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PARABOLIC NETWORKS

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Abstract. In an infinite network N without positive potentials (called *a parabolic network*), a generalized version of the Dirichlet problem is solved in an arbitrary subset of N. This solution is shown to play a pivotal role in the potential-theoretic study in N, such as the construction of an analogue of the logarithmic potential in N, balayage, harmonic and superharmonic extensions, condenser principle etc.

1. Introduction

An infinite electrical network is characterized mathematically by imposing a specific analytic structure on an infinite graph, to each branch of which is associated several electrical parameters. Unrelated to an electrical network, a Markov chain is determined by a countable state space X and a stochastic transition matrix $(p(x, y))$, x, y in X and $p(x, y)$ being the probability of transition from x to y. However, the similarities between these two structures are striking (see Zemanian [9] and Woess [6]).

In these two cases, the common feature is to start with a countable set X of nodes and a countable set Y of edges, each edge $[x, y]$ joining a pair of nodes x and y; further, with each edge $[x, y]$ is associated a real number $t(x, y) \geq 0$ called the conductance. We say that $N = \{X, Y, t\}$ determines an infinite network. Then follows the classification of infinite networks into two classes, hyperbolic and parabolic networks, which correspond to the transient and the recurrent Markov chains (see Yamasaki [7] and [8]). A tree as defined by Cartier [3] can be considered as a special case of an infinite network without circuits. There are many results proved in the frame work of a hyperbolic network or tree, but not so in parabolic networks. In this note, pursuing the analogy of a parabolic network N with a parabolic Riemann surface R, we construct a function on N similar to the logarithmic potential in R, and study the Dirichlet problem, balayage, minimum principle and condenser principle in this context. The proofs of the theorems in this article are so arranged as to show the important role played by the Dirichlet solution in the

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potential-theoretic study of networks. Using the fact that there are no circuits in a tree, some of the results found here can be proved differently in the context of trees (see [1] and Bajunaid et al.[2]).

2. Preliminaries

Let X be a countable set of points (here called *nodes*), some of them pairwise joined by *edges*; we say that the edge $[x, y]$ joins the nodes x and y. Let Y denote the set of edges which are assumed to be countable. Denote $x \sim y$ to mean that there is an edge $[x, y]$ joining x and y, in which case the nodes x and y are said to be *neighbours*. A path joining x and y is a collection of vertices $\{x = x_0, x_1, \ldots, x_n = y\}$ where $x_i \sim x_{i+1}$ if $0 \leq i \leq n-1$; for this path, the length is n. The shortest length between x and y is called *the distance between x and y*. We also assume that given any two nodes x and y, there exists *an associated real number* $t(x, y) \geq 0$ such that $t(x, y) > 0$ if and only if $x \sim y$. Then $N = \{X, Y, t\}$ is called *an infinite network* if the following conditions are satisfied.

(1) There is no self-loop in N, that is no edge of the form $[x, x]$ in Y.
(2) Given any vertices x and y in X, there is a path connecting x and y.
(3) Every $x \in X$ has only a finite number of neighbours.

For any subset E of X, we write $\overset{0}{E} = \{x : x \text{ and all its neighbours are in } E\}$ and $\partial E = E \backslash \overset{0}{E}$. Note that for a subset E, we have $E = \overset{0}{E}$ if and only if $E = X$. An arbitrary set E in X is said to be *circled* if every node in ∂E has at least one neighbour in $\overset{0}{E}$. Example: Let e be a fixed node. For any node x, let $|x|$ denote the distance between e and x. Then $B_m = \{x : |x| \leq m\}$ is circled. Write $V(E)$ to denote the union of E and all the neighbours of each node of E, that is $V(E) = E \cup \{y : y \sim x \text{ for some } x \in E\}$. In particular, $V(x)$ denotes the set consisting of x and all its neighbours. Remark that if E is connected, $V(E)$ also is connected. Also note that for any set E, $V(E)$ is circled. For, if $F = V(E)$, then $E \subset \overset{0}{F}$. Hence if $z \in \partial F$, then by definition z has a neighbour in E and hence in $\overset{0}{F}$.

Proposition 2.1. *Let A be circled, and $B = \overset{0}{A}{}^c$. Then $\partial B = \partial A$ and $\overset{0}{B} = A^c$; also, B is circled.*

Proof. Note $B = \overset{0}{A}{}^c = A^c \cup \partial A$. Let $z \in \partial A$. Then, for some $y \in \overset{0}{A}$, $y \sim z$; thus $z \in \partial A \subset B$, but a neighbour y of z is not in B, hence $z \in \partial B$. Conversely, let $b \in \partial B$. Then, $b \sim a$ for some $a \in X \backslash B = \overset{0}{A}$. Since $a \in \overset{0}{A}$ and $a \sim b$, we should have $b \in A \backslash \overset{0}{A}$, which means $b \in \partial A$. Consequently, $\partial B = \partial A$ and $\overset{0}{B} = A^c$.

To show that B is circled, take $b \in \partial B$. Since $\partial B = \partial A$, b should have a neighbour $a \in A^c = \overset{0}{B}$. Hence B is circled. □

Proposition 2.2. *Let E be an arbitrary set and $F = V(\overset{0}{E})$. Then $\overset{0}{E} = \overset{0}{F}$ and $\partial F \subset \partial E$; also F is the largest circled set contained in E.*

Proof. By definition, $\overset{0}{E} \subset \overset{0}{F}$. Since $F \subset E$, we also have $\overset{0}{F} \subset \overset{0}{E}$. Hence $\overset{0}{E} = \overset{0}{F}$. To see $\partial F \subset \partial E$, first note that $\partial F \cap \overset{0}{E} = \partial F \cap \overset{0}{F} = \emptyset$. Then, $\overset{0}{E} \cup \partial F = \overset{0}{F} \cup \partial F = F \subset E = \overset{0}{E} \cup \partial E$ implies that $\partial F \subset \partial E$. $V(\overset{0}{E})$ is circled by definition and $V(\overset{0}{E}) \subset E$.

Suppose the circled set $A \subset E$. Then $\overset{0}{A} \subset \overset{0}{E}$. Let $x \in \partial A$. Then there exists $z \in \overset{0}{A}$ such that $x \sim z$. Since $z \in \overset{0}{E}$ and $z \sim x$, we find that $x \in F$. Hence $\partial A \subset F$. Then $A = \overset{0}{A} \cup \partial A \subset \overset{0}{E} \cup \partial F = \overset{0}{F} \cup \partial F = F$. □

We consider now functions defined on subsets of X. All functions are real-valued. Given a function f on $V(x)$, define $\Delta f(x) = \sum_{x \sim x_i} t(x, x_i)\,[f(x_i) - f(x)] = -t(x)f(x) + \sum_{x \sim x_i} t(x, x_i) f(x_i)$, where $t(x) = \sum_{x \sim x_i} t(x, x_i)$ and note $t(x) > 0$ for any $x \in X$. We say that f is *harmonic* (resp. *superharmonic*) at x if $\Delta f(x) = 0$ (resp. $\Delta f(x) \leq 0$). *A function f defined on an arbitrary set E is said to be harmonic (resp. superharmonic) on E if and only if $\Delta f(x) = 0$ (resp. $\Delta f(x) \leq 0$) for every $x \in \overset{0}{E}$.*

Note. Some authors prefer to define a real-valued function u as a harmonic function on E provided its Laplacian is 0 at every node of E. This presupposes that u is defined on $V(E)$. But for the topics we discuss here, such as the Dirichlet problem, the minimum principle, the condenser principle etc. u is either not defined outside E or its value outside E is not of consequence. Hence, it becomes important to distinguish between the interior and the boundary nodes of E. Thus, the Laplacian can be defined only for the interior nodes of E and a similar operator at the boundary nodes is the inner normal derivative [1].

Mimimum Principle. *Let E be circled and $\overset{0}{E}$ be connected. If s is a lower bounded superharmonic function on E and attains its minimum at a node in $\overset{0}{E}$, then s is constant.*

A variation of this principle is the following.

Proposition 2.3. *Let E be an arbitrary proper subset of X. Let s be a superharmonic function on E, attaining its minimum on E. Then $\inf_{\partial E} s = \inf_E s$.*

Proof. Let $\alpha = \inf_{\partial E} s$ and $\beta = \inf_E s$. Then, $\alpha \geq \beta > -\infty$. Suppose $\alpha > \beta$. Then $s(z) = \beta$ for some $z \in \overset{0}{E}$, by hypothesis. Choose $y \notin E$. There is a path $\{z = x_0, x_1, \ldots, x_n = y\}$ connecting z and y. Let i be such that $x_k \in \overset{0}{E}$ for all $k \leq i$ and $x_{i+1} \notin \overset{0}{E}$. Then $i < n$.

Now, $\beta t(z) = t(z)s(z) \geq \sum_{z_i \sim z} t(z, z_i) s(z_i) \geq \beta \sum t(z, z_i) = \beta t(z)$.

It is clear then $s(z_j) = \beta$ for every $z_j \sim z$. In particular $s(x_1) = \beta$. Continuing this process, we see that $s(x_{i+1}) = \beta$. Now $x_{i+1} \notin \overset{0}{E}$, but $x_{i+1} \sim x_i \in \overset{0}{E}$. Hence $x_{i+1} \in \partial E$. Consequently, $\inf_{x \in \partial E} s(x) \leq \beta$, which is a contradiction. This proves $\alpha = \beta$. □

Definition 2.4. An infinite network $N = \{x, y, t\}$ is called parabolic if and only if any lower bounded function s on X is a constant if $\Delta s \leq 0$ on X; otherwise X is called a hyperbolic network.

The minimum principle stated in Proposition 2.3 is mostly useful in case of a finite set E where the condition that the superharmonic function s attains its minimum on E is superfluous. However in the case of a parabolic network, we have the following Minimum Principle.

Theorem 2.5. *Let A be an arbitrary proper subset in a parabolic network. Let u be a lower bounded superharmonic function on A such that $u \geq \alpha$ on ∂A. Then $u \geq \alpha$ on A.*

Proof. Let $v = \inf(u, \alpha)$. Then v is superharmonic on A and $v = \alpha$ on ∂A. Suppose for some $z \in \overset{0}{A}$, $v(z) = \beta < \alpha$. Define $s(x) = v(x)$ if $x \in A$, and $= \alpha$ if $x \in X \backslash A$. Then s is superharmonic on X and since it is lower bounded on X also, s is a constant which should be α, contradicting the fact that $s(z) = v(z) = \beta < \alpha$. □

Remark 2.6. The above minimum principle is a characteristic property of parabolic networks. For, if N is a hyperbolic network, there exist superharmonic functions $p > 0$ on X such that $\inf_X p = 0$. Hence if A is the complement of a finite set, then $\inf_A p = 0$ while $\inf_{\partial A} p > 0$.

3. Dirichlet problem on an arbitrary set

In this section, the solution to the Dirchlet problem on an arbitrary subset of a network is obtained. This becomes an important tool in proving, in the next sections, many of the basic properties of superharmonic functions in a parabolic network.

Theorem 3.1 (Dirichlet problem). *Let E be a proper subset (finite or not) of a parabolic or hyperbolic network X. Let $F = V(E)$. Suppose f is a real valued function on $F \backslash E$. Let u (respectively, v) be superharmonic (respectively, subharmonic) on F such that $v \leq f \leq u$ on $F \backslash E$. Then there exists a function h such that $v \leq h \leq u$ on F, $h = f$ on $F \backslash E$ and h is harmonic at every node of E.*

Proof. Let $p = f$ on $F \backslash E$, and $p = u$ on E. Let $q = f$ on $F \backslash E$, and $q = v$ on E. Recall that $E \subset \overset{0}{F}$. Then p is superharmonic and q is subharmonic at each node of E; and $q \leq p$ on F. Let $\mathcal{F}$ be the family of functions s on F such that $s = f$ on $F \backslash E$, s is subharmonic at each node of E and $s \leq p$ on F. Let $z \in E$. Then for any $s \in \mathcal{F}$, $t(z)s(z) \leq \sum t(z, z_i)s(z_i) \leq \sum t(z, z_i)p(z_i) \leq t(z)p(z)$.

Define $s_1(x) = s(x)$ if $x \neq z$, and $s_1(x) = \sum \frac{t(z,z_i)}{t(z)} s(z_i)$ if $x = z$. Then $s_1(x)$ is subharmonic at each node of E, $s_1(x)$ is harmonic at $x = z$; $s_1 = f$ on $F \backslash E$; and

$s_1 \leq p$ on F. Hence $s_1 \in \mathcal{F}$. Consequently, if we define $h(x) = \sup s(x)$, for all $s \in \mathcal{F}$, then $h(x)$ is harmonic at each node of E. Moreover, $h = f$ on $F \backslash E$ and $v \leq q \leq h \leq p \leq u$ on F. □

Corollary 3.2. *Let E be an arbitrary subset of a parabolic network X. Let $F = V(E)$. Let f be a bounded real-valued function on $F \backslash E$. Then there exists a unique bounded function h on F such that $h = f$ on $F \backslash E$ and h is harmonic at each node of E.*

Proof. Let $\alpha \leq f \leq \beta$ on $F \backslash E$. Then take $u = \beta$ and $v = \alpha$ in the above theorem to arrive at the existence of h with the stated properties.

To prove the uniqueness, suppose h_1 is another bounded function on F such that $h_1 = f$ on $F \backslash E$ and h_1 is harmonic at every node of E. Then take $\phi = h - h_1$ so that ϕ is bounded on F, harmonic at each node of E and $\phi = 0$ on $F \backslash E$. Let $\phi_0 = \phi^+$ on F and $\phi_0 = 0$ on $X \backslash F$. Then ϕ_0 is bounded subharmonic on X, so that ϕ_0 is a constant which should be 0. Hence $\phi \leq 0$ on F; similarly $\phi \geq 0$ on F, so that $\phi \equiv 0$. □

Corollary 3.3 (See Theorem 2.3 [1], Classical Dirichlet Problem)**.** *Let E be an arbitrary subset of a parabolic network X, such that ∂E is finite (in particular, E is any finite set). Suppose f is a real valued function on ∂E. Then there exists a uniquely determined bounded function h on E such that $h = f$ on ∂E and h is harmonic at every node of $\overset{0}{E}$.*

Remark 3.4. Since the proof of the above Theorem 3.1 does not convincingly show that $h(x)$ is harmonic at each node of E, the referee suggests the inclusion of a note on the Perron family. Lemma 3.4 and Theorem 3.5 are the referee's. See also Constantinescu and Cornea [4, Folgesatz 1.3] and Premalatha and Kalyani [5, Theorem 3.3].

Recall that $V(x) = \{y \in E : y \sim x\} \cup \{x\}$. For a real-valued function u on E, and an arbitrary node $a \in \overset{0}{E}$, the discrete analogue $P_a u$ of the Poisson integral is denoted by $P_a u(x) = u(x)$ if $x \neq a$ and $P_a u(a) = \sum_{z \in X} \frac{t(z,a)}{t(a)} u(z)$.

Lemma 3.5. *Assume that u is superharmonic on E. Let $a \in \overset{0}{E}$. Then $P_a u$ is superharmonic on E, harmonic at a and $P_a u(x) \leq u(x)$ on E.*

Proof. Since u is superharmonic at a, we have $P_a u(a) \leq u(a)$. For $x \notin V(a)$, we have $P_a u = u$ on $E \cap V(x)$, so that $\Delta P_a u(x) = \Delta u(x) \leq 0$ at each $x \in \overset{0}{E} \backslash V(a)$. In case, $x \in \overset{0}{E} \cap V(a)$, we have for $x \neq a$, $\Delta P_a u(x) = -t(x) P_a u(x) + \sum_z t(z,x) P_a u(x) \leq -t(x) u(x) + \sum_z t(z,x) u(z) = \Delta u(x) \leq 0$; for $x = a$, we have $\Delta P_a u(a) = -t(a) P_a u(a) + \sum_z t(z,a) u(a) = 0$. □

A non-empty subset $\mathcal{J}$ of superharmonic functions on E is said to be *a Perron family* if it satisfies the following conditions.

(1) For any $v_1, v_2 \in \mathcal{J}$, there exists $v \in \mathcal{J}$ such that $v \leq \min(v_1, v_2)$.

(2) $P_a u \in \mathcal{J}$ for every $u \in \mathcal{J}$ and $a \in \overset{0}{E}$.
(3) There exists a real-valued function u_0 on $\mathcal{J}$ such that $v \geq u_0$ for all $v \in \mathcal{J}$.

Theorem 3.6. *If $\mathcal{J}$ is a Perron family on E, then $h(x) = \inf\{v(x) : v \in \mathcal{J}\}$ is harmonic on E.*

Proof. By (3), we have $h(x) \geq u_0(x)$ on E. Let $a \in \overset{0}{E}$ be fixed arbitrarily. Since $V(a)$ is a finite set, we can find a sequence $\{v_x^{(n)}\}$ in $\mathcal{J}$ for every $x \in V(a)$ such that $v_x^{(n)} \to h(x)$ as $n \to \infty$. By (1), there exists $u_n \in \mathcal{J}$ such that $u_n \leq \min\{v_x^{(n)}, x \in V(a)\}$. Then, $u_n(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$. Let $u_n^* = P_a u_n$. Then $u_n^* \in \mathcal{J}$ and $u_n^*(x) \to h(x)$ as $n \to \infty$ for every $x \in V(a)$ and u_n^* is harmonic at a. Consequently, we have $\Delta h(a) = \lim_{n\to\infty} \Delta u_n^*(a) = 0$. Thus h is harmonic at a. □

4. Some potential-theoretic results in a parabolic network

By making use of the Dirichlet solution in a parabolic network N, an analogue of the logarithmic potential in N is constructed in this section. Also by the same method, balayage, harmonic and superharmonic extentions, condenser principle etc. are investigated.

Theorem 4.1 (Green function, see Yamasaki [7, Theorem 3.2]). *Given any finite subset E of X and a node $e \in \overset{0}{E}$, there exists a unique (non-harmonic) superharmonic function $g \geq 0$ on E such that $g = 0$ on ∂E, and $\Delta g(x) = -\delta_e(x)$ on $\overset{0}{E}$.*

Proof. Let B be the set of neighbours $\{z_i\}$ of e. Let $|x|$ denote the distance of a node x from e. Choose m large so that $B_m = \{x : |x| \leq m\}$ is such that $\overset{0}{B}_m \supset E$. Let $A = B_m \backslash \{e\}$. Then $\partial A = B \cup \partial B_m$. Find the Dirichlet solution ϕ on A with boundary values 1 on B and 0 on ∂B_m. Extend ϕ to B_m by setting $\phi(e) = 1$. Then ϕ is superharmonic on B_m, with $0 \leq \phi \leq 1$.

Let $\mathcal{F}$ be the family of superharmonic functions $s \geq 0$ on B_m such that $0 \leq s \leq 1$ and $s(e) = 1$. Let $u(x) = \inf s(x)$, for all s in $\mathcal{F}$. Since $\phi \in \mathcal{F}$, it is clear (following the proof of Theorem 3.1) that $0 \leq u \leq 1$, $u(e) = 1$, $u(x) = 0$ on ∂B_m and u is harmonic at every node of $\overset{0}{B}_m \backslash \{e\}$; $u(x)$ is not harmonic at $x = e$, for otherwise $u \equiv 0$ by the minimum principle on B_m.

Now, let h be the Dirichlet solution on E, with boundary values u on ∂E. Let $v(x) = u(x) - h(x)$ on E. Then $v(x)$ is harmonic at every node in $\overset{0}{E} \backslash \{e\}$, $\Delta v(e) < 0$ and $v \geq 0$ on E. Define $g(x) = \frac{v(x)}{-\Delta v(e)}$ on E. Then $g(x) \geq 0$ on E, $g = 0$ on ∂E and $\Delta g(x) = -\delta_e(x)$ on $\overset{0}{E}$.

To prove the uniqueness, suppose there exists another function $g_1 \geq 0$ on E such that $g_1(x) = 0$ on ∂E and $\Delta g_1(x) = -\delta_e(x)$ on $\overset{0}{E}$. Let $\phi(x) = g(x) - g_1(x)$ on E. Then $\Delta \phi(x) = 0$ on $\overset{0}{E}$, $\phi = 0$ on ∂E and ϕ is bounded. Hence, by the minimum principle, $\phi \equiv 0$ on E. □

Lemma 4.2 (Harmonic Extension). *Let h be a harmonic function defined outside a finite set in a network. Let e be a fixed node. Then there exists a harmonic function u on $X\backslash\{e\}$ such that $(h-u)$ is bounded outside a finite set.*

Proof. For a node x, let $|x|$ denote the distance of x from the fixed node e. Let m be sufficiently large so that h is defined on $\overset{0}{B}{}_m^{c}$ where $B_m = \{x : |x| \leq m\}$. Let f be the bounded Dirichlet solution outside B_m with boundary values h on ∂B_m (Corollary 3.3). Write $\phi = h - f$ which is harmonic on $\overset{0}{B}{}_m^{c}$ with boundary values 0.

Let $p \geq 0$ be the superharmonic function on B_m with harmonic point singularity $\{e\}$ and $p = 0$ on ∂B_m (Theorem 4.1). By the minimum principle, $p > 0$ on $\overset{0}{B}_m$. For a large $\alpha > 0$, let $v_1 = \phi$ on $\overset{0}{B}{}_m^{c}$, and $v_1 = \alpha p$ on $\overset{0}{B}_m$. On ∂B_m, $t(y)v_1(y) = 0 \leq \sum t(y, y_i)v_1(y_i)$, since $y \in \partial B_m$ has a neighbour in $\overset{0}{B}_m$ where v_1 can be made to take an arbitrarily large value since α is large and arbitrary. Then, v_1 is subharmonic on $X\backslash\{e\}$, harmonic outside ∂B_m. Similarly, if $v_2 = \phi$ on $\overset{0}{B}{}_m^{c}$ and $v_2 = -\alpha p$ on $\overset{0}{B}_m$, then v_2 is superharmonic on $X\backslash\{e\}$ and harmonic outside ∂B_m. Choose now $\beta > 0$ large, so that $\alpha p \leq -\alpha p + \beta$ on $\overset{0}{B}_m$.

Then, on $X\backslash\{e\}$, $v_1 \leq v_2 + \beta$. Use now the method of proof of Theorem 3.1 to determine a harmonic function u on $X\backslash\{e\}$ such that $v_1 \leq u \leq v_2 + \beta$ on $X\backslash\{e\}$. Clearly, $(u - h)$ is bounded outside $\overset{0}{B}_m$. □

Remark 4.3. The above Lemma 4.2 is not of much interest in a hyperbolic network. For, by using the Dirichlet solution on a finite set, it is easy to see that if h is harmonic outside a finite set in a hyperbolic network N, then there exists a unique harmonic function H and two bounded potentials p_1 and p_2 on N such that $h = H + p_1 - p_2$ outside a finite set in N.

Recall that a superharmonic function s in a network is said to be *admissible* if and only if it has a harmonic minorant outside a finite set. It is immediate (using Lemma 4.2) that if s is admissible and A is any finite set, then s has a harmonic minorant outside A. Two admissible superharmonic functions in a network are said to be *equivalent*, if the difference between their greatest harmonic minorants outside a finite set is bounded.

Theorem 4.4. *Let s be an admissible superharmonic function on X. Let A be any proper subset of X. Then the Dirichlet problem is solvable on A with boundary values s on ∂A.*

Proof. Let u be a harmonic minorant of s outside a finite set in X. Let $e \notin A$. Then there exists a harmonic function v on $X\backslash e$ such that $(u - v)$ is bounded outside a finite set. Hence s has a harmonic minorant on A. Let g be the greatest harmonic minorant (g.h.m.) of s on A.

Define $\phi(x) = s(x)$ on ∂A, and $\phi(x) = g(x)$ on $\overset{0}{A}$. Then ϕ is subharmonic on A, $\phi \leq s$ on A and $\phi = s$ on ∂A. Hence by Theorem 3.1, there exists a harmonic function h on A such that $h = s$ on ∂A. Note that $h = g$ is the g.h.m. of s on A. □

Remark 4.5. (See Theorem 3.13 [1], Balayage) Let u be an admissible superharmonic function on X. Let E be any proper subset of X. Let h be the Dirichlet solution on E with boundary values u on ∂E. Define $B_u^E = h$ on E, and $= u$ on E^c. Then, B_u^E *is an admissible superharmonic function on* X *such that* $B_u^E \leq u$ *on* X, $B_u^E = u$ *on* $\overset{0}{E}{}^c$; B_u^E *is harmonic on E; and* B_u^E *is equivalent to u.*

Theorem 4.6 (Condenser Principle). *Let A and B be two disjoint (finite or not) subsets of X. Then there exists a function* ϕ, $0 \leq \phi \leq 1$ *on* X, *such that*

(i) $\phi = 0$ *on A and subharmonic at every node of A;*
(ii) $\phi = 1$ *on B and superharmonic at every node of B;*
(iii) ϕ *is harmonic at every node of* $X \backslash (A \cup B)$.

Proof. Let $E = [X \backslash (A \cup B)]$. Let $F = V(E)$. Let $f = 0$ on $(F \backslash E) \cap A$ and $f = 1$ on $(F \backslash E) \cap B$. Then (Theorem 3.1), there exists a function h (the Dirichlet solution) such that $h = f$ on $F \backslash E$, and $\Delta h = 0$ at every node of E. Extend h by 0 on A, and by 1 on B. Let the thus extended function be denoted by ϕ. Then ϕ satisfies the conditions stated in the theorem. □

Theorem 4.7. *Let E be a connected infinite subset of a parabolic network. Let* $F = V(E)$. *Then there exists an unbounded function* $H \geq 0$ *on F such that H is positive harmonic at every node of E and* $H = 0$ *on* $F \backslash E$.

Proof. Since E is connected, $F = V(E)$ is connected also. Fix a vertex e and let $|x|$ denote the distance of x from e; write $B_n = \{x : |x| \leq n\}$ and $S_n = \{x : |x| = n\}$. Choosing a subsequence of B_n, if necessary, we can assume without loss of generality that $E \cap S_n \neq \emptyset$.

Let $f_n = 0$ on $(F \backslash E) \cap B_{n-1}$, and $f_n = 1$ on S_n. Let h_n be the function on $F \cap B_n$ such that $h_n = f_n$ on $[(F \backslash E) \cap B_n] \cup [E \cap S_n]$ and $\Delta h_n = 0$ on $E \cap B_{n-1}$. (Note that $E \cap B_{n-1}$ is in the interior of $F \cap B_n$. For, if $x \in E \cap B_{n-1}$ and $x \sim y$, then $|y| \leq n$ and $y \in F$, so that $y \in F \cap B_n$.)

Let $u_n = h_n$ on $F \cap B_n$, and $= 1$ on $F \cap B_n^c$. Then u_n is defined on F such that $\Delta u_n \leq 0$ on E. Note $u_n > 0$ on $E \cap B_{n-1}$. For, suppose $u_n(z) = h_n(z) = 0$ for some $z \in E \cap B_{n-1}$. Let $a \in E \cap S_n$. Since E is connected, there exists a path connecting z to a; call it $\{z = z_0, z_1, z_2, \ldots, z_m = a\}$. Here $|z| \leq n-1$ and $|z_m| = n$. It is possible that there is some other node in this path lying on S_n. Let i be the smallest index such that $|z_i| = n$ and $\Delta h_n(z_j) = 0$ for $0 \leq j \leq i-1$. Since $h_n \geq 0$ on $F \cap B_n$, $h_n(z) = 0$ and $\Delta h_n(z) = 0$, by the minimum principle we should have $h_n(x) = 0$ for all $x \sim z$. In particular, $h_n(z_1) = 0$; again since $\Delta h_n(z_1) = 0$, $h_n(z_2) = 0$. Repeating this process, we arrive at the conclusion $h_n(z_i) = 0$; but this is a contradiction, for $h_n(z_i) = 1$ since $|z_i| = n$.

Consequently, $u_n > 0$ and $\Delta u_n \leq 0$ on E for all n. Define $v_n(x) = \frac{u_n(x)}{u_n(x_0)}$ on E, where x_0 is a node fixed in E. Then on the connected set E, $v_n > 0$, $\Delta v_n \leq 0$ and $v_n(x_0) = 1$ for all $n \geq$ some N. Hence by the Harnack property, $\{v_n(z)\}$ is bounded for any $z \in E$. This enables us to extract a subsequence $\{v'_n\}$ from $\{v_n\}$

such that $\lim v'_n(x) = H(x)$ exists for any $x \in E$. Since for each x in E, $\Delta v'_n(x) = 0$ except for a finite number of n, we conclude that $H(x)$ is harmonic at every node of E, $H > 0$ on E and $H = 0$ on $F\backslash E$.

By the minimum principle in a parabolic network (Theorem 2.4), note that H should be unbounded on E. □

Remark 4.8. Fix a node e in a parabolic network X. Let $\{x_i\}$ be the neighbours of e. Let $E_i = \{x :$ the path connecting x and e passes through $x_i\}$. We shall place x_i also in E_i but $e \notin E_i$. Then E_i is connected and the E_i's are finite in number, so at least one of them, say E_j, contains an infinite number of nodes. Take $E = \cup_k (E_j \cup E_k)$ if $E_j \cap E_k \neq \emptyset$. Then E is an infinite connected component which contains all the nodes $y \in X\backslash\{e\}$ such that there is a path in $X\backslash\{e\}$ connecting y and x_j. (The situation is somewhat like in the Euclidean plane where the complement of a non-empty compact set consists of disjoint connected components, one of which is unbounded.) Let $F = V(E)$. Note that $F = E \cup \{e\}$. Construct a function $H \geq 0$ on F as in the theorem, which is harmonic at every node of E, $H > 0$ on E and $H = 0$ on $F\backslash E$. Extend H to the whole of X, by giving values 0 to nodes outside F. If we denote this extension by H_e, it has the following properties: $H_e \geq 0$ on X, $H_e(e) = 0$, $H_e(x)$ is harmonic at every node $x \neq e$; and note that $H_e(x)$ should be unbounded on X, for otherwise H_e being subharmonic and bounded on a parabolic network should be a constant in X, which is a contradiction. Clearly such functions H_e may be many and non-proportional. In the sequel, we shall fix a node e and one such function H_e.

Theorem 4.9. *For any $z \in X$, there exists a unique superharmonic function $q_z(x)$ on a parabolic network X such that (i) $\Delta q_z(x) = -\delta_z(x)$, (ii) $q_z(z) = 0$, and (iii) for a (unique) $\alpha \geq 0$, $(q_z + \alpha H_e)$ is bounded on X.*

Proof. Let A be a finite circled set such that e and z are in $\overset{0}{A}$. Let h be the bounded harmonic function on A^c with values H_e on ∂A (Corollary 3.3), so that $h - H_e \leq 0$ on A^c. Let $\mathcal{F}$ be the family of all superharmonic functions s on X such that $s(z) \geq 0$ and $s \geq h - H_e$ on A^c. Clearly, the superharmonic function $s \equiv 0$ is in $\mathcal{F}$. Let $u(x) = \inf s(x)$, for all s in $\mathcal{F}$.

For any $y \neq z$ and any $s \in \mathcal{F}$, if we define $s_1(x) = s(x)$ if $x \neq y$, and $s_1(y) = \sum \frac{t(y,y_i)}{t(y)} s(y_i)$, then $s_1 \leq s$, $s_1(x)$ is harmonic at $x = y$ and $s_1 \in \mathcal{F}$. Consequently, $u(x)$ is harmonic on $X\backslash\{z\}$ and $u(x) \geq h(x) - H_e(x)$ on A^c. Let $v(x) = h(x) - H_e(x)$ on A^c, extended by 0 on A. Then $v \in \mathcal{F}$. Hence, $u(x) \leq h(x) - H_e(x)$ on A^c and $u(z) = 0$. Also, since $y \sim z$ implies that $y \in A$, we have $\sum \frac{t(z,y)}{t(z)} u(y) \leq 0 = u(z)$, so that $u(x)$ is superharmonic at $x = z$. Note that u being upper bounded and non-constant, $u(x)$ cannot be harmonic at $x = z$. Hence $\Delta u(z) < 0$. Write $q_z(x) = -\frac{u(x)}{\Delta u(z)}$. Then, $\Delta q_z(x) = -\delta_z(x)$ on X, $q_z(z) = 0$ and $\left[q_z(x) - \frac{H_e(x)}{\Delta u(z)}\right]$ is a bounded harmonic function on A^c. That is, $q_z(x)$ has all the properties stated in the theorem.

To prove the uniqueness of q_z, suppose Q is another function on X, such that $\Delta Q(x) = -\delta_z(x)$, $Q(z) = 0$ and $(Q + \beta H_e)$ is bounded on X. Then, let $l(x) =$

$q_z(x) - Q(x)$. Since $\Delta l(x) \equiv 0$, $l(x)$ is harmonic on X. Moreover, $l(x) = (\beta - \alpha)H_e(x) + f(x)$, where $f(x)$ is bounded on X and harmonic outside a finite set. Suppose $\beta - \alpha \neq 0$. Then, since H_e is positive, $l(x)$ should be bounded on one side on X. This means that $l(x)$ is a constant, since X is a parabolic network and $l(x)$ is a harmonic function on X bounded on one side. Consequently, $(\beta - \alpha)H_e$ should be bounded on X. But this is a contradiction since H_e is unbounded and $\beta - \alpha \neq 0$. We conclude therefore that $\alpha = \beta$, in which case $l(x)$ is a constant that should be 0. □

Proposition 4.10. *Given a superharmonic function s outside a finite set in a X, there exist two superharmonic functions Q and Q_c on X such that Q_c has finite harmonic support and $s = Q - Q_c$ outside a finite set.*

Proof. Choose a finite circled set A such that s is defined on ∂A. Let h be the harmonic function on A with boundary values s on ∂A. Let $p = s$ on A^c, and $= h$ on A. Then, $\Delta p = 0$ on $\overset{0}{A}$, and $\Delta p \leq 0$ on A^c. It is possible that $\Delta p > 0$ at some nodes on ∂A. Let them be $y_1, y_2, \ldots, y_i$. Consider $Q(x) = p(x) + \Delta p(y_1)q_{y_1}(x) + \ldots + \Delta p(y_i)q_{y_i}(x)$ where $q_z(x)$ denotes the superharmonic function on X with $\Delta q_z(x) = -\delta_z(x)$. Then $\Delta Q(y_j) = 0$ for $1 \leq j \leq i$, that is Q is superharmonic on X. Moreover, if we write $Q_c(x) = \sum_{j=1}^{i} \Delta p(y_j)q_j(x)$, then $s(x) = p(x) = Q(x) - Q_c(x)$ on A^c. This proves the proposition. □

In the above proposition, Q_c can be removed in some cases, as the following proposition shows. That is, p extends as a superharmonic function on X. To see that, write $X = \cup A_n$ where A_n is an increasing sequence of finite circled sets such that $A_{n-1} \subset \overset{0}{A_n}$ and for large n, take s_n as the Dirichlet solution in A_n with boundary values s on ∂A_n.

Proposition 4.11. *Given a superharmonic function s outside a finite set in a parabolic network, there exists a (non-harmonic) superharmonic function Q on X such that $s = Q$ outside a finite set if and only if s_n tends to $-\infty$ uniformly on finite sets.*

Proof. The network being parabolic, if $s = Q$ outside a finite set, then $s_n = Q_n$ for large n so that s_n tends to $-\infty$ uniformly on finite sets. Conversely, assume that s is defined outside A_{m-1}. Since s_n tends to $-\infty$ uniformly on A_m, choose n large so that $s_n(x) \leq \inf_{\partial A_m} s$ on A_m. Then by the minimum principle, $s_n \leq s$ on $A_n \backslash A_m$. Define $Q = s$ on A_n^c, and $= s_n$ on A_n. Then Q is superharmonic on X and $s = Q$ outside A_n. □

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