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# HOMOGENEOUS CLOSED CURVES ON GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE FROM THE VIEWPOINT OF SUBMANIFOLD THEORY

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Abstract. It is known that every geodesic on each geodesic sphere in a complex projective space is a homogeneous curve, that is, every geodesic is an orbit under a certain one-parameter subgroup of the isometry group of the geodesic sphere. In this paper, we give a family of homogeneous non-geodesic closed curves on this geodesic sphere through submanifold theory.

## 1. Introduction

In this paper we consider some homogeneous curves on geodesic spheres in a complex $n(\geqq 2)$-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(>0)$. Geodesic spheres of radius smaller than the injectivity radius of $\mathbb{C}P^n(c)$ are nice objects in differential geometry. They are typical examples of naturally reductive Riemannian homogeneous manifolds (see [11]). Every geodesic on a geodesic sphere in $\mathbb{C}P^n(c)$ is hence an orbit under some one-parameter subgroup of the isometry group of this geodesic sphere. We shall say such a curve to be *homogeneous*.

It is also well-known that some geodesic spheres in $\mathbb{C}P^n(c)$ are so-called *Berger spheres*. That is, when the radius $r$ $(0 < r < \pi/\sqrt{c}\,)$ of a geodesic sphere in $\mathbb{C}P^n(c)$ satisfies $\tan^2(\sqrt{c}\,r/2) > 2$, sectional curvatures of this geodesic sphere lie in the interval $[\delta K, K]$ $(K = c\{4+\cot^2(\sqrt{c}\,r/2)\}/4)$ with some $\delta \in (0, 1/9)$. But it has closed geodesics of length of $(2\pi/\sqrt{c})\sin(\sqrt{c}\,r)$ which is shorter than $4\pi/\sqrt{c\{4+\cot^2(\sqrt{c}\,r/2)\}}$. These closed geodesics are integral curves of the characteristic vector field $\xi$ of this geodesic sphere which is defined by $\xi = -J\mathcal{N}$ with

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the outward unit normal vector field $\mathcal{N}$ of this geodesic sphere and the complex structure $J$ on $\mathbb{C}P^n(c)$ ([17]). Futhermore, every its geodesic which is not congruent to such integral curves has length longer than $4\pi/\sqrt{c\{4+\cot^2(\sqrt{c}\,r/2)\}}$ (see Corollary 2.8 in [4]). In this context it is natural to pay attention to integral curves of the characteristic vector field $\xi$ of a geodesic sphere in $\mathbb{C}P^n(c)$.

We study curves on geodesic spheres in $\mathbb{C}P^n(c)$ through the parallel isometric embedding $f$ of $\mathbb{C}P^n(c)$ into Euclidean space $\mathbb{R}^{n(n+2)}$ which is a composition of the first standard minimal embedding of $\mathbb{C}P^n(c)$ into some standard sphere and a totally umbilic embedding of this sphere into $\mathbb{R}^{n(n+2)}$ (for details, see section 2). If we denote by $\iota$ the inclusion of a geodesic sphere into $\mathbb{C}P^n(c)$, through the isometric embedding $f\circ\iota$, we can treat geodesic spheres in $\mathbb{C}P^n(c)$ as Riemannian submanifolds in $\mathbb{R}^{n(n+2)}$. Hence we can consider curves on geodesic spheres as curves in $\mathbb{R}^{n(n+2)}$. It is known that the shape of each integral curve of the characteristic vector field $\xi$ on this geodesic sphere through $f\circ\iota$ is a circle in $\mathbb{R}^{n(n+2)}$ in the sense of Euclidean geometry (see [6]). Motivated by this fact, we classify all curves on a geodesic sphere whose shapes through $f\circ\iota$ are circles in $\mathbb{R}^{n(n+2)}$ and show that they are homogeneous.

## 2. Circles and the parallel isometric embedding

Let $M$ be a Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$ and Riemannian connection $\nabla$. A smooth regular curve $\gamma=\gamma(s)$ on $M$ parameterized by its arclength $s$ is said to be a *circle* if there exist a constant $k(\geqq 0)$ and a field of unit vectors $Y$ along $\gamma$ which satisfy the ordinary differential equations:

$$\nabla_{\dot\gamma}\dot\gamma = kY \quad \text{and} \quad \nabla_{\dot\gamma}Y=-k\dot\gamma. \tag{2.1}$$

We call the constant $k$ its *curvature* and $\{\dot\gamma, Y\}$ its Fenet frame. A circle of null curvature is nothing but a geodesic. Given a point $x\in M$, an orthonormal pair of vectors $(u,v)\in T_xM\times T_xM$ and a positive constant $k$, there exists locally a unique circle $\gamma:(-\epsilon,\epsilon)\to M$ of curvature $k$ whose initial frame at $\gamma(0)=x$ is $(u,v)$. It is known that on a complete Riemannian manifold the domain of each circle can be extended to $\mathbb{R}$ (cf. [13]).

The following canonical isometric embedding $f$ of $\mathbb{C}P^n(c)$ into a Euclidean space $\mathbb{R}^{n(n+2)}$ plays a key role in this paper. We consider the first standard minimal embedding of $f_1$ of $\mathbb{C}P^n(c)$ into an $\big(n(n+2)-1\big)$-dimensional standard sphere $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ of constant sectional curvature $(n+1)c/(2n)$, which is constructed by eigenfunctions with respect to the first eigenvalue of the Laplacian on $\mathbb{C}P^n(c)$, and a totally umbilic embedding $f_2$ of $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ into $\mathbb{R}^{n(n+2)}$ (for details on standard embeddings, see [16]). The composition

$$f=f_2\circ f_1:\mathbb{C}P^n(c)\xrightarrow{f_1}S^{n(n+2)-1}\big((n+1)c/(2n)\big)\xrightarrow{f_2}\mathbb{R}^{n(n+2)}$$

has the following nice geometric properties:

1) The second fundamental form $\sigma$ of $f$ is parallel and satisfies $\sigma(JX,JY)=\sigma(X,Y)$ for all vectors $X,Y$ on the submanifold $\mathbb{C}P^n(c)$ with complex structure $J$ (see [7]);

2) It is $\sqrt{c}$-isotropic, namely the second fundamental form $\sigma$ of $f$ satisfies $\langle \sigma(X,X), \sigma(X,X)\rangle = c$ for each unit vector $X$ on $\mathbb{C}P^n(c)$ (cf. [9]);
3) When $n = 1$, this embedding is congruent to a natural totally umbilic embedding of $S^2(c)$ into $\mathbb{R}^3$.

Through this isometric embedding some circles on $\mathbb{C}P^n(c)$ can be seen as circles on $\mathbb{R}^{n(n+2)}$.

**Lemma 1** ([15]). *For each geodesic $\gamma$ on $\mathbb{C}P^n(c)$, the curve $f \circ \gamma$ is a circle of curvature $\sqrt{c}$ in a Euclidean space $\mathbb{R}^{n(n+2)}$.*

We here classify circles on a Kähler manifold $M$ with complex structure $J$ by another invariant. For a circle $\gamma$ satisfying (2.1), we set $\tau(s) = \langle \dot\gamma(s), JY(s)\rangle$ and call it its *complex torsion*. It should be noted that the complex torsion $\tau$ of a circle $\gamma$ is a constant function with $-1 \leqq \tau \leqq 1$:

$$\nabla_{\dot\gamma}\langle \dot\gamma, JY\rangle = \langle \nabla_{\dot\gamma}\dot\gamma, JY\rangle + \langle \dot\gamma, J\nabla_{\dot\gamma}Y\rangle = k\langle Y, JY\rangle - k\langle \dot\gamma, J\dot\gamma\rangle = 0.$$

We say a circle with complex torsion $\tau = \pm 1$ to be a *Kähler circle* and a circle with null complex torsion to be *totally real circle*. A Kähler circle of curvature $k$ on a Kähler manifold is hence a curve satisfying either

$$\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma \qquad \text{or} \qquad \nabla_{\dot\gamma}\dot\gamma = -kJ\dot\gamma. \tag{2.2}$$

On $\mathbb{C}P^n(c)$, a circle is Kähler if and only if it lies on some totally geodesic complex line $\mathbb{C}P^1(c)$, and is totally real if and only if it lies on some totally geodesic totally real real projective plane $\mathbb{R}P^2(c/4)$ of constant sectional curvature $c/4$. We regard geodesics as Kähler circles of null curvature.

Through the parallel embedding $f : \mathbb{C}P^n(c) \to \mathbb{R}^{n(n+2)}$ we can say the following which is an extension of Lemma 1:

**Lemma 2** ([6]). *A curve $\gamma$ on $\mathbb{C}P^n(c)$ is a Kähler circle if and only if the curve $f \circ \gamma$ is a circle in Euclidean space $\mathbb{R}^{n(n+2)}$. When $\gamma$ is a Kähler circle of curvature $k$ ($\geqq 0$), the curvature of the circle $f \circ \gamma$ is $\sqrt{k^2 + c}$.*

This lemma is a generalization of the fact that a curve on a standard sphere $S^2(c)$ is a circle, namely either a great circle or a small circle, if and only if the curve is a circle of positive curvature as a curve in a Euclidean 3-space. By this lemma our problem is reduced to find curves on a geodesic sphere which can be seen as Kähler circles in a complex projective space.

## 3. Structure torsions

Since two geodesic spheres in $\mathbb{C}P^n(c)$ are isometric to each other if their radii are the same, we denote by $G(r)$ a geodesic sphere of radius $r$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. It is well-known that each geodesic sphere has an almost contact metric structure induced by the complex structure $J$ on $\mathbb{C}P^n(c)$. This structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ on $G(r)$ is a quartet of the vector field $\xi = -J\mathcal{N}$, the induced metric $\langle\ ,\ \rangle$, the function $\eta : TG(r) \to \mathbb{R}$ defined by $\eta(v) = \langle v, \xi\rangle$ and the map $\phi : TG(r) \to TG(r)$ given by $\phi(v) = J\big(v - \eta(v)\xi\big)$. The Riemannian connections

$\widetilde{\nabla}$ of $\mathbb{C}P^n(c)$ and $\nabla$ of $G(r)$ are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N}, \tag{3.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{3.2}$$

for vector fields $X$ and $Y$ tangent to $G(r)$, where $A$ is the shape operator of $G(r)$ in $\mathbb{C}P^n(c)$. As $\widetilde{\nabla} J = 0$ we see

$$\nabla_X \xi = \phi AX. \tag{3.3}$$

Indeed,

$$\begin{aligned}\nabla_X \xi &= \widetilde{\nabla}_X \xi - \langle AX, \xi\rangle \mathcal{N} = J\widetilde{\nabla}_X(-\mathcal{N}) + \langle AX, J\mathcal{N}\rangle \mathcal{N} \\ &= JAX - \langle JAX, \mathcal{N}\rangle \mathcal{N} = \phi AX.\end{aligned}$$

We now study curves on $G(r)$ whose shapes through $\iota$ are Kähler circles in $\mathbb{C}P^n(c)$.

**Lemma 3** ([10]). *A smooth curve $\gamma$ on $G(r)$ can be seen as a Kähler circle of curvature $k(\geqq 0)$ through the inclusion $\iota$ if and only if it satisfies both of the following equations*:

$$\nabla_{\dot\gamma}\dot\gamma = \pm k\phi\dot\gamma, \tag{3.4}$$

$$\langle A\dot\gamma, \dot\gamma\rangle = \pm k\rho_\gamma, \tag{3.5}$$

*where double signs take the same signatures.*

*Proof.* For a smooth curve $\gamma$ on $G(r)$ we have

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle \mathcal{N} \quad \text{and} \quad J\dot\gamma = \phi\dot\gamma + \eta(\dot\gamma)\mathcal{N} = \phi\dot\gamma + \rho_\gamma \mathcal{N},$$

hence get the conclusion. □

We here give the definition of Sasakian curves on an odd dimensional Riemannian manifold $M$ furnished with an alomost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. A curve $\gamma$ on $M$ with Riemannian connection $\nabla$ is said to be a *Sasakian curve* if it satisfies $\nabla_{\dot\gamma}\dot\gamma = k\phi\dot\gamma$ with some constant $k$.

Sasakian curves were treated in [2] and were called trajectories for canonical magnetic fields on a geodesic sphere. Kähler circles were originally called trajectories for Kähler magnetic fields, which are uniform magnetic fields on a Kähler manifold. As corresponding objects to these Kähler magnetic fields on geodesic spheres, the second author considered canonical magnetic fields. Though canonical magnetic fields are not uniform, each trajectory on a geodesic sphere in $\mathbb{C}P^n(c)$ has constant first curvature $\|\nabla_{\dot\gamma}\dot\gamma\|$ by the following lemma. Given a Sasakian curve $\gamma = \gamma(s)$ on $G(r)$ in $\mathbb{C}P^n(c)$ we define its *structure torsion* $\rho_\gamma$ by $\rho_\gamma = \langle\dot\gamma, \xi\rangle$. We then have $\|\phi\dot\gamma\| = \sqrt{1-\rho_\gamma^2}$ and the first curvature of a Sasakian curve satisfying $\nabla_{\dot\gamma}\dot\gamma = k\phi\dot\gamma$ is $|k|\sqrt{1-\rho_\gamma^2}$.

**Lemma 4.** *The structure torsion of a Sasakian curve $\gamma$ on $G(r)$ is constant along $\gamma$.*

*Proof.* By use of (3.3) we find

$$\rho'_\gamma = \nabla_{\dot\gamma}\langle\dot\gamma,\xi\rangle = \langle k\phi\dot\gamma,\xi\rangle + \langle\dot\gamma,\phi A\dot\gamma\rangle = \langle\dot\gamma,\phi A\dot\gamma\rangle = \langle\dot\gamma, A\phi\dot\gamma\rangle = -\langle\phi A\dot\gamma,\dot\gamma\rangle,$$

which shows $\rho'_\gamma = 0$ with the fact that $\phi A = A\phi$ holds on $G(r)$. □

The notion of structure torsions of Sasakian curves on a geodesic sphere is a quite important invariant. It is deeply related to the congruence of Sasakian curves. In order to state precisely the congrunce theorem for Sasakian curves on geodesic spheres, we review the definition on congruence for curves in a Riemannian manifold $M$. Two curves $\gamma_1, \gamma_2$ on $M$ are congruent (in the usual sense) if there exist an isometry $\varphi$ of $M$ and a constant $s_0$ with $\gamma_2(s) = (\varphi\circ\gamma_1)(s+s_0)$ for each $s$. We call two curves $\gamma_1, \gamma_2$ on $M$ *strongly congruent* to each other if there is an isometry $\varphi$ of $M$ with $\gamma_2(s) = (\varphi\circ\gamma_1)(s)$ for each $s$.

The following is a strongly congruence theorem for Sasakian curves.

**Lemma 5** ([2]). *Two Sasakian curves $\gamma_1$, $\gamma_2$ on $G(r)$ in $\mathbb{C}P^n(c)$ satisfying $\nabla_{\dot\gamma_i}\dot\gamma_i = k_i\phi\dot\gamma_i$ with structure torsions $\rho_{\gamma_i}$ $(i = 1, 2)$ are strongly congruent to each other if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$,
ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|k_1| = |k_2|$,
iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $k_1\rho_{\gamma_1} = k_2\rho_{\gamma_2}$.

As a consequence of this lemma we can show the homogenity of Sasakian curves.

**Corollary 1.** *Every Sasakian curve on a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ is homogeneous.*

*Proof.* Let $\gamma$ be a Sasakian curve. For each fixed $t \in \mathbb{R}$ we set a curve $\mu_t$ by $\mu_t(s) = \gamma(s+t)$ for every $s$. Clearly they are strongly congruent to each other by Lemma 5. We hence have an isometry $\varphi_t$ of $G(r)$ with $\gamma(t) = \varphi_t(\gamma(0))$ for each $t$ and get the conclusion. □

## 4. Curves whose shapes in $\mathbb{R}^{n(n+1)}$ are circles

We now show our results on Sasakian curves on a geodesic sphere in a complex projective space from the viewpoint of submanifold theory. For this purpose we consider the isometric embedding $f\circ\iota$ given by

$$f\circ\iota:\ G(r) \xrightarrow{\ \iota\ } \mathbb{C}P^n(c) \xrightarrow{\ f\ } \mathbb{R}^{n(n+2)},$$

where $f$ is the parallel embedding of $\mathbb{C}P^n(c)$ into $\mathbb{R}^{n(n+2)}$ (for details, see section 2) and $\iota$ is an inclusion mapping of $G(r)$ into $\mathbb{C}P^n(c)$.

We study curves on a geodesic sphere $G(r)$ whose shapes through the isometring embedding $f\circ\iota$ are circles in the ambient Euclidean space $\mathbb{R}^{n(n+2)}$.

**Theorem 1.** *Let $G(r)$ be a geodesic sphere of radius $0 < r \leqq \pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$.*

(1) *For $0 \leqq \kappa < \sqrt{c}/\sin(\sqrt{c}\,r)$, there are no curves on $G(r)$ whose shape thorugh $f\circ\iota$ is a circle of curvature $\kappa$.*

(2) *When* $\kappa = \sqrt{c}/\sin(\sqrt{c}\,r)$, *the shape of a curve on* $G(r)$ *through* $f \circ \iota$ *is a circle of curvature* $\kappa$ *if and only it a geodesic with structure torsion* $\rho_\gamma = \pm 1$, *namely it is an integral curve of* $\xi$ *on* $G(r)$.

(3) *When* $\kappa > \sqrt{c}/\sin(\sqrt{c}\,r)$, *the shape of a curve* $\gamma$ *on* $G(r)$ *through* $f \circ \iota$ *is a circle of curvature* $\kappa$ *if and only if it is a Sasakian curve satisfying* $\nabla_{\dot\gamma}\dot\gamma = \pm\sqrt{\kappa^2 - c}\,\phi\dot\gamma$ *and whose structure torsion is*

$$\rho_\gamma = \pm c^{-1/2}\big(\kappa - \sqrt{\kappa^2 - c}\big)\cot(\sqrt{c}\,r/2),$$

*where double signs take the same signatures.*

*Trivially these curves in* (2), (3) *are closed with length* $2\pi/\kappa$.

**Theorem 2.** *Let* $G(r)$ *be a geodesic sphere of radius* $r$ *with* $\pi/(2\sqrt{c}) < r < \pi/\sqrt{c}$ *in* $\mathbb{C}P^n(c)$.

(1) *For* $0 \leqq \kappa < \sqrt{c}$, *there are no curves on* $G(r)$ *whose shape thorugh* $f \circ \iota$ *is a circle of curvature* $\kappa$.

(2) *When* $\kappa = \sqrt{c}$, *the shape of a curve on* $G(r)$ *through* $f \circ \iota$ *is a circle of curvature* $\kappa$ *if and only it is a geodesic with structure torsion* $\rho_\gamma = \pm\cot(\sqrt{c}\,r/2)$.

(3) *When* $\sqrt{c} < \kappa < \sqrt{c}/\sin(\sqrt{c}\,r)$, *the shape of a curve* $\gamma$ *on* $G(r)$ *through* $f \circ \iota$ *is a circle of curvature* $\kappa$ *if and only if it is a Sasakian curve satisfying* $\nabla_{\dot\gamma}\dot\gamma = \pm\sqrt{\kappa^2 - c}\,\phi\dot\gamma$ *and whose structure torsion is*

$$\rho_\gamma = \pm c^{-1/2}\big(\kappa - \sqrt{\kappa^2 - c}\big)\cot(\sqrt{c}\,r/2),$$

*where double signs take either the same signatures or the opposite signatures.*

(4) *When* $\kappa = \sqrt{c}/\sin(\sqrt{c}\,r)$, *the shape of a curve on* $G(r)$ *through* $f \circ \iota$ *is a circle of curvature* $\kappa$ *if and only it a geodesic with structure torsion* $\rho_\gamma = \pm 1$.

(5) *When* $\kappa > \sqrt{c}/\sin(\sqrt{c}\,r)$, *the shape of a curve* $\gamma$ *on* $G(r)$ *through* $f \circ \iota$ *is a circle of curvature* $\kappa$ *if and only if it is a Sasakian curve satisfying* $\nabla_{\dot\gamma}\dot\gamma = \pm\sqrt{\kappa^2 - c}\,\phi\dot\gamma$ *and whose structure torsion is*

$$\rho_\gamma = \pm c^{-1/2}\big(\kappa - \sqrt{\kappa^2 - c}\big)\cot(\sqrt{c}\,r/2),$$

*where double signs take the same signatures.*

*Trivially these curves in* (2), (3), (4), (5) *are closed with length* $2\pi/\kappa$.

*Proof of Theorems* 1 *and* 2. The shape operator $A$ of $G(r)$ on $\mathbb{C}P^n(c)$ satisfies

$$A\xi = \sqrt{c}\cot(\sqrt{c}\,r)\,\xi \quad \text{and} \quad Au = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)\,u$$

for every tangent vector $u \in TG(r)$ orthogonal to $\xi$. By Lemmas 2 and 3, we are hence enough to consider Sasakian curves satisfying both $\nabla_{\dot\gamma}\dot\gamma = \pm\sqrt{\kappa^2 - c}\,\phi\dot\gamma$ and

$$\rho_\gamma^2\sqrt{c}\cot(\sqrt{c}\,r) + (1 - \rho_\gamma^2)\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}\,r}{2}\right) = \pm\sqrt{\kappa^2 - c}\,\rho_\gamma. \tag{4.1}$$

Regarding this equality (4.1) as a quadratic equation with respect to $\rho_\gamma$, we obtain the conclusions by the same discussion in [2] or [10]. $\square$

By these theorems we find that there are infinitely many homogeneous curves on a geodesic sphere in $\mathbb{C}P^n(c)$ whose shapes in $\mathbb{R}^{n(n+2)}$ through $f \circ \iota$ are circles. We can say that two of those curves are not congruent if their shapes through $f \circ \iota$ do not have the same curvatures, because $f \circ \iota$ is an equivariant mapping. But the converse does not hold in general. The following is an immediate consequence of Lemma 5 and Theorem 2, which shows that information on curvature for shapes in $\mathbb{R}^{n(n+2)}$ does not give a sufficient condition for non-congruency of those curves on $G(r)$.

**Corollary 2.** *On a geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ of radius $r$ with $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$, for each $\kappa$ with $\sqrt{c} < \kappa < \sqrt{c}/\sin(\sqrt{c}\,r)$, there are two congruence classes of homogeneous curves with respect to the full isometry group of $G(r)$ whose shapes in $\mathbb{R}^{n(n+2)}$ through $f \circ \iota$ are circles of the same curvature $\kappa$.*

On $\mathbb{C}P^n(c)$ two circles are congruent to each other with respect to the full isometry group of $\mathbb{C}P^n(c)$ if and only if they have the same curvatures and the same absolute values of complex torsions. This, together with Lemma 2, implies that those Sasakian curves in Corollary 2 are congruent to each other with respect to the full isometry group of $\mathbb{C}P^n(c)$ if we consider them as curves in $\mathbb{C}P^n(c)$.

## 5. Shapes of Sasakian curves in $\mathbb{C}P^n(c)$

In this paper we devote ourselves to study Sasakian curves whose shapes in $\mathbb{C}P^n(c)$ are Kähler circles. We here make mention of shapes of other Sasakian curves on geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ through the inclusion $\iota$.

To do this, we recall the definition of helices in Riemannian geometry. A smooth curve $\gamma$ on $\mathbb{C}P^n(c)$ parameterized by its arclength is called a helix of proper order $d$ if there exist positive constants $\kappa_1, \dots, \kappa_{d-1}$ and a field of of orthonormal frames $\{Y_1 = \dot\gamma, Y_2, \dots, Y_d\}$ along $\gamma$ satisfying

$$\widetilde{\nabla}_{\dot\gamma} Y_j = -\kappa_{j-1} Y_{j-1} + \kappa_j Y_{j+1} \qquad (j = 1, 2, \dots, d),$$

where $\kappa_0 = \kappa_d = 0$ and $Y_0, Y_{d+1}$ are null vector fields along $\gamma$. When it is an orbit of some one-parameter subgroup of the isometry group of $\mathbb{C}P^n(c)$, we say that it is *Killing helix*. That is, such a Killing helix is a homogeneous curve on $\mathbb{C}P^n(c)$.

For about geodesics we can say the following.

**Proposition 1** ([4]). *Let $\gamma$ be a geodesic on a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. The curve $\iota \circ \gamma$ through the inclusion $\iota : G(r) \to \mathbb{C}P^n(c)$ is as follows:*

(1) *When the radius $r$ satisfies $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$, if $\rho_\gamma = \pm\cot(\sqrt{c}\,r/2)$, then the curve $\iota \circ \gamma$ is a geodesic;*
(2) *If $\rho_\gamma = \pm 1$, that is $\gamma$ is an integral curve of $\xi$, then the curve $\iota \circ \gamma$ is a Kähler circle of curvature $\sqrt{c}|\cot(\sqrt{c}\,r)|$;*
(3) *If $\rho_\gamma = 0$, the curve $\iota\circ\gamma$ is a totally real circle of curvature $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$;*
(4) *Otherwise, the curve $\iota \circ \gamma$ is a Killing helix of proper order 4.*

For about Sasakian curves which are not geodesics, their features are a bit complicated.

**Proposition 2** ([2]). *Let $\gamma$ be a Sasakian curve with $\rho_\gamma = 0$ on $G(r)$ in $\mathbb{C}P^n(c)$ satisfying $\nabla_{\dot\gamma}\dot\gamma = k\phi\dot\gamma$ $(k \neq 0)$. Then its shape in $\mathbb{C}P^n(c)$ through inclusion $\iota$ is a Killing helix of proper order 4 and lies on some totally geodesic $\mathbb{C}P^2(c)$.*

**Proposition 3** ([2]). *Let $\gamma$ be Sasakian curve with $0 < |\rho_\gamma| < 1$ on $G(r)$ in $\mathbb{C}P^n(c)$ satisfying $\nabla_{\dot\gamma}\dot\gamma = k\phi\dot\gamma$ $(k \neq 0)$.*

(1) *When $2k\rho_\gamma = 2\rho_\gamma^2\sqrt{c}\cot(\sqrt{c}\,r) + (1-\rho_\gamma^2)\sqrt{c}\cot(\sqrt{c}\,r/2)$, which is the case of our theorems, its shape in $\mathbb{C}P^n(c)$ is a Kähler circle.*

(2) *When $2k = \rho_\gamma\sqrt{c}\tan(\sqrt{c}\,r/2)$, its shape in $\mathbb{C}P^n(c)$ is a non-Kähler circle of positive curvature.*

(3) *When $2k\rho_\gamma\{1-\tan^2(\sqrt{c}\,r/2)\} = 2\rho_\gamma^2\sqrt{c}\cot(\sqrt{c}\,r) + (1-\rho_\gamma^2)\sqrt{c}\cot(\sqrt{c}\,r/2)$, its shape in $\mathbb{C}P^n(c)$ is a Killing helix of proper order 3.*

(4) *Otherwise, its shape in $\mathbb{C}P^n(c)$ is a Killing helix of proper order 4.*

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