sin(2kn\pi/j)}{k\pi} - \frac{\sin^2(n\pi/j)}{2\left(1+\sin^2(n\pi/j)\right)^k}\right).$$

Proof. By the well-known property [1, p. 264]

$$\sum_{j=1}^{n} d(j) = \sum_{j=1}^{n}\left[\frac{n}{j}\right],$$

the corollary follows immediately. □

References

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