Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
40 (2007), pp. 33–62

A CHRACTERIZATION OF PARALLEL ISOMETRIC IMMERSIONS OF A QUATERNIONIC SPACE FORM INTO A REAL SPACE FORM

HIROMASA TANABE

Communicated by S. Maeda

(Received: September 25, 2006)

Dedicated to Professor Tetsuo Furumochi on the occasion of his sixtieth birthday.

Abstract. We give a chracterization of all parallel isometric immersions of a quaternionic space form into a real space form by the extrinsic shape of some kind of Frenet curves of order 2 which is closely related to the quaternionic Kähler structure.

1. Introduction

Let $f : M \to \widetilde{M}$ be an isometric immersion of a Riemannian manifold M into an ambient Riemannian manifold $\widetilde{M}$. It is possible in some cases to know the shape of the submanifold M by examining the extrinsic shape of curves in the submanifold M.

A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M parametrized by its arclength s is called a *Frenet curve of proper order* 2 if there exist a field of orthonormal frames $\{V_1 = \dot{\gamma}, V_2\}$ along γ and a positive smooth function $\kappa = \kappa(s)$ satisfying that

$$\nabla_{\dot{\gamma}} V_1(s) = \kappa(s) V_2(s) \quad \text{and} \quad \nabla_{\dot{\gamma}} V_2(s) = -\kappa(s) V_1(s), \tag{1.1}$$

where $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The function κ and the orthonormal frame $\{V_1, V_2\}$ are called the *curvature* and the *Frenet frame* of γ, respectively. A Frenet curve of proper order 2 with constant curvature $k(> 0)$ is called a *circle* of curvature k. We regard a geodesic as a circle of null curvature. A curve is said to be a Frenet curve *of order* 2 if it is either a geodesic or a Frenet curve of proper order 2.

In their paper [6], K. Nomizu and K. Yano proved that a submanifold M is an extrinsic sphere of $\widetilde{M}$, namely M is a totally umbilic submanifold with parallel

2000 *Mathematics Subject Classification.* 53B25, 53C40.

Key words and phrases. quaternionic space forms, quaternionic Frenet curves, parallel isometric immersions, real space forms.

mean curvature vector in $\widetilde{M}$, if and only if there exists some positive number k such that all circles of curvature k in M are circles in the ambient space $\widetilde{M}$.

Motivated by this result, the author showed that M is a totally geodesic submanifold of $\widetilde{M}$ if and only if there exists some nonconstant positive smooth function $\kappa = \kappa(s)$ such that all Frenet curves of proper order 2 of curvature κ in M are Frenet curves of order 2 in the ambient space $\widetilde{M}$([11]). Moreover, in [5, 12], S. Maeda and the author considered a class of Frenet curves of order 2 in a Kähler manifold M which satisfies the condition that the Frenet frame $\{V_1, V_2\}$ spans a holomorphic plane, that is, $V_2 = JV_1$ or $V_2 = -JV_1$ for the complex structure J of M. By using such a class of curves they characterized all totally geodesic Kähler immersions of Kähler manifolds into an ambient Kähler manifold and all parallel isometric immersions of a complex space form into a real space form.

The purpose of the present paper is to provide a characterization of every parallel isometric immersion of a quaternionic space form $M^n(c\,;\mathbb{H})$ into a real space form $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ by observing the extrinsic shape of *quaternionic Frenet curves* in $M^n(c\,;\mathbb{H})$, which is a particular class of Frenet curves of order 2 closely related to the quaternionic Kähler structure. That is, we shall prove the following theorem:

Theorem. *Let M^n be a quaternionic Kähler manifold of quaternionic dimension n and f an isometric immersion of M^n into a real space form $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$. If there exists a positive smooth function κ satisfying that f maps every quaternionic Frenet curve γ of curvature κ in M^n to a plane curve in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$, then the immersion f is rigid. Moreover, f is a parallel immersion and locally congruent to one of the following*:

(1) *f is a totally geodesic immersion of $M^n = \mathbb{H}^n = \mathbb{R}^{4n}$ into $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R}) = \mathbb{R}^{4n+p}$, where $\tilde{c} = 0$,*
(2) *f is a totally umbilic immersion of $M^n = \mathbb{H}^n = \mathbb{R}^{4n}$ into $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R}) = \mathbb{R}H^{4n+p}(\tilde{c})$, where $\tilde{c} < 0$,*
(3) *f is a parallel immersion defined by*

$$f = f_2 \circ f_1 : M^n = \mathbb{H}P^n(c) \xrightarrow{f_1} S^{2n^2+3n-1}((n+1)c/(2n)) \xrightarrow{f_2} \widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R}),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion and $(n+1)c/(2n) \geq \tilde{c}$.

This theorem is a quaternionic version of the preceding results in [5, 12]. The accurate definition of a quaternionic Frenet curve will be given in section 2. In section 3, we review fundamental equations in submanifold theory and the notion of isotropic immersions. In section 4, we discuss the rigidity of constant isotropic parallel immersions of a quaternionic space form $M^n(c\,;\mathbb{H})$ into a real space form $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$. The proof of our theorem will be given in section 5.

Acknowledgement. The author wishes to express his appreciation to Professor S. Maeda for his constant encouragement and valuable suggestions.

2. Quaternionic Frenet curves in quaternionic Kähler manifolds

Let M be a Riemannian manifold of real dimension $4n$ with Riemannian metric $\langle\ ,\ \rangle$. A *quaternionic Kähler structure* $\mathcal{J}$ on M is a rank 3 vector subbundle of the bundle of endmorphism of the tangent bundle TM with the following properties.

(1) For each point $x \in M$ there exist an open neighborhood U of x in M and sections J_1, J_2, J_3 of the restriction $\mathcal{J}|_U$ over U such that
   (a) each J_r $(r = 1, 2, 3)$ is an almost Hermitian structure on U, that is,

$$J_r{}^2 = -id \quad \text{and} \quad \langle J_rX, Y\rangle + \langle X, J_rY\rangle = 0$$

   for all vector fields X and Y on U.
   (b) $J_rJ_{r+1} = J_{r+2} = -J_{r+1}J_r$ $(r \bmod 3)$ for $r = 1, 2, 3$.
(2) The condition that $\nabla_X J$ is a section of $\mathcal{J}$ holds for each vector field X on M and section J of the bundle $\mathcal{J}$.

This triple $\{J_1, J_2, J_3\}$ is called a *canonical local basis* of $\mathcal{J}$. For each canonical local basis of quaternionic Kähler structure, there exist three 1-forms q_1, q_2 and q_3 on U satisfying

$$\nabla_X J_r = q_{r+2}(X)J_{r+1} - q_{r+1}(X)J_{r+2} \quad (r \bmod 3) \tag{2.1}$$

for each vector field X on U and $r = 1, 2, 3$. A Riemannian manifold M of real dimension $4n$ with a quaternionic Kähler structure $\mathcal{J}$ is called an n-dimensional *quaternionic Kähler manifold*.

We say that an n-dimensional connected quaternionic Kähler manifold M is an *n-dimensional quaternionic space form* of quaternionic sectional curvature c $(\in \mathbb{R})$ if the Riemannian sectional curvature of M is equal to c for all tangent 2-planes spanned by $X \in T_xM$ and JX with $J \in \mathcal{J}_x$ at each point $x \in M$. We donote it by $M^n(c\,;\mathbb{H})$. The standard model of a quaternionic space form is locally congruent to one of a quaternionic projective space $\mathbb{H}P^n(c)$ of quaternionic sectional curvature c (> 0), a quaternionic Euclidean space $\mathbb{H}^n$ and a quaternionic hyperbolic space $\mathbb{H}H^n(c)$ of quaternionic sectional curvature c (< 0).

Let $\gamma = \gamma(s)$ be a Frenet curve of proper order 2 in a quaternionic Kähler manifold M which satisfies (1.1). For this curve γ we put

$$\tau_\gamma = \sqrt{\sum_{r=1}^{3} \langle V_1, J_rV_2\rangle^2}\,.$$

Since we have from (1.1) and (2.1)

$$\begin{aligned}
\frac{d}{ds}\tau_\gamma^2 &= 2\sum_{r=1}^{3}\langle V_1, J_rV_2\rangle\big\{\langle \nabla_{\dot\gamma}V_1, J_rV_2\rangle + \langle V_1, (\nabla_{\dot\gamma}J_r)V_2\rangle + \langle V_1, J_r(\nabla_{\dot\gamma}V_2)\rangle\big\}\\
&= 2\sum_{r=1}^{3}\langle V_1, J_rV_2\rangle\big\{\langle \kappa V_2, J_rV_2\rangle\\
&\qquad\qquad + \langle V_1, q_{r+2}(\dot\gamma)J_{r+1}V_2 - q_{r+1}(\dot\gamma)J_{r+2}V_2\rangle + \langle V_1, -\kappa J_rV_1\rangle\big\}\\
&= 2\sum_{r=1}^{3} q_{r+2}(\dot\gamma)\langle V_1, J_rV_2\rangle\langle V_1, J_{r+1}V_2\rangle - 2\sum_{r=1}^{3} q_{r+1}(\dot\gamma)\langle V_1, J_rV_2\rangle\langle V_1, J_{r+2}V_2\rangle\\
&= 0,
\end{aligned}$$

we see that τ_γ is constant along γ. We call τ_γ *structure torsion* of γ (see [1]). Then it is easy to prove that for the structure torsion τ_γ of γ satisfying (1.1) the following two conditions are mutually equivalent:

(1) $\tau_\gamma = 1$,

(2) there exist a smooth section J of $\mathcal{J}$ with $J^2 = -id$ such that

$$V_2(s) = J_{\gamma(s)}V_1(s) \quad \text{for each } s.$$

A Frenet curve γ of proper order 2 in a quaternionic Kähler manifold M is said to be a *quaternionic Frenet curve* if it satisfies one (hence both) of the conditions above. A quaternionic Frenet curve of constant curvature $k(>0)$ is called a *quaternionic circle* of curvature k. We regard a geodesic as a quaternionic circle of null curvature. Thus the notion of quaternionic Frenet curves is a natural extension of that of quaternionic circles(see [9]).

We note that for an arbitrary unit tangent vector X at any point x of M and for an arbitrary $J \in \mathcal{J}_x$ with $J^2 = -id$, there exists a unique quaternionic Frenet curve $\gamma = \gamma(s)$ defined on some open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ such that

$$\gamma(0) = x, \quad \dot\gamma(0) = V_1(0) = X \quad \text{and} \quad V_2(0) = JX.$$

3. Isotropic immersions

We recall a few fundamental notions in submanifold theory. Let M, $\widetilde{M}$ be Riemannian manifolds and $f : M \to \widetilde{M}$ an isometric immersion. We identify a vector X of M with a vector $f_*(X)$ of $\widetilde{M}$. The Riemannian metrics on M, $\widetilde{M}$ are denoted by the same notation $\langle\ ,\ \rangle$. We denote by ∇ and $\widetilde\nabla$ the covariant differentiations of M and $\widetilde{M}$, respectively. Then the formulae of Gauss and Weingarten are

$$\widetilde\nabla_X Y = \nabla_X Y + \sigma(X, Y),$$

$$\widetilde\nabla_X \xi = -A_\xi X + D_X\xi,$$

where σ, A_ξ and D denote the second fundamental form of f, the shape operator in the direction of ξ and the covariant differentiation in the normal bundle, respectively. We define the covariant differentiation $\bar\nabla$ of the second fundamental form σ

with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) as follows:

$$(\bar{\nabla}_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{3.1}$$

If $\bar{\nabla}\sigma = 0$, we say that the isometric immersion f is *parallel*.

A *real space form* $\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$ is an n-dimensional Riemannian manifold of constant sectional curvature $\tilde{c}$, which is locally congruent to either a Euclidean space $\mathbb{R}^n$, a standard sphere $S^n(\tilde{c})$ or a real hyperbolic space $H^n(\tilde{c})$ according as the curvature $\tilde{c}$ is zero, positive or negative. In case that the ambient manifold is a real space form $\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$, the *equations of Gauss, Codazzi* and *Ricci* for an isometric immersion $f : M \to \widetilde{M}^n(\tilde{c}\,;\mathbb{R})$ can be written as follows:

$$\begin{aligned}\langle R(X,Y)Z,\, W\rangle =&\tilde{c}\,(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)\\ &+ \langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle,\end{aligned} \tag{3.2}$$

$$(\bar{\nabla}_X\sigma)(Y,Z) = (\bar{\nabla}_Y\sigma)(X,Z), \tag{3.3}$$

$$\langle R^{\perp}(X,Y)\zeta,\, \xi\rangle = \langle[A_\zeta, A_\xi]X,\, Y\rangle \tag{3.4}$$

for vector fields X, Y, Z, W of M and normal vector fields ζ, ξ, where R, $R^{\perp}$ represent the curvature tensors for ∇, D respectively.

For a unit vector $X \in T_xM$, $\sigma(X,X)$ is called the *normal curvature vector* determined by $X \in T_xM$. An isometric immersion $f : M \to \widetilde{M}$ is said to be (λ_x-)*isotropic* at $x \in M$ if there exists a nonnegative constant λ_x such that $\|\sigma_x(X,X)\| = \lambda_x$ for every unit tangent vector $X \in T_xM$. If there exists a nonnegative constant λ satisfying that $\|\sigma_x(X,X)\| = \lambda$ for every point $x \in M$ and for every unit tangent vector $X \in T_xM$, then $f : M \to \widetilde{M}$ is called a *constant* (λ-)*isotropic immersion*.

The *first normal space* at the point x of M is defined as the subspace $N^1_x(M)$ of N_xM spanned by the image of the second fundamental form at x, that is,

$$N^1_x(M) = Span_{\mathbb{R}}\{\sigma(X,Y); X, Y \in T_xM\} \subset N_xM.$$

If the dimension of the first normal space does not depend on $x \in M$, then the first normal space

$$N^1(M) = \bigcup_{x\in M} N^1_x(M)$$

is a subbundle of the normal bundle NM.

The *discriminant* Δ_x at $x \in M$ is defined by

$$\Delta_x = K(X,Y) - \widetilde{K}(X,Y),$$

where $K(X,Y)$ (resp. $\widetilde{K}(X,Y)$) represents the sectional curvature of the plane spanned by orthonormal vectors $X, Y \in T_xM$ for M (resp. for $\widetilde{M}$).

The following two lemmas are due to B. O'Neill ([7]):

Lemma 1. *Let M, $\widetilde{M}$ be Riemannian manifolds and $f : M \to \widetilde{M}$ an isometric immersion. Then the following are mutually equivalent.*

(1) *f is λ_x-isotropic at $x \in M$ for some $\lambda_x(\geq 0)$.*
(2) *$\langle\sigma(X,X),\sigma(X,Y)\rangle = 0$ for an arbitrary orthogonal pair $X, Y \in T_xM$.*

(3) $\langle\sigma(X,Y),\sigma(Z,W)\rangle + \langle\sigma(X,Z),\sigma(W,Y)\rangle + \langle\sigma(X,W),\sigma(Y,Z)\rangle$
$= \lambda_x^2(\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle)$
for some $\lambda_x(\geq 0)$ *and for any vectors* $X,Y,Z,W \in T_xM$.

Lemma 2. *Let* f *be an isometric immersion of* M *into* $\widetilde{M}$. *Suppose that* f *is* $\lambda_x\,(>0)$*-isotropic at each point* x *of* M *and the discriminant* Δ_x *at* $x \in M$ *is constant. Then we have*

$$-\frac{n+2}{2(n-1)}\,\lambda_x^2 \leq \Delta_x \leq \lambda_x^2.$$

Moreover,

(1) $\Delta_x = \lambda_x^2 \iff f$ *is umbilic at* $x \in M \iff \dim N_x^1(M) = 1$,

(2) $\Delta_x = -\dfrac{n+2}{2(n-1)}\lambda_x^2 \iff f$ *is minimal at* $x \in M \iff \dim N_x^1(M) = \dfrac{n(n+1)}{2} - 1$,

(3) $-\dfrac{n+2}{2(n-1)}\lambda_x^2 < \Delta_x < \lambda_x^2 \iff \dim N_x^1(M) = \dfrac{n(n+1)}{2}$.

We note that these lemmas are obtained by quite algebraic argument on the linear mapping of T_xM into $T_{f(x)}\widetilde{M}$.

4. The rigidity of constant isotropic parallel immersions of a quaternionic space form into a real space form

We have the following Lemma ([3, 4, 10]):

Lemma 3. *Let* M *be a quaternionic* n*-dimensional connected quaternionic Käher manifold with canonical local basis* $\{J_1, J_2, J_3\}$ *which is isometrically immersed into a real space form* $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ *through an immersion* f. *Then the following two conditions are equivalent*:

(1) *The isometric immersion* f *is parallel.*
(2) $\sigma(J_rX, J_rY) = \sigma(X,Y)$ *for all* $X, Y \in TM$ *and* $r = 1,2,3$.

Now, we shall prove the following proposition:

Proposition 1. *Let* M *be a connected open submanifold of an* n*-dimentional quaternionic space form* $M^n(c\,;\mathbb{H})$ *of constant quaternionic sectional curvature* c *and* f *a constant* $(\lambda\text{-})$*isotropic parallel immersion of* M *into a real space form* $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ *of constant sectional curvature* $\tilde{c}$. *Then the immersion* f *is locally rigid.*

Proof. The rigidity of totally umbilic submanifolds of a real space form $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ is well-known. Hence we may assume that f is not totally umbilic. Moreover, we have only to study in case that M is a connected open dense subset of $M^n(c\,;\mathbb{H})$ which has no umbilic point because of continuity. We denote by σ the second fundamental form of f. Since the immersion f is parallel, the first normal space $N^1(M)$ is invariant under parallel translation with respect to the connection in the normal bundle and the dimension of $N^1(M)$ is constant on M. In fact, let x and y be arbitrary two points of M. Let γ be a curve from x to y in M. Take an orthonormal basis $\{X_1, \dots, X_{4n}\}$ for T_xM and parallel translate this frame to y along γ with respect to the Riemannian connection ∇ of M. Thus we have orthonormal frame field parallel along γ, which is denoted by $\{Y_1, \dots, Y_{4n}\}$. Then $\sigma(Y_i, Y_j)$ is parallel along γ with respect to the normal connection D, because

$$D_{\dot\gamma}(\sigma(Y_i, Y_j)) = (\bar\nabla_{\dot\gamma}\sigma)(Y_i, Y_j) + \sigma(\nabla_{\dot\gamma}Y_i, Y_j) + \sigma(Y_i, \nabla_{\dot\gamma}Y_j) = 0.$$

Noting that the set $\{\sigma(Y_i(y), Y_j(y));\ i, j = 1, \dots, 4n\}$ spans $N^1_y(M)$, we see that the parallel translation along any γ from x to y with respect to D gives a linear isomorphism of $N^1_x(M)$ to $N^1_y(M)$. Therefore the dimension of $N^1(M)$ is constant and $N^1(M)$ is invariant under parallel translation. By celebrated Theorem of J. Erbacher [2], we may assume that the first normal space $N^1_x(M)$ coinsides with the normal space at any point $x \in M$.

Let R denote the curvature tensor of M. From the equation of Gauss (3.2) and Lemma 1(3) we have

$$\begin{aligned} &3\langle\sigma(X,Y), \sigma(Z,W)\rangle \\ &\qquad = \langle R(Z,X)Y, W\rangle + \langle R(Z,Y)X, W\rangle \\ &\qquad\quad - \tilde c\{2\langle X,Y\rangle\langle Z,W\rangle - \langle Z,Y\rangle\langle X,W\rangle - \langle Z,X\rangle\langle Y,W\rangle\} \\ &\qquad\quad + \lambda^2\{\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\} \end{aligned} \tag{4.1}$$

for all vector fields X, Y, Z and W tangent to M. On the other hand, the curvature tensor R of M is given by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\Big\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y \\ &+ \sum_{r=1}^{3}\big(\langle J_rY, Z\rangle J_rX - \langle J_rX, Z\rangle J_rY + 2\langle X, J_rY\rangle J_rZ\big)\Big\} \end{aligned} \tag{4.2}$$

for all vector fields X, Y and Z tangent to M, where $\{J_1, J_2, J_3\}$ denotes the canonical local basis of quaternionic Kähler structure of $M \subset M^n(c\,;\mathbb{H})$. The equations (4.1) and (4.2) yield the following:

$$\begin{aligned} &\langle\sigma(X,Y), \sigma(Z,W)\rangle \\ &\qquad = \frac{1}{3}\left\{\lambda^2 + 2\left(\frac{c}{4} - \tilde c\right)\right\}\langle X,Y\rangle\langle Z,W\rangle \\ &\qquad\quad + \frac{1}{3}\left\{\lambda^2 - \left(\frac{c}{4} - \tilde c\right)\right\}\{\langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\} \\ &\qquad\quad + \frac{c}{4}\sum_{r=1}^{3}\{\langle J_rX, Z\rangle\langle W, J_rY\rangle + \langle J_rX, W\rangle\langle J_rY, Z\rangle\} \end{aligned} \tag{4.3}$$

for all vector fields X, Y, Z and W tangent to M.

Now we investigate the first normal space $N^1_x(M)$ at the point $x \in M$. We choose an orthonormal basis

$$\{e_1, \ldots, e_n, e_{n+1} = J_1 e_1, \ldots, e_{2n} = J_1 e_n,$$
$$e_{2n+1} = J_2 e_1, \ldots, e_{3n} = J_2 e_n, e_{3n+1} = J_3 e_1, \ldots, e_{4n} = J_3 e_n\}$$

of $T_x M$. As our immersion f is parallel, Lemma 3 implies that the $2n^2 - n$ vectors

$$\begin{aligned} &\sigma(e_k, e_k) && \text{for } k = 1, \ldots, n, \\ &\sigma(e_i, e_j) && \text{for } 1 \leq i < j \leq n \\ \text{and} \quad &\sigma(e_i, e_{rn+j}) && \text{for } 1 \leq i < j \leq n,\ r = 1, 2, 3 \end{aligned} \tag{4.4}$$

span the first normal space $N^1_x(M)$. The equaion (4.3) yeilds the following orthogonal relations:

$$\langle \sigma(e_i, e_i), \sigma(e_j, e_j) \rangle = \frac{1}{3}\left\{\lambda^2 + 2\left(\frac{c}{4} - \tilde{c}\right)\right\} + \frac{2}{3}\left\{\lambda^2 - \left(\frac{c}{4} - \tilde{c}\right)\right\}\delta_{ij} \tag{4.5}$$
$$\text{for} \quad i, j = 1, \ldots, n.$$

$$\langle \sigma(e_i, e_i), \sigma(e_k, e_l) \rangle = 0 \tag{4.6}$$
$$\text{for} \quad i = 1, \ldots, n,\ 1 \leq k < l \leq n.$$

$$\langle \sigma(e_i, e_i), \sigma(e_k, e_{rn+l}) \rangle = 0 \tag{4.7}$$
$$\text{for} \quad i = 1, \ldots, n,\ 1 \leq k < l \leq n,\ r = 1, 2, 3.$$

$$\langle \sigma(e_i, e_j), \sigma(e_k, e_l) \rangle = \frac{1}{3}\left\{\lambda^2 - \left(\frac{c}{4} - \tilde{c}\right)\right\}\delta_{ik}\delta_{jl} \tag{4.8}$$
$$\text{for} \quad 1 \leq i < j \leq n,\ 1 \leq k < l \leq n.$$

$$\langle \sigma(e_i, e_j), \sigma(e_k, e_{rn+l}) \rangle = 0 \tag{4.9}$$
$$\text{for} \quad 1 \leq i < j \leq n,\ 1 \leq k < l \leq n,\ r = 1, 2, 3.$$

$$\langle \sigma(e_i, e_{rn+j}), \sigma(e_k, e_{sn+l}) \rangle = \frac{1}{3}\left\{\lambda^2 - \left(\frac{c}{4} - \tilde{c}\right)\right\}\delta_{ik}\delta_{rn+j,sn+l} \tag{4.10}$$
$$\text{for} \quad 1 \leq i < j \leq n,\ 1 \leq k < l \leq n,\ r, s = 1, 2, 3.$$

It follows from (4.2) that we have

$$\langle R(e_i, e_j)e_j, e_i \rangle = \frac{c}{4} \qquad \text{for} \quad 1 \leq i < j \leq n,$$

so that

$$\Delta = \Delta_x = \frac{c}{4} - \tilde{c} \tag{4.11}$$

at $x \in M$. Hence we may apply Lemma 2 to the linear subspace of T_xM which is generated by $\{e_1, \ldots, e_n\}$ because $\lambda > 0$. Suppose that $\Delta = \frac{c}{4} - \tilde{c} = \lambda^2$. Then the equations (4.5), (4.8) and (4.10) reduce to

$$\langle \sigma(e_i, e_i), \sigma(e_j, e_j) \rangle = \lambda^2 \qquad \text{for } \ i, j = 1, \ldots, n$$

and

$$\langle \sigma(e_i, e_j), \sigma(e_k, e_l) \rangle = \langle \sigma(e_i, e_{rn+j}), \sigma(e_k, e_{sn+l}) \rangle = 0$$
$$\text{for } \ 1 \leq i < j \leq n, \ 1 \leq k < l \leq n, \ r, s = 1, 2, 3.$$

Hence we have

$$\sigma(e_i, e_i) = \lambda \xi \qquad \text{for } \ i = 1, \ldots, n$$

and

$$\sigma(e_i, e_j) = \sigma(e_i, e_{rn+j}) = 0 \qquad \text{for } \ 1 \leq i < j \leq n, \ r = 1, 2, 3,$$

where ξ is a unit normal vector of M in $\widetilde{M}^{4n+p}(\tilde{c}\,; \mathbb{R})$. These equalities imply that the immersion $f : M \to \widetilde{M}^{4n+p}(\tilde{c}\,; \mathbb{R})$ is totally umbilic. Thus we have a contradiction. Hence we have

$$\lambda^2 - \left(\frac{c}{4} - \tilde{c} \right) \neq 0 \tag{4.12}$$

and our discussion is divided into the following two cases: (A) $\Delta = -\dfrac{n+2}{2(n-1)}\lambda^2$, (B) $-\dfrac{n+2}{2(n-1)}\lambda^2 < \Delta < \lambda^2$.

First, we investigate the case (B). In this case, by Lemma 2, we have

$$\dim Span_{\mathbb{R}} \{\sigma(e_i, e_j)\}_{i,j=1,\ldots,n} = \frac{n(n+1)}{2}.$$

This, combined with (4.5), ..., (4.10) and (4.12), means that the $2n^2 - n$ vectors (4.4) form a basis of the normal space at each point x of M in $\widetilde{M}^{4n+p}(\tilde{c}\,; \mathbb{R})$. Besides, we have by Lemma 3

$$\sum_{a=1}^{4n} \sigma(e_a, e_a) = 4 \sum_{i=1}^{n} \sigma(e_i, e_i) \neq 0,$$

so we find the immersion $f : M \to \widetilde{M}^{4n+p}(\tilde{c}\,; \mathbb{R})$ is not minimal. Let us use the Gram-Schmidt orthonormalization for the linearly indpendent system of vectors $\{\sigma(e_i, e_i)\}_{i=1,\ldots,n}$ in order to obtain an orthonormal basis of the first normal space at each point of M. We denote by θ ($0 < \theta < \pi$) the angle between the normal curvature vectors $\sigma(e_i, e_i)$ and $\sigma(e_j, e_j)$ ($i \neq j$), so that by (4.5)

$$\langle \sigma(e_i, e_i), \sigma(e_j, e_j) \rangle = \frac{1}{3} \left\{ \lambda^2 + 2 \left(\frac{c}{4} - \tilde{c} \right) \right\} = \lambda^2 \cos \theta. \tag{4.13}$$

The angle θ dose not depend on the choice of i, j ($i \neq j$) and $x \in M$, because the immersion f is constant isotropic. Put

$$f_1 = \sigma(e_1, e_1), \qquad e_{\tilde{1}} = \frac{f_1}{\|f_1\|} = \frac{1}{\lambda} \sigma(e_1, e_1)$$

and

$$f_k = \sigma(e_k, e_k) - \sum_{l=1}^{k-1} \langle \sigma(e_k, e_k),\, e_{\tilde{l}} \rangle e_{\tilde{l}}\,, \qquad e_{\tilde{k}} = \frac{f_k}{\|f_k\|} \tag{4.14}$$

for $2 \leq k \leq n$. Then, by induction we can verify

$$\begin{aligned} f_k &= \sigma(e_k, e_k) - \frac{\cos\theta}{1 + (k-2)\cos\theta} \sum_{l=1}^{k-1} \sigma(e_l, e_l), \\ \|f_k\|^2 &= \frac{\lambda^2 (1 - \cos\theta)\{1 + (k-1)\cos\theta\}}{1 + (k-2)\cos\theta} \end{aligned} \tag{4.15}$$

and hence

$$\begin{aligned} e_{\tilde{k}} = \frac{1}{\lambda} & \sqrt{\frac{1 + (k-2)\cos\theta}{(1 - \cos\theta)\{1 + (k-1)\cos\theta\}}} \\ & \times \left\{ \sigma(e_k, e_k) - \frac{\cos\theta}{1 + (k-2)\cos\theta} \sum_{l=1}^{k-1} \sigma(e_l, e_l) \right\} \end{aligned} \tag{4.16}$$

for $k = 1, \ldots, n$. On the other hand, since $\dfrac{c}{4} - \tilde{c} = \dfrac{\lambda^2}{2}(3\cos\theta - 1)$ from (4.13), we get

$$\|\sigma(e_i, e_j)\|^2 = \|\sigma(e_i, e_{rn+j})\|^2 = \frac{\lambda^2 (1 - \cos\theta)}{2}$$

by (4.8) and (4.10). Therefore relations (4.5), ..., (4.10) show that the vectors

$$e_{\tilde{k}}, \quad \frac{\sqrt{2}}{\lambda\sqrt{1 - \cos\theta}}\,\sigma(e_i, e_j) \quad \text{and} \quad \frac{\sqrt{2}}{\lambda\sqrt{1 - \cos\theta}}\,\sigma(e_i, e_{rn+j})$$

for $k = 1, \ldots, n,\ 1 \leq i < j \leq n,\ r = 1, 2, 3$ form an orthonormal system.

Now, using above vectors, we choose a local field of orthonormal frames

$$e_1, \ldots, e_{4n}, e_{\tilde{1}}, \ldots, e_{\tilde{p}} \quad (p = 2n^2 - n)$$

in $\widetilde{M}^{4n+p}(\tilde{c}; \mathbb{R})$ in such a way that, restricted to M, $e_1, \ldots, e_n$, $e_{n+1} = J_1 e_1, \ldots,$ $e_{2n} = J_1 e_n$, $e_{2n+1} = J_2 e_1, \ldots, e_{3n} = J_2 e_n$, $e_{3n+1} = J_3 e_1, \ldots, e_{4n} = J_3 e_n$ are tangent to M, $e_{\tilde{1}}, \ldots, e_{\tilde{n}}$ are defined by (4.16), and $e_{\widetilde{n+1}}, \ldots, e_{\tilde{p}}$ are defined by

$$e_{\widetilde{(i,j)}} = \frac{\sqrt{2}}{\lambda\sqrt{1 - \cos\theta}}\,\sigma(e_i, e_j) \qquad \text{for } 1 \leq i < j \leq n, \tag{4.17}$$

$$e_{\widetilde{(i,rn+j)}} = \frac{\sqrt{2}}{\lambda\sqrt{1 - \cos\theta}}\,\sigma(e_i, e_{rn+j}) \qquad \text{for } 1 \leq i < j \leq n,\ r = 1, 2, 3, \tag{4.18}$$

where we set

$$(i, j) = i + \frac{1}{2}(j - i)\{2n + 1 - (j - i)\}$$

and

$$(i, rn + j) = i + \frac{1}{2}(j - i)\{2n + 1 - (j - i)\} + \frac{r}{2} n(n - 1)$$

for $i < j$, $r = 1, 2, 3$.

We shall compute connection forms of the normal bundle to see the rigidity. We use the following convention on the ranges of indices unless otherwise stated:

$$A, B, C = 1, \ldots, 4n, \tilde{1}, \ldots, \tilde{p} \quad (p = 2n^2 - n); \quad i, j, k, l, m, q = 1, \ldots, n;$$
$$a, b, c, d = 1, \ldots, 4n; \quad \alpha, \beta = \tilde{1}, \ldots, \tilde{p}; \quad r, s, t = 1, 2, 3.$$

With respect to the frame field of $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ chosen as above, let

$$\omega^1, \ldots, \omega^{4n}, \omega^{\tilde{1}}, \ldots, \omega^{\tilde{p}} \quad (p = 2n^2 - n)$$

be the field of dual frames. Then the structure equations of $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ are given by

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B,$$
$$\omega^A_B + \omega^B_A = 0,$$
$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \tilde{c}\,\omega^A \wedge \omega^B.$$

Restricting these forms to M, we have the structure equations of the immersion:

$$\omega^\alpha = 0,$$
$$\omega^\alpha_a = \sum h^\alpha_{ab}\omega^b,$$
$$h^\alpha_{ab} = h^\alpha_{ba},$$
$$d\omega^a = -\sum \omega^a_b \wedge \omega^b,$$
$$\omega^a_b + \omega^b_a = 0,$$
$$d\omega^a_b = -\sum \omega^a_c \wedge \omega^c_b + \Omega^a_b,$$
$$\Omega^a_b = \frac{1}{2}\sum R^a_{bcd}\omega^c \wedge \omega^d,$$
$$R^a_{bcd} = \tilde{c}\,(\delta^a_c\delta_{bd} - \delta^a_d\delta_{bc}) + \sum(h^\alpha_{ac}h^\alpha_{bd} - h^\alpha_{ad}h^\alpha_{bc}).$$

The second fundamental form σ can be described as

$$\begin{aligned}\sigma(e_a, e_b) &= \sum_\alpha h^\alpha_{ab}e_\alpha \\ &= \sum_k h^{\tilde{k}}_{ab}e_{\tilde{k}} + \sum_{l<m} h^{\widetilde{(l,m)}}_{ab} e_{\widetilde{(l,m)}} + \sum_r\sum_{l<m} h^{\widetilde{(l,rn+m)}}_{ab} e_{\widetilde{(l,rn+m)}}\,.\end{aligned} \tag{4.19}$$

The equation (3.1) has the representation

$$\sum h^\alpha_{abc}\omega^c = dh^\alpha_{ab} - \sum h^\alpha_{ac}\omega^c_b - \sum h^\alpha_{cb}\omega^c_a + \sum h^\beta_{ab}\omega^\alpha_\beta. \tag{4.20}$$

Then we have

$$h^\alpha_{abc} = h^\alpha_{acb}. \tag{4.21}$$

Furthermore, the following relations hold:

$$\omega^k_{rn+i} = -\omega^{rn+k}_i = \omega^i_{rn+k}\,, \tag{4.22}$$

$$\omega^{rn+k}_{rn+i} = \omega^k_i\,, \tag{4.23}$$

$$\omega_{rn+i}^{(r+1)n+k} = q_{r+2}\delta_i^k - \omega_i^{(r+2)n+k} \quad (r \bmod 3), \tag{4.24}$$

where q_{r+2} are the 1-forms given in (2.1),

$$\omega_{rn+i}^{(r+1)n+k} = \omega_{rn+k}^{(r+1)n+i} = -\omega_{(r+1)n+i}^{rn+k} \quad (r \bmod 3). \tag{4.25}$$

In fact, we have

$$\nabla e_{rn+i} = \sum_a \omega_{rn+i}^a e_a = \sum_k \omega_{rn+i}^k e_k + \sum_{s,k} \omega_{rn+i}^{sn+k} e_{sn+k}. \tag{4.26}$$

On the other hand, by using (2.1) we see

$$\begin{aligned}
\nabla(J_r e_i) &= (\nabla J_r)\, e_i + J_r \nabla e_i \\
&= q_{r+2} J_{r+1} e_i - q_{r+1} J_{r+2} e_i + J_r \left(\sum_a \omega_i^a e_a \right) \\
&= q_{r+2} e_{(r+1)n+i} - q_{r+1} e_{(r+2)n+i} \\
&\qquad + \sum_k \omega_i^k J_r e_k + \sum_k \omega_i^{rn+k} J_r e_{rn+k} \\
&\qquad + \sum_k \omega_i^{(r+1)n+k} J_r e_{(r+1)n+k} + \sum_k \omega_i^{(r+2)n+k} J_r e_{(r+2)n+k} \\
&= q_{r+2} e_{(r+1)n+i} - q_{r+1} e_{(r+2)n+i} \\
&\qquad + \sum_k \omega_i^k e_{rn+k} - \sum_k \omega_i^{rn+k} e_k \\
&\qquad + \sum_k \omega_i^{(r+1)n+k} e_{(r+2)n+k} - \sum_k \omega_i^{(r+2)n+k} e_{(r+1)n+k} \\
&= -\sum_k \omega_i^{rn+k} e_k + \sum_k \omega_i^k e_{rn+k} \\
&\qquad + \sum_k \left(q_{r+2}\delta_i^k - \omega_i^{(r+2)n+k} \right) e_{(r+1)n+k} \\
&\qquad + \sum_k \left(-q_{r+1}\delta_i^k + \omega_i^{(r+1)n+k} \right) e_{(r+2)n+k} \qquad (r \bmod 3).
\end{aligned} \tag{4.27}$$

Thus the relations (4.22), (4.23) and (4.24) are derived from (4.26) and (4.27). The relation (4.25) follows from (4.22) and (4.24).

We here determine all h_{ab}^α. Thanks to Lemma 3, we have

$$\sigma(J_r e_i, J_r e_j) = \sigma(e_i, e_j),$$

$$\sigma(e_i, J_r e_j) = -\sigma(J_r e_i, e_j),$$

$$\sigma(J_r e_i, J_{r+1} e_j) = -\sigma(e_i, J_r J_{r+1} e_j) = -\sigma(e_i, J_{r+2} e_j) \quad (r \bmod 3),$$

that is,

$$h_{rn+i,rn+j}^\alpha = h_{ij}^\alpha\,, \tag{4.28}$$

$$h_{i,rn+j}^\alpha = -h_{rn+i,j}^\alpha\,, \tag{4.29}$$

$$h^{\alpha}_{rn+i,(r+1)n+j} = -h^{\alpha}_{i,(r+2)n+j} \quad (r \bmod 3). \tag{4.30}$$

For i, j $(i < j)$ we see from (4.6), (4.16)

$$h^{\tilde{k}}_{ij} = \langle \sigma(e_i, e_j),\, e_{\tilde{k}} \rangle = 0.$$

Using (4.14) and (4.15), we have

$$h^{\tilde{k}}_{kk} = \langle \sigma(e_k, e_k),\, e_{\tilde{k}} \rangle = \|f_k\| = \lambda \sqrt{\frac{(1-\cos\theta)\{1+(k-1)\cos\theta\}}{1+(k-2)\cos\theta}}.$$

For i, k with $i > k$, by (4.13) and (4.15)

$$\begin{aligned}
h^{\tilde{k}}_{ii} &= \langle \sigma(e_i, e_i),\, e_{\tilde{k}} \rangle \\
&= \frac{1}{\|f_k\|} \langle \sigma(e_i, e_i),\, \sigma(e_k, e_k) - \frac{\cos\theta}{1+(k-2)\cos\theta} \sum_{l=1}^{k-1} \sigma(e_l, e_l) \rangle \\
&= \frac{1}{\|f_k\|} \left\{ \lambda^2 \cos\theta - \frac{(k-1)\lambda^2 \cos^2\theta}{1+(k-2)\cos\theta} \right\} \\
&= \frac{1}{\lambda} \sqrt{\frac{1+(k-2)\cos\theta}{(1-\cos\theta)\{1+(k-1)\cos\theta\}}} \left\{ \frac{\lambda^2 \cos\theta(1-\cos\theta)}{1+(k-2)\cos\theta} \right\} \\
&= \frac{\lambda\cos\theta\sqrt{1-\cos\theta}}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}.
\end{aligned}$$

Hence we get

$$h^{\tilde{k}}_{ij} = \begin{cases} \lambda \sqrt{\dfrac{(1-\cos\theta)\{1+(k-1)\cos\theta\}}{1+(k-2)\cos\theta}} & \text{for } k = i = j, \\[2ex] \dfrac{\lambda\cos\theta\sqrt{1-\cos\theta}}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}} & \text{for } k < i = j, \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

Similar computation gives the following:

$$h^{\widetilde{(l,m)}}_{ij} = \begin{cases} \dfrac{\lambda\sqrt{1-\cos\theta}}{\sqrt{2}}\, \delta^l_i \delta^m_j & \text{for } i < j,\ l < m, \\[2ex] 0 & \text{for } i = j,\ l < m, \end{cases} \tag{4.32}$$

$$h^{\widetilde{(l,sn+m)}}_{i,rn+j} = \begin{cases} \dfrac{\lambda\sqrt{1-\cos\theta}}{\sqrt{2}}\, \delta^l_i \delta^{sn+m}_{rn+j} & \text{for } i < j,\ l < m, \\[2ex] 0 & \text{for } i = j,\ l < m, \end{cases} \tag{4.33}$$

$$h_{ij}^{\widetilde{(l,rn+m)}} = 0 \quad \text{for } i \leq j,\ l < m, \tag{4.34}$$

$$h_{i,rn+j}^{\tilde{k}} = h_{i,rn+j}^{\widetilde{(l,m)}} = 0 \quad \text{for } i < j,\ l < m. \tag{4.35}$$

Since our immersion f is parallel and the angle θ is constant on M, the equation (4.20) reduces to

$$\sum h_{ab}^{\beta}\omega_{\beta}^{\alpha} = \sum h_{ac}^{\alpha}\omega_{b}^{c} + \sum h_{cb}^{\alpha}\omega_{a}^{c},$$

that is,

$$\begin{aligned}\sum_{\beta} h_{ij}^{\beta}\omega_{\beta}^{\alpha} =& \sum_{q} h_{iq}^{\alpha}\omega_{j}^{q} + \sum_{t,q} h_{i,tn+q}^{\alpha}\omega_{j}^{tn+q} \\ &+ \sum_{q} h_{qj}^{\alpha}\omega_{i}^{q} + \sum_{t,q} h_{tn+q,j}^{\alpha}\omega_{i}^{tn+q},\end{aligned} \tag{4.36}$$

$$\begin{aligned}\sum_{\beta} h_{i,sn+j}^{\beta}\omega_{\beta}^{\alpha} =& \sum_{q} h_{iq}^{\alpha}\omega_{sn+j}^{q} + \sum_{t,q} h_{i,tn+q}^{\alpha}\omega_{sn+j}^{tn+q} \\ &+ \sum_{q} h_{q,sn+j}^{\alpha}\omega_{i}^{q} + \sum_{t,q} h_{tn+q,sn+j}^{\alpha}\omega_{i}^{tn+q},\end{aligned} \tag{4.37}$$

$$\begin{aligned}\sum_{\beta} h_{rn+i,sn+j}^{\beta}\omega_{\beta}^{\alpha} =& \sum_{q} h_{rn+i,q}^{\alpha}\omega_{sn+j}^{q} + \sum_{t,q} h_{rn+i,tn+q}^{\alpha}\omega_{sn+j}^{tn+q} \\ &+ \sum_{q} h_{q,sn+j}^{\alpha}\omega_{rn+i}^{q} + \sum_{t,q} h_{tn+q,sn+j}^{\alpha}\omega_{rn+i}^{tn+q}.\end{aligned} \tag{4.38}$$

By using (4.22), (4.23), (4.24), (4.25), (4.28), (4.29), (4.30) we can find that it is sufficient to investigate (4.36) for $i \leq j$ and (4.37) for $i \leq j$. First we consider the case that $\alpha = \tilde{k}$ and $i < j$ in (4.36). For the left-hand side, we see from (4.31), (4.32), (4.34)

$$\begin{aligned}\sum_{\beta} h_{ij}^{\beta}\omega_{\beta}^{\tilde{k}} =& \sum_{q} h_{ij}^{\tilde{q}}\omega_{\tilde{q}}^{\tilde{k}} + \sum_{\widetilde{(l,m)}} h_{ij}^{\widetilde{(l,m)}}\omega_{\widetilde{(l,m)}}^{\tilde{k}} + \sum_{\widetilde{(l,tn+m)}} h_{ij}^{\widetilde{(l,tn+m)}}\omega_{\widetilde{(l,tn+m)}}^{\tilde{k}} \\ =& \sum_{\widetilde{(l,m)}} \frac{\lambda\sqrt{1-\cos\theta}}{\sqrt{2}}\,\delta_{i}^{l}\delta_{j}^{m}\omega_{\widetilde{(l,m)}}^{\tilde{k}} \\ =& \frac{\lambda\sqrt{1-\cos\theta}}{\sqrt{2}}\omega_{\widetilde{(i,j)}}^{\tilde{k}}.\end{aligned}$$

For the right-hand side, by (4.29), (4.31) and (4.35) we have

$$\begin{aligned}&\sum_{q} h_{iq}^{\tilde{k}}\omega_{j}^{q} + \sum_{t,q} h_{i,tn+q}^{\tilde{k}}\omega_{j}^{tn+q} + \sum_{q} h_{qj}^{\tilde{k}}\omega_{i}^{q} + \sum_{t,q} h_{tn+q,j}^{\tilde{k}}\omega_{i}^{tn+q} \\ &= \sum_{q} h_{iq}^{\tilde{k}}\omega_{j}^{q} + \sum_{q} h_{qj}^{\tilde{k}}\omega_{i}^{q} \\ &= h_{ii}^{\tilde{k}}\omega_{j}^{i} + h_{jj}^{\tilde{k}}\omega_{i}^{j}\end{aligned}$$

$$= \begin{cases} 0 & \text{for } i < j < k \text{ or } k < i < j, \\ \dfrac{\lambda \cos\theta\sqrt{1-\cos\theta}}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}\omega_i^j & \text{for } i < k < j, \\ \Biggl(\lambda\sqrt{\dfrac{(1-\cos\theta)\{1+(k-1)\cos\theta\}}{1+(k-2)\cos\theta}} \\ \qquad -\dfrac{\lambda\cos\theta\sqrt{1-\cos\theta}}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}\Biggr)\omega_j^k & \text{for } k = i < j, \\ \lambda\sqrt{\dfrac{(1-\cos\theta)\{1+(k-1)\cos\theta\}}{1+(k-2)\cos\theta}}\omega_i^k & \text{for } i < j = k. \end{cases}$$

Hence, we obtain

$$\omega^{\tilde{k}}_{\widetilde{(i,j)}} = 0 \qquad \text{for } \ k < i < j \quad \text{or} \quad i < j < k, \tag{4.39}$$

$$\omega^{\tilde{k}}_{\widetilde{(i,j)}} = \frac{\sqrt{2}\cos\theta}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}\,\omega_i^j \qquad \text{for } \ i < k < j, \tag{4.40}$$

$$\omega^{\tilde{k}}_{\widetilde{(k,j)}} = \sqrt{\frac{2\{1+(k-2)\cos\theta\}}{1+(k-1)\cos\theta}}\,\omega_j^k \qquad \text{for } \ k < j, \tag{4.41}$$

$$\omega^{\tilde{k}}_{\widetilde{(i,k)}} = \sqrt{\frac{2\{1+(k-1)\cos\theta\}}{1+(k-2)\cos\theta}}\,\omega_i^k \qquad \text{for } \ i < k. \tag{4.42}$$

If $\alpha = \tilde{k}$ and $i = j$ in (4.36), from (4.31), (4.32), (4.34) the left-hand side becomes

$$\begin{aligned} \sum_{\beta} h_{ii}^{\beta}\omega_{\beta}^{\tilde{k}} &= \sum_{q} h_{ii}^{\tilde{q}}\omega_{\tilde{q}}^{\tilde{k}} + \sum_{\widetilde{(l,m)}} h_{ii}^{\widetilde{(l,m)}}\omega^{\tilde{k}}_{\widetilde{(l,m)}} + \sum_{\widetilde{(l,tn+m)}} h_{ii}^{\widetilde{(l,tn+m)}}\omega^{\tilde{k}}_{\widetilde{(l,tn+m)}} \\ &= \sum_{q<i} h_{ii}^{\tilde{q}}\omega_{\tilde{q}}^{\tilde{k}} + h_{ii}^{\tilde{i}}\omega_{\tilde{i}}^{\tilde{k}} + \sum_{q>i} h_{ii}^{\tilde{q}}\omega_{\tilde{q}}^{\tilde{k}} \\ &= \sum_{q<i} \frac{\lambda\cos\theta\sqrt{1-\cos\theta}}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde{q}}^{\tilde{k}} \\ &\qquad\qquad + \lambda\sqrt{\frac{(1-\cos\theta)\{1+(i-1)\cos\theta\}}{1+(i-2)\cos\theta}}\,\omega_{\tilde{i}}^{\tilde{k}}. \end{aligned}$$

The right-hand side is

$$2\sum_{q} h_{iq}^{\tilde{k}}\omega_i^q + 2\sum_{t,q} h_{i,tn+q}^{\tilde{k}}\omega_i^{tn+q} = 2\sum_{q\neq i} h_{iq}^{\tilde{k}}\omega_i^q + 2h_{ii}^{\tilde{k}}\omega_i^i = 0.$$

It follows that

$$\sum_{q<i}\frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\tilde k} + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde i}^{\tilde k}=0. \tag{4.43}$$

Particularly, if $k=i$,

$$\sum_{q<k}\frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\tilde k}=0. \tag{4.44}$$

Similarly, settting $\alpha=\widetilde{(k,l)}$ and $(k,\widetilde{rn+l})$ in (4.36), we can get the following relations:

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,l)}}=0 \qquad \text{for } \ i<j,\ k<l, \tag{4.45}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(j,l)}}=\omega_i^l \qquad \text{for } \ i<j<l, \tag{4.46}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,j)}}=\omega_i^k \qquad \text{for } \ i,k<j, \tag{4.47}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(i,l)}}=\omega_j^l \qquad \text{for } \ i<j,l, \tag{4.48}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,i)}}=\omega_j^k \qquad \text{for } \ k<i<j, \tag{4.49}$$

$$\sum_{q<i}\frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\widetilde{(k,l)}} + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde i}^{\widetilde{(k,l)}}=0, \tag{4.50}$$

$$\sum_{q<i}\frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\widetilde{(i,l)}} + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde i}^{\widetilde{(i,l)}}=\sqrt{2}\,\omega_i^l \qquad \text{for } \ i<l, \tag{4.51}$$

$$\sum_{q<i}\frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\widetilde{(k,i)}} + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde i}^{\widetilde{(k,i)}}=\sqrt{2}\,\omega_i^k \qquad \text{for } \ k<i, \tag{4.52}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,rn+l)}} = 0 \qquad \text{for } \; i<j,\; k<l, \tag{4.53}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(j,rn+l)}} = \omega_i^{rn+l} \qquad \text{for } \; i<j<l, \tag{4.54}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,rn+j)}} = -\omega_i^{rn+k} \qquad \text{for } \; i,k<j, \tag{4.55}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(i,rn+l)}} = \omega_j^{rn+l} \qquad \text{for } \; i<j,l, \tag{4.56}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,rn+i)}} = -\omega_j^{rn+k} \qquad \text{for } \; k<i<j, \tag{4.57}$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(i,rn+j)}} = \omega_j^{rn+j} - \omega_i^{rn+i} \qquad \text{for } \; i<j, \tag{4.58}$$

$$\begin{aligned} &\sum_{q<i} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde{q}}^{\widetilde{(k,rn+l)}} \\ &\qquad + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde{i}}^{\widetilde{(k,rn+l)}} = 0, \end{aligned} \tag{4.59}$$

$$\begin{aligned} &\sum_{q<i} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde{q}}^{\widetilde{(i,rn+l)}} \\ &\qquad + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde{i}}^{\widetilde{(i,rn+l)}} = \sqrt{2}\,\omega_i^{rn+l} \qquad \text{for } \; i<l, \end{aligned} \tag{4.60}$$

$$\begin{aligned} &\sum_{q<i} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde{q}}^{\widetilde{(k,rn+i)}} \\ &\qquad + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde{i}}^{\widetilde{(k,rn+i)}} = -\sqrt{2}\,\omega_i^{rn+k} \qquad \text{for } \; k<i, \end{aligned} \tag{4.61}$$

where different indices indicate different numbers. We also obtain the following by putting $\alpha = \tilde{k}$ and $\widetilde{(k, rn+l)}$ in (4.37) (if $\alpha = \widetilde{(k,l)}$, we get nothing but the same relations as above):

$$\omega_{\widetilde{(i,rn+j)}}^{\tilde{k}} = 0 \qquad \text{for } \; k<i<j \quad \text{or} \quad i<j<k, \tag{4.62}$$

$$\omega_{\widetilde{(i,rn+j)}}^{\tilde{k}} = \frac{\sqrt{2}\cos\theta}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}\,\omega_i^{rn+j} \quad \text{for } \; i<k<j, \tag{4.63}$$

$$\omega_{\widetilde{(k,rn+j)}}^{\tilde{k}} = -\sqrt{\frac{2\{1+(k-2)\cos\theta\}}{1+(k-1)\cos\theta}}\,\omega_k^{rn+j} \qquad \text{for } \; k<j, \tag{4.64}$$

$$\omega^{\tilde{k}}_{\widetilde{(i,rn+k)}} = \sqrt{\frac{2\{1+(k-1)\cos\theta\}}{1+(k-2)\cos\theta}}\,\omega_i^{rn+k} \qquad \text{for } \ i<k, \tag{4.65}$$

$$\omega^{\widetilde{(k,rn+l)}}_{\widetilde{(i,rn+j)}} = 0 \qquad \text{for } \ i<j, \ k<l, \tag{4.66}$$

$$\omega^{\widetilde{(j,rn+l)}}_{\widetilde{(i,rn+j)}} = -\omega_i^l \qquad \text{for } \ i<j<l, \tag{4.67}$$

$$\omega^{\widetilde{(k,rn+j)}}_{\widetilde{(i,rn+j)}} = \omega_i^k \qquad \text{for } \ i,k<j, \tag{4.68}$$

$$\omega^{\widetilde{(i,rn+l)}}_{\widetilde{(i,rn+j)}} = \omega_j^l \qquad \text{for } \ i<j,l, \tag{4.69}$$

$$\omega^{\widetilde{(k,rn+i)}}_{\widetilde{(i,rn+j)}} = -\omega_j^k \qquad \text{for } \ k<i<j, \tag{4.70}$$

$$\omega^{\widetilde{(k,(r+1)n+l)}}_{\widetilde{(i,rn+j)}} = 0 \quad (r \bmod 3) \qquad \text{for } \ i<j, \ k<l, \tag{4.71}$$

$$\omega^{\widetilde{(j,(r+1)n+l)}}_{\widetilde{(i,rn+j)}} = \omega_i^{(r+2)n+l} \quad (r \bmod 3) \qquad \text{for } \ i<j<l, \tag{4.72}$$

$$\omega^{\widetilde{(k,(r+1)n+j)}}_{\widetilde{(i,rn+j)}} = -\omega_i^{(r+2)n+k} \quad (r \bmod 3) \qquad \text{for } \ i,k<j, \tag{4.73}$$

$$\omega^{\widetilde{(i,(r+1)n+l)}}_{\widetilde{(i,rn+j)}} = -\omega_j^{(r+2)n+l} \quad (r \bmod 3) \qquad \text{for } \ i<j,l, \tag{4.74}$$

$$\omega^{\widetilde{(k,(r+1)n+i)}}_{\widetilde{(i,rn+j)}} = \omega_j^{(r+2)n+k} \quad (r \bmod 3) \qquad \text{for } \ k<i<j, \tag{4.75}$$

$$\omega^{\widetilde{(i,(r+1)n+j)}}_{\widetilde{(i,rn+j)}} = \omega_{rn+j}^{(r+1)n+j} - \omega_i^{(r+2)n+i} \quad (r \bmod 3) \qquad \text{for } \ i<j, \tag{4.76}$$

where different indices also indicate different numbers.

We can see that both of (4.43) and (4.44) are equivalent to

$$\omega_{\tilde{l}}^{\tilde{k}} = 0. \tag{4.77}$$

Indeed, evidently (4.77) implies (4.43) and (4.44). Conversely, put $i = k-1$ in (4.43):

$$\sum_{q<k-1} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde{q}}^{\tilde{k}} + \sqrt{\frac{1+(k-2)\cos\theta}{1+(k-3)\cos\theta}}\,\omega_{\widetilde{k-1}}^{\tilde{k}} = 0.$$

On the other hand, owing to (4.44) we have

$$\begin{aligned}\sum_{q<k-1} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\tilde k}\\ = -\frac{\cos\theta}{\sqrt{\{1+(k-2)\cos\theta\}\{1+(k-3)\cos\theta\}}}\,\omega_{\widetilde{k-1}}^{\tilde k}.\end{aligned}$$

These relations give

$$\begin{aligned}&\left\{\sqrt{\frac{1+(k-2)\cos\theta}{1+(k-3)\cos\theta}} - \frac{\cos\theta}{\sqrt{\{1+(k-2)\cos\theta\}\{1+(k-3)\cos\theta\}}}\right\}\omega_{\widetilde{k-1}}^{\tilde k}\\ &= \sqrt{\frac{1+(k-3)\cos\theta}{1+(k-2)\cos\theta}}\,\omega_{\widetilde{k-1}}^{\tilde k}\\ &= 0,\end{aligned}$$

so that

$$\omega_{\widetilde{k-1}}^{\tilde k} = 0$$

and

$$\sum_{q<k-1} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\tilde k} = 0.$$

Thus, inductively we have $\omega_{\tilde l}^{\tilde k} = 0$ $(l < k)$.

Besides, we can derive (4.50), (4.51), (4.52), (4.59), (4.60), (4.61) from the other relations. For instance, we see (4.61) is valid as follows. From (4.62), (4.63), (4.64), (4.65) the left-hand side is equal to

$$\begin{aligned}&\sum_{q<k} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\widetilde{(k,rn+i)}}\\ &\quad+ \frac{\cos\theta}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}\,\omega_{\tilde k}^{\widetilde{(k,rn+i)}}\\ &\quad+ \sum_{k<q<i} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_{\tilde q}^{\widetilde{(k,rn+i)}}\\ &\quad+ \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}}\,\omega_{\tilde i}^{\widetilde{(k,rn+i)}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos\theta}{\sqrt{\{1+(k-1)\cos\theta\}\{1+(k-2)\cos\theta\}}}\sqrt{\frac{2\{1+(k-2)\cos\theta\}}{1+(k-1)\cos\theta}}\,\omega_k^{rn+i} \\
&\quad + \sum_{k<q<i} \frac{\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}} \\
&\qquad\qquad \times \left(\frac{-\sqrt{2}\cos\theta}{\sqrt{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}}}\,\omega_k^{rn+i} \right) \\
&\quad + \sqrt{\frac{1+(i-1)\cos\theta}{1+(i-2)\cos\theta}} \left(-\sqrt{\frac{2\{1+(i-1)\cos\theta\}}{1+(i-2)\cos\theta}}\,\omega_k^{rn+i} \right) \\
&= \left\{ \frac{\sqrt{2}\cos\theta}{1+(k-1)\cos\theta} - \sum_{k<q<i} \frac{\sqrt{2}\cos^2\theta}{\{1+(q-1)\cos\theta\}\{1+(q-2)\cos\theta\}} \right. \\
&\qquad\qquad \left. - \frac{\sqrt{2}\{1+(i-1)\cos\theta\}}{1+(i-2)\cos\theta} \right\} \omega_k^{rn+i} \\
&= \left\{ \frac{\sqrt{2}\cos\theta}{1+(k-1)\cos\theta} - \sum_{k<q<i} \left(\frac{\sqrt{2}\cos\theta}{1+(q-2)\cos\theta} - \frac{\sqrt{2}\cos\theta}{1+(q-1)\cos\theta} \right) \right. \\
&\qquad\qquad \left. - \frac{\sqrt{2}\{1+(i-1)\cos\theta\}}{1+(i-2)\cos\theta} \right\} \omega_k^{rn+i} \\
&= \left\{ \frac{\sqrt{2}\cos\theta}{1+(i-2)\cos\theta} - \frac{\sqrt{2}\{1+(i-1)\cos\theta\}}{1+(i-2)\cos\theta} \right\} \omega_k^{rn+i} \\
&= -\sqrt{2}\,\omega_k^{rn+i}.
\end{aligned}
$$

This becomes $-\sqrt{2}\,\omega_i^{rn+k}$ by (4.22).

The relations (4.39), ..., (4.42), (4.45), ..., (4.49), (4.53), ..., (4.58), (4.62), ..., (4.77) mean that all connection forms of the normal bundle NM in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ ($p = 2n^2 - n$) are uniquely determined by connection forms of M. Hence, according to the fundamental theorem of submanifolds, we conclude that our immersion $f : M \to \widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ is rigid in case of (B).

Next, we study the case (A). By the same argument as in the case (B), we can see the immersion f is rigid. Thanks to Lemma 2, we have

$$
\dim Span_{\mathbb{R}}\{\sigma(e_i, e_j)\}_{i,j=1,\dots,n} = \frac{n(n+1)}{2} - 1,
$$

so that the orthgonal relations (4.5), ..., (4.10) with (4.12) imply that the $2n^2-n-1$ vectors $\sigma(e_k, e_k)$ $(k = 1, \dots, n-1)$, $\sigma(e_i, e_j)$ $(1 \le i < j \le n)$, $\sigma(e_i, e_{rn+j})$ $(1 \le i < j \le n,\ r = 1, 2, 3)$ form a basis of the normal space at each point x of M in

$\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$. Lemma 2 (2) and Lemma 3 give

$$\sum_{a=1}^{4n} \sigma(e_a, e_a) = 4\sum_{i=1}^{n} \sigma(e_i, e_i) = 0,$$

that is, the immersion $f : M \to \widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ is minimal. Since

$$\Delta = \frac{c}{4} - \tilde{c} = -\frac{n+2}{2(n-1)}\lambda^2,$$

we have

$$\lambda = \sqrt{\frac{2\,(n-1)\left(\tilde{c} - \dfrac{c}{4}\right)}{n+2}}$$

and from (4.13)

$$\cos\theta = \frac{-1}{n-1}$$

for the angle θ $(0 < \theta < \pi)$ between the vectors $\sigma(e_i, e_i)$ and $\sigma(e_j, e_j)$ $(i \neq j)$. We find that $2n^2 - n - 1$ vectors

$$\sqrt{\frac{n+2}{2n(n-k)(n-k+1)\left(\tilde{c} - \dfrac{c}{4}\right)}}\left\{(n-k+1)\sigma(e_k, e_k) + \sum_{l=1}^{k-1}\sigma(e_l, e_l)\right\},$$

$$\sqrt{\frac{n+2}{n\left(\tilde{c} - \dfrac{c}{4}\right)}}\,\sigma(e_i, e_j) \quad \text{and} \quad \sqrt{\frac{n+2}{n\left(\tilde{c} - \dfrac{c}{4}\right)}}\,\sigma(e_i, e_{rn+j})$$

for $k = 1, \ldots, n-1$, $1 \leq i < j \leq n$, $r = 1, 2, 3$ form an orthonormal system. Choose a local field of orthonormal frames

$$e_1, \ldots, e_{4n},\ e_{\tilde{1}}, \ldots, e_{\widetilde{n-1}},\ e_{\tilde{n}} = e_{\widetilde{(1,2)}},\ \ldots,\ e_{\tilde{p}} = e_{\widetilde{(1,4n)}} \quad (p = 2n^2 - n - 1)$$

in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ such that, restricted to M, $e_1, \ldots, e_{4n}$ are tangent to M, and

$$e_{\tilde{k}} = \sqrt{\frac{n+2}{2n(n-k)(n-k+1)\left(\tilde{c} - \dfrac{c}{4}\right)}}\left\{(n-k+1)\sigma(e_k, e_k) + \sum_{l=1}^{k-1}\sigma(e_l, e_l)\right\}$$

$$\text{for } k = 1, \ldots, n-1,$$

$$e_{\widetilde{(i,j)}} = \sqrt{\frac{n+2}{n\left(\tilde{c} - \dfrac{c}{4}\right)}}\,\sigma(e_i, e_j) \qquad \text{for } 1 \leq i < j \leq n,$$

$$e_{\widetilde{(i,rn+j)}} = \sqrt{\frac{n+2}{n\left(\tilde{c} - \dfrac{c}{4}\right)}}\,\sigma(e_i, e_{rn+j}) \qquad \text{for } 1 \leq i < j \leq n,\ r = 1, 2, 3,$$

where

$$(i, j) = i + \frac{1}{2}(j - i)\,\{2n + 1 - (j - i)\} - 1$$

and

$$(i, rn+j) = i + \frac{1}{2}(j-i)\{2n+1-(j-i)\} + \frac{r}{2}n(n-1) - 1$$

for $i < j$, $r = 1, 2, 3$. Let

$$\omega^1, \ldots, \omega^{4n}, \omega^{\tilde{1}}, \ldots, \omega^{\tilde{p}} \quad (p = 2n^2 - n - 1)$$

be the field of dual frames with respect to this frame field of $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$. Then, noting the range of α, we have from (4.19)

$$h_{ij}^{\tilde{k}} = \begin{cases} \sqrt{\dfrac{2n(n-k)\left(\tilde{c} - \dfrac{c}{4}\right)}{(n+2)(n-k+1)}} & \text{for } 1 \leq k = i = j \leq 1-n, \\ -\sqrt{\dfrac{2n\left(\tilde{c} - \dfrac{c}{4}\right)}{(n+2)(n-k)(n-k+1)}} & \text{for } 1 \leq k < i = j \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$h_{ij}^{\widetilde{(l,m)}} = \begin{cases} \sqrt{\dfrac{n\left(\tilde{c} - \dfrac{c}{4}\right)}{n+2}}\,\delta_i^l \delta_j^m & \text{for } 1 \leq i < j \leq n,\ 1 \leq l < m \leq n, \\ 0 & \text{for } 1 \leq i = j \leq n,\ 1 \leq l < m \leq n, \end{cases}$$

$$h_{i,rn+j}^{\widetilde{(l,sn+m)}} = \begin{cases} \sqrt{\dfrac{n\left(\tilde{c} - \dfrac{c}{4}\right)}{n+2}}\,\delta_i^l \delta_{rn+j}^{sn+m} \\ \qquad \text{for } 1 \leq i < j \leq n,\ 1 \leq l < m \leq n,\ r,s = 1,2,3, \\ 0 \qquad \text{for } 1 \leq i = j \leq n,\ 1 \leq l < m \leq n,\ r,s = 1,2,3, \end{cases}$$

$$h_{ij}^{\widetilde{(l,rn+m)}} = 0 \qquad \text{for } 1 \leq i \leq j \leq n,\ 1 \leq l < m \leq n,\ r = 1,2,3,$$

$$h_{i,rn+j}^{\tilde{k}} = h_{i,rn+j}^{\widetilde{(l,m)}} = 0 \qquad \text{for } 1 \leq i < j \leq n,\ 1 \leq l < m \leq n,\ r = 1,2,3.$$

Hence, by using (4.36) and (4.38) we obtain

$$\omega_{\widetilde{(i,j)}}^{\tilde{k}} = 0 \qquad \text{for } \ 1 \leq k < i < j \leq n \quad \text{or} \quad 1 \leq i < j < k \leq n-1,$$

$$\omega_{\widetilde{(i,j)}}^{\tilde{k}} = -\frac{\sqrt{2}}{\sqrt{(n-k)(n-k+1)}}\,\omega_i^j \qquad \text{for } \ 1 \leq i < k < j \leq n,$$

$$\omega_{\widetilde{(k,j)}}^{\tilde{k}} = \sqrt{\frac{2(n-k+1)}{n-k}}\,\omega_j^k \qquad \text{for } \ 1 \leq k < j \leq n,$$

$$\omega^{\tilde{k}}_{\widetilde{(i,k)}} = \sqrt{\frac{2(n-k)}{n-k+1}}\,\omega^k_i \qquad \text{for } 1 \le i < k \le n-1,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\,\omega^{\tilde{k}}_{\tilde{q}} - \sqrt{\frac{n-i}{n-i+1}}\,\omega^{\tilde{k}}_{\tilde{i}} = 0 \tag{4.78}$$

$$\text{for } 1 \le i, k \le n-1,$$

$$\sum_{q<k} \frac{1}{\sqrt{(n-q)(n-q+1)}}\,\omega^{\tilde{k}}_{\tilde{q}} = 0 \qquad \text{for } 1 \le k \le n-1, \tag{4.79}$$

$$\omega^{\widetilde{(k,l)}}_{\widetilde{(i,j)}} = 0 \qquad \text{for } 1 \le i < j \le n,\ 1 \le k < l \le n,$$

$$\omega^{\widetilde{(j,l)}}_{\widetilde{(i,j)}} = \omega^l_i \qquad \text{for } 1 \le i < j < l \le n,$$

$$\omega^{\widetilde{(k,j)}}_{\widetilde{(i,j)}} = \omega^k_i \qquad \text{for } 1 \le i, k < j \le n,$$

$$\omega^{\widetilde{(i,l)}}_{\widetilde{(i,j)}} = \omega^l_j \qquad \text{for } 1 \le i < j, l \le n,$$

$$\omega^{\widetilde{(k,i)}}_{\widetilde{(i,j)}} = \omega^k_j \qquad \text{for } 1 \le k < i < j \le n,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\,\omega^{\widetilde{(k,l)}}_{\tilde{q}} - \sqrt{\frac{n-i}{n-i+1}}\,\omega^{\widetilde{(k,l)}}_{\tilde{i}} = 0 \tag{4.80}$$

$$\text{for } 1 \le k < l \le n,\ 1 \le i \le n-1,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\,\omega^{\widetilde{(i,l)}}_{\tilde{q}} - \sqrt{\frac{n-i}{n-i+1}}\,\omega^{\widetilde{(i,l)}}_{\tilde{i}} = -\sqrt{2}\,\omega^l_i \tag{4.81}$$

$$\text{for } 1 \le i < l \le n,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\,\omega^{\widetilde{(k,i)}}_{\tilde{q}} - \sqrt{\frac{n-i}{n-i+1}}\,\omega^{\widetilde{(k,i)}}_{\tilde{i}} = -\sqrt{2}\,\omega^k_i \tag{4.82}$$

$$\text{for } 1 \le k < i \le n-1,$$

$$\omega^{\widetilde{(k,rn+l)}}_{\widetilde{(i,j)}} = 0 \qquad \text{for } 1 \le i < j \le n,\ 1 \le k < l \le n,$$

$$\omega^{\widetilde{(j,rn+l)}}_{\widetilde{(i,j)}} = \omega^{rn+l}_i \qquad \text{for } 1 \le i < j < l \le n,$$

$$\omega^{\widetilde{(k,rn+j)}}_{\widetilde{(i,j)}} = -\omega^{rn+k}_i \qquad \text{for } 1 \le i, k < j \le n,$$

$$\omega^{\widetilde{(i,rn+l)}}_{\widetilde{(i,j)}} = \omega^{rn+l}_j \qquad \text{for } 1 \le i < j, l \le n,$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(k,rn+i)}} = -\omega_j^{rn+k} \qquad \text{for } \ 1 \le k < i < j \le n,$$

$$\omega_{\widetilde{(i,j)}}^{\widetilde{(i,rn+j)}} = \omega_j^{rn+j} - \omega_i^{rn+i} \qquad \text{for } \ 1 \le i < j \le n,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\, \omega_{\tilde{q}}^{\widetilde{(k,rn+l)}} - \sqrt{\frac{n-i}{n-i+1}}\, \omega_{\tilde{i}}^{\widetilde{(k,rn+l)}} = 0 \tag{4.83}$$

$$\text{for } \ 1 \le k < l \le n, \ 1 \le i \le n-1,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\, \omega_{\tilde{q}}^{\widetilde{(i,rn+l)}} - \sqrt{\frac{n-i}{n-i+1}}\, \omega_{\tilde{i}}^{\widetilde{(i,rn+l)}} = -\sqrt{2}\,\omega_i^{rn+l} \tag{4.84}$$

$$\text{for } \ 1 \le i < l \le n,$$

$$\sum_{q<i} \frac{1}{\sqrt{(n-q)(n-q+1)}}\, \omega_{\tilde{q}}^{\widetilde{(k,rn+i)}} - \sqrt{\frac{n-i}{n-i+1}}\, \omega_{\tilde{i}}^{\widetilde{(k,rn+i)}} = \sqrt{2}\,\omega_i^{rn+k} \tag{4.85}$$

$$\text{for } \ 1 \le k < i \le n-1,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\tilde{k}} = 0 \qquad \text{for } \ 1 \le k < i < j \le n \quad \text{or} \quad 1 \le i < j < k \le n-1,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\tilde{k}} = -\frac{\sqrt{2}}{\sqrt{(n-k)(n-k+1)}}\, \omega_i^{rn+j} \quad \text{for } \ 1 \le i < k < j \le n,$$

$$\omega_{\widetilde{(k,rn+j)}}^{\tilde{k}} = -\sqrt{\frac{2(n-k+1)}{n-k}}\, \omega_k^{rn+j} \qquad \text{for } \ 1 \le k < j \le n,$$

$$\omega_{\widetilde{(i,rn+k)}}^{\tilde{k}} = \sqrt{\frac{2(n-k)}{n-k+1}}\, \omega_i^{rn+k} \qquad \text{for } \ 1 \le i < k \le n-1,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(k,rn+l)}} = 0 \qquad \text{for } \ 1 \le i < j \le n, \ 1 \le k < l \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(j,rn+l)}} = -\omega_i^l \qquad \text{for } \ 1 \le i < j < l \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(k,rn+j)}} = \omega_i^k \qquad \text{for } \ 1 \le i, k < j \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(i,rn+l)}} = \omega_j^l \qquad \text{for } \ 1 \le i < j, l \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(k,rn+i)}} = -\omega_j^k \qquad \text{for } \ 1 \le k < i < j \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(k,(r+1)n+l)}} = 0 \quad (r \bmod 3) \qquad \text{for } \ 1 \le i < j \le n, \ 1 \le k < l \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{\widetilde{(j,(r+1)n+l)}} = \omega_i^{(r+2)n+l} \quad (r \bmod 3) \qquad \text{for } \ 1 \le i < j < l \le n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{(k,\widetilde{(r+1)n+j})} = -\omega_i^{(r+2)n+k} \quad (r \bmod 3) \qquad \text{for } 1 \leq i, k < j \leq n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{(i,\widetilde{(r+1)n+l})} = -\omega_j^{(r+2)n+l} \quad (r \bmod 3) \qquad \text{for } 1 \leq i < j, l \leq n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{(k,\widetilde{(r+1)n+i})} = \omega_j^{(r+2)n+k} \quad (r \bmod 3) \qquad \text{for } 1 \leq k < i < j \leq n,$$

$$\omega_{\widetilde{(i,rn+j)}}^{(i,\widetilde{(r+1)n+j})} = \omega_{rn+j}^{(r+1)n+j} - \omega_i^{(r+2)n+i} \quad (r \bmod 3) \qquad \text{for } 1 \leq i < j \leq n,$$

where different indices also indicate different numbers and $r = 1, 2, 3$. Relations (4.78) and (4.79) are equivalent to

$$\omega_{\tilde{l}}^{\tilde{k}} = 0 \qquad \text{for } 1 \leq k, l \leq n-1.$$

Relations (4.80), ..., (4.85) follow from the others. Consequently, the immersion f is rigid in case of (A). $\square$

5. Proof of Theorem

First, we recall the notion of plane curves. A curve $\gamma = \gamma(s)$ in a Riemannian manifold M is said to be a *plane curve* if the curve γ is locally contained in some real 2-dimensional totally geodesic submanifold of M. As a matter of course, every plane curve with positive curvature is a Frenet curve of proper order 2. But in general, the converse does not hold. In case that the space M is a real space form $\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$ of constant sectional curvature $\tilde{c}$, a curve γ is a Frenet curve of proper order 2 if and only if the curve γ is a plane curve with positive curvature. In fact, let x be an arbitrary point of $\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$ and X, Y an orthonormal pair of vectors in $T_x\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$. Let $\gamma = \gamma(s)$ be a Frenet curve of proper order 2 with curvature $\kappa(s)$ in $\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$ satisfying the Frenet formulas (1.1) with initial condition

$$\gamma(0) = x, \quad \dot{\gamma}(0) = X \quad \text{and} \quad V_2(0) = Y. \tag{5.1}$$

Then there exists a 2-dimensional totally geodesic submanifold $M^2(\tilde{c}\,;\mathbb{R})$ passing through the point x of $\widetilde{M}^n(\tilde{c}\,;\mathbb{R})$ with $T_xM^2(\tilde{c}\,;\mathbb{R}) = Span_{\mathbb{R}}\{X, Y\}$. We consider the curve $\gamma_1 = \gamma_1(s)$ in $M^2(\tilde{c}\,;\mathbb{R})$ satisfying the same differential equations (1.1) and the initial condition (5.1). By the uniqueness of solution for a system of ordinary differential equations, we have $\gamma_1(s) = \gamma(s)$ $(s \in (-\varepsilon, \varepsilon))$ for some $\varepsilon > 0$. Thus γ is a plane curve.

We shall now prove Theorem. We denote by ∇ and $\widetilde{\nabla}$ the covariant differentiations of M^n and $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$, respectively. Let x be an arbitrary point of M^n, X an arbitrary unit vector in T_xM^n and J an arbitrary element of $\mathcal{J}_x$ with $J^2 = -id$. We consider a quaternionic Frenet curve $\gamma = \gamma(s)$ $(s \in (-\varepsilon, \varepsilon))$ in M^n satisfying

$$\nabla_{\dot{\gamma}}\dot{\gamma}(s) = \kappa(s)V(s), \quad \nabla_{\dot{\gamma}}V(s) = -\kappa(s)\dot{\gamma}(s) \tag{5.2}$$

and the initial condition

$$\gamma(0) = x, \quad \dot{\gamma}(0) = X \quad \text{and} \quad V(0) = JX.$$

Since the curve $f \circ \gamma$ is a plane curve in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ by assumption, there exist a (nonnegative) function $\tilde{\kappa} = \tilde{\kappa}(s)$ and a field of unit vectors $\widetilde{V} = \widetilde{V}(s)$ along $f \circ \gamma$ in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ which satisfy

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma(s) = \tilde{\kappa}(s)\widetilde{V}(s), \quad \widetilde{\nabla}_{\dot\gamma}\widetilde{V}(s) = -\tilde{\kappa}(s)\dot\gamma(s). \tag{5.3}$$

Then by the formula of Gauss we have

$$\tilde{\kappa}\widetilde{V} = \kappa V + \sigma(\dot\gamma, \dot\gamma), \tag{5.4}$$

so that

$$\tilde{\kappa}^2 = \kappa^2 + \|\sigma(\dot\gamma, \dot\gamma)\|^2. \tag{5.5}$$

We here note that the function $\tilde{\kappa}$ is positive because $\kappa > 0$.

Now we shall compute the covariant differentiation of (5.4): For the left-hand side, by use of (5.3) and (5.4) we see

$$\begin{aligned} \widetilde{\nabla}_{\dot\gamma}(\tilde{\kappa}\widetilde{V}) &= \dot{\tilde{\kappa}}\widetilde{V} + \tilde{\kappa}\widetilde{\nabla}_{\dot\gamma}\widetilde{V} \\ &= \frac{\dot{\tilde{\kappa}}}{\tilde{\kappa}}\{\kappa V + \sigma(\dot\gamma, \dot\gamma)\} - \tilde{\kappa}^2\dot\gamma. \end{aligned} \tag{5.6}$$

For the right-hand side, by the formulae of Gauss and Weingarten we obtain

$$\begin{aligned} &\widetilde{\nabla}_{\dot\gamma}\{\kappa V + \sigma(\dot\gamma, \dot\gamma)\} \\ &= \dot\kappa V + \kappa\widetilde{\nabla}_{\dot\gamma}V - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma, \dot\gamma)) \\ &= \dot\kappa V + \kappa\{\nabla_{\dot\gamma}V + \sigma(\dot\gamma, V)\} - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) \\ &= \dot\kappa V - \kappa^2\dot\gamma + 3\kappa\sigma(\dot\gamma, V) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma). \end{aligned} \tag{5.7}$$

We compare the tangential components and the normal components for the submanifold M^n in (5.6) and (5.7), respectively. Then we get the following:

$$\dot{\tilde{\kappa}}\kappa V - \tilde{\kappa}^3\dot\gamma = \tilde{\kappa}\left\{\dot\kappa V - \kappa^2\dot\gamma - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma\right\}, \tag{5.8}$$

$$\dot{\tilde{\kappa}}\sigma(\dot\gamma, \dot\gamma) = \tilde{\kappa}\left\{3\kappa\sigma(\dot\gamma, V) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma)\right\}. \tag{5.9}$$

The equation (5.9) implies

$$\tilde{\kappa}\dot{\tilde{\kappa}}\sigma(\dot\gamma, \dot\gamma) = \tilde{\kappa}^2\left\{3\kappa\sigma(\dot\gamma, V) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma)\right\}. \tag{5.10}$$

On the other hand, from (5.5) we have

$$\begin{aligned} \tilde{\kappa}\dot{\tilde{\kappa}} &= \frac{1}{2}\frac{d}{ds}\tilde{\kappa}^2 \\ &= \kappa\dot\kappa + \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &= \kappa\dot\kappa + \langle D_{\dot\gamma}(\sigma(\dot\gamma, \dot\gamma)), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &= \kappa\dot\kappa + \langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle + 2\kappa\langle\sigma(V, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle. \end{aligned} \tag{5.11}$$

Substituting (5.5) and (5.11) into (5.10), at $s=0$ we obtain

$$\begin{aligned}&\Big\{\kappa(0)\dot\kappa(0)+\langle(\bar\nabla_X\sigma)(X,X),\sigma(X,X)\rangle\\ &\qquad\qquad+2\kappa(0)\langle\sigma(X,X),\sigma(X,JX)\rangle\Big\}\sigma(X,X)\\ &\qquad\qquad=\Big\{\kappa(0)^2+\|\sigma(X,X)\|^2\Big\}\Big\{3\kappa(0)\sigma(X,JX)+(\bar\nabla_X\sigma)(X,X)\Big\}.\end{aligned}\tag{5.12}$$

Here we consider anther quaternionic Frenet curve $\gamma_0=\gamma_0(s)$ ($s\in(-\varepsilon_0,\varepsilon_0)$) of the same curvature κ in M^n satisfying the equations

$$\nabla_{\dot\gamma_0}\dot\gamma_0(s)=\kappa(s)V_0(s),\quad \nabla_{\dot\gamma_0}V_0(s)=-\kappa(s)\dot\gamma_0(s)$$

with initial condition

$$\gamma_0(0)=x,\quad \dot\gamma_0(0)=X\quad\text{and}\quad V_0(0)=-JX.$$

Since the curve $f\circ\gamma_0$ is a plane curve in $\widetilde{M}^{4n+p}(\tilde c\,;\mathbb{R})$ by assumption, we can apply the above discussion to this curve γ_0. Then the equality (5.12) for γ_0 turns to

$$\begin{aligned}&\Big\{\kappa(0)\dot\kappa(0)+\langle(\bar\nabla_X\sigma)(X,X),\sigma(X,X)\rangle\\ &\qquad\qquad-2\kappa(0)\langle\sigma(X,X),\sigma(X,JX)\rangle\Big\}\sigma(X,X)\\ &\qquad\qquad=\Big\{\kappa(0)^2+\|\sigma(X,X)\|^2\Big\}\Big\{-3\kappa(0)\sigma(X,JX)+(\bar\nabla_X\sigma)(X,X)\Big\}.\end{aligned}\tag{5.12'}$$

Therefore, from (5.12) and (5.12') we have

$$2\kappa(0)\langle\sigma(X,X),\sigma(X,JX)\rangle\sigma(X,X)=3\kappa(0)\Big\{\kappa(0)^2+\|\sigma(X,X)\|^2\Big\}\sigma(X,JX),$$

so that

$$2\langle\sigma(X,X),\sigma(X,JX)\rangle\sigma(X,X)=3\Big\{\kappa(0)^2+\|\sigma(X,X)\|^2\Big\}\sigma(X,JX).\tag{5.13}$$

Taking the inner product of both sides of this with $\sigma(X,X)$, we get

$$\begin{aligned}&2\langle\sigma(X,X),\sigma(X,JX)\rangle\|\sigma(X,X)\|^2\\ &\qquad=3\Big\{\kappa(0)^2+\|\sigma(X,X)\|^2\Big\}\langle\sigma(X,X),\sigma(X,JX)\rangle\end{aligned}$$

hence

$$\Big\{3\kappa(0)^2+\|\sigma(X,X)\|^2\Big\}\langle\sigma(X,X),\sigma(X,JX)\rangle=0.$$

So we have

$$\langle\sigma(X,X),\sigma(X,JX)\rangle=0,$$

because $3\kappa(0)^2+\|\sigma(X,X)\|^2>0$. This, combined with (5.13), shows that

$$\sigma(X,JX)=0\tag{5.14}$$

for any $X\in T_xM^n$ at any point $x\in M^n$ and any $J\in\mathcal{J}_x$ with $J^2=-id$. Replacing X by $JX+Y$ in (5.14), we get

$$\sigma(JX,JY)=\sigma(X,Y)\tag{5.15}$$

for each $X, Y \in T_xM^n$. Thus, by virtue of Lemma 3, we see that the immersion f is parallel: $\bar{\nabla}\sigma = 0$.

Next, taking the inner product of both sides of (5.8) with V, we have

$$\begin{aligned}\dot{\tilde{\kappa}}\kappa &= \tilde{\kappa}\dot{\kappa} - \tilde{\kappa}\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma, V\rangle \\ &= \tilde{\kappa}\dot{\kappa} - \tilde{\kappa}\langle \sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, V)\rangle.\end{aligned}$$

On the other hand, from (5.14) we know that

$$\sigma(\dot\gamma, V) = 0 \quad \text{for each } s \in (-\varepsilon, \varepsilon). \tag{5.16}$$

Hence the above equation becomes

$$\dot{\tilde{\kappa}}\kappa = \tilde{\kappa}\dot{\kappa}, \tag{5.17}$$

so that the equation (5.8) reduces to

$$A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma = (\tilde{\kappa}^2 - \kappa^2)\dot\gamma.$$

Therefore

$$\langle \sigma(X,X), \sigma(X,Y)\rangle = \langle A_{\sigma(X,X)}X, Y\rangle = 0$$

for any orthonormal pair of vectors $X, Y \in T_xM^n$ at each point $x \in M^n$. Thus, by virtue of Lemma 1, the immersion f is isotropic at each point $x \in M^n$. Moreover, we can see that f is constant isotropic as follows: Let $c = c(s)$ be an arbitrary geodesic in M^n parametrized by its arclength s. Then, since $\bar{\nabla}\sigma = 0$, we have

$$\frac{d}{ds}\|\sigma(\dot c,\dot c)\|^2 = 2\langle(\bar{\nabla}_{\dot c}\sigma)(\dot c,\dot c), \sigma(\dot c,\dot c)\rangle + 4\langle\sigma(\nabla_{\dot c}\dot c,\dot c), \sigma(\dot c,\dot c)\rangle = 0.$$

Thus $\|\sigma(\dot c,\dot c)\|$ is constant along the curve $c = c(s)$. This fact implies that the immersion f is constant isotropic.

Let R denote the curvature tensor of M^n. For arbitrary $J \in \mathcal{J}_x$ with $J^2 = -id$, from (5.14), (5.15) and the equation of Gauss (3.2), we have

$$\begin{aligned}\langle R(X,JX)JX, X\rangle &= \tilde{c} + \langle\sigma(X,X), \sigma(JX,JX)\rangle - \|\sigma(X,JX)\|^2 \\ &= \tilde{c} + \|\sigma(X,X)\|^2\end{aligned}$$

for an arbitrary unit vector $X \in T_xM$ at any point x of M^n. Since M^n is constant isotropic, this shows that M^n is a quaternonic space form. Then, by Proposition 1, we can see that the submanifold M^n is one of (1), (2) and (3) in the statement of Theorem.

In order to prove our assertion, we must check the examples (1), (2) and (3) satisfy the hypothesis of Theorem. In case of (1), the hypothesis is obviously satisfied. In case of (2), for each circle γ of curvature $k(>0)$ in M^n the curve $f \circ \gamma$ is a circle of curvature $\sqrt{k^2 - \tilde{c}}$ (see page 169 in [6]), hence it is a plane curve in the ambient space $\widetilde{M}^{4n+p}(\tilde{c}; \mathbb{R})$.

In case of (3), the isometric immersion $f : M^n = \mathbb{H}P^n(c) \to \widetilde{M}^{4n+p}(\tilde{c}; \mathbb{R})$ given by (3) is constant $\sqrt{c - \tilde{c}}$-isotropic and parallel (see for example [8]). Denote by ∇ and $\widetilde{\nabla}$ the covariant differentiations of M^n and $\widetilde{M}^{4n+p}(\tilde{c}; \mathbb{R})$, respectively. Let $\gamma = \gamma(s)$ be a quaternionic circle of curvature k (>0) satisfying

$$\nabla_{\dot\gamma}\dot\gamma(s) = kV(s), \quad \nabla_{\dot\gamma}V(s) = -k\dot\gamma(s).$$

Then we can see that the curve $f \circ \gamma$ is a circle of curvature $\sqrt{k^2 + c - \tilde{c}}$ in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ as follows: The curve $f \circ \gamma$ satisfies

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = kV + \sigma(\dot\gamma, \dot\gamma),$$

so that

$$\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\| = \sqrt{k^2 + \|\sigma(\dot\gamma,\dot\gamma)\|^2} = \sqrt{k^2 + c - \tilde{c}}.$$

We write

$$\widetilde{V} = \frac{1}{\sqrt{k^2 + c - \tilde{c}}}\{kV + \sigma(\dot\gamma,\dot\gamma)\}.$$

Since f is parallel, from Lemma 3 we obtain

$$\sigma(\dot\gamma, V) = \sigma(\dot\gamma, J\dot\gamma) = 0.$$

Moreover, as f is constant isotropic, by using (2) of Lemma 1 we get

$$A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma = \|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma.$$

So, by the formulae of Gauss and Weingarten we have

$$\begin{aligned}
\widetilde{\nabla}_{\dot\gamma}\widetilde{V} &= \frac{1}{\sqrt{k^2 + c - \tilde{c}}}\widetilde{\nabla}_{\dot\gamma}\{kV + \sigma(\dot\gamma,\dot\gamma)\} \\
&= \frac{1}{\sqrt{k^2 + c - \tilde{c}}}\Big\{k\left(\nabla_{\dot\gamma}V + \sigma(\dot\gamma, V)\right) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma))\Big\} \\
&= \frac{1}{\sqrt{k^2 + c - \tilde{c}}}\Big\{-k^2\dot\gamma - \|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma)\Big\} \\
&= \frac{1}{\sqrt{k^2 + c - \tilde{c}}}\Big\{-(k^2 + c - \tilde{c})\dot\gamma + 2k\sigma(V,\dot\gamma)\Big\} \\
&= -\sqrt{k^2 + c - \tilde{c}}\;\dot\gamma\,.
\end{aligned}$$

Thus the curve $f \circ \gamma$ is a plane curve in $\widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$. Hence our assertion follows. $\square$

Remark 1. Theorem also holds under the condition $\kappa \equiv 0$ (see [8]).

Remark 2. Assuming that the curvature function κ in the statement of Theorem is not constant, we obtain only the case (1). In fact, suppose that there exists some $s_0 \in (-\varepsilon, \varepsilon)$ with $\dot\kappa(s_0) \neq 0$. Then we find $\dot{\tilde\kappa}(s_0) \neq 0$ from (5.17) because $\kappa, \tilde\kappa > 0$. We know the fact that $\bar\nabla\sigma = 0$. So the equation (5.9), combined with (5.16), yields $\sigma(\dot\gamma(s_0), \dot\gamma(s_0)) = 0$. Moreover, we can see that $\|\sigma(\dot\gamma,\dot\gamma)\|$ is constant along the curve γ because

$$\frac{d}{ds}\|\sigma(\dot\gamma,\dot\gamma)\|^2 = 2\langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma), \sigma(\dot\gamma,\dot\gamma)\rangle + 4\kappa\langle\sigma(V,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle = 0.$$

Thus we conclude $\sigma(X, X) = 0$ for an arbitrary unit vector $X \in T_xM^n$ at each point $x \in M^n$. Consequently our immersion $f : M^n \to \widetilde{M}^{4n+p}(\tilde{c}\,;\mathbb{R})$ is a totally geodesic immersion.

Remark 3. In [13], the author obtained a similar theorem to ours by using the results of D. Ferus [3] and M. Takeuchi [10]. In the classification theory of parallel submanifolds due to M. Takeuchi, we need a global condition that the submanifold is complete. However, our theorem is a local version. So, in this paper, we gave a proof which does not depend on results of [3, 10].

References

[1] T. Adachi, *Circles on quaternionic space forms,* J. Math. Soc. Japan, **48** (1996), 205–227.
[2] J. Erbacher, *Reduction of the codimension of an isotropic immersion,* J. Diff. Geometry **5** (1971), 333–340.
[3] D. Ferus, *Immersions with parallel second fundamental form,* Math. Z. **140** (1974), 87–92.
[4] S. Maeda, *Remarks on isotropic immersions,* Yokohama Math. J. **34** (1986), 83–90.
[5] S. Maeda and H. Tanabe, *Totally geodesic immersions of Kähler manifolds and Kähler Frenet curves,* Math. Z. **252** (2006), 787–795.
[6] K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry,* Math. Ann. **210** (1974), 163–170.
[7] B. O'Neill, *Isotropic and Kaehler immersions,* Candian J. Math. **17** (1965), 905–915.
[8] K. Sakamoto, *Planar geodesic immersions,* Tôhoku Math. J. **29** (1977), 25–56.
[9] K. Suizu, S. Maeda and T. Adachi, *Study of submanifolds by curves of order 2,* Mem. Fac. Sci. Eng. Shimane Univ. Seies B: Mathematical Science **35** (2002), 1–21.
[10] M. Takeuchi, *Parallel submanifolds of space forms,* Manifolds and Lie groups, in honor of Y. Matsushima, Birkhäuser, Boston, 1981, 429–447.
[11] H. Tanabe, *Characterization of totally geodesic submanifolds in terms of Frenet curves,* Sci. Math. Japonicae, **63** (2006), 83–88, :**e2005**, 557–562.
[12] H. Tanabe, *Parallel isometric immersions and Kähler Frenet curves,* Far East J. Math. Sci. **20** (2006), 1–19.
[13] H. Tanabe, *Quaternionic Frenet curves and totally geodesic immersions,* Yokohama Math. J. **52** (2005), 1–10.

Yonago Higashi Highschool, 1 Kanda-machi, Yonago Tottori Pref. 683-0852, Japan
E-mail address: `tanabe_hr1@mailk.torikyo.ed.jp`