Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
40 (2007), pp.15–32

SUBMANIFOLD THEORY FROM THE VIEWPOINT OF CIRCLES

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(Received: April 13, 2006)

Abstract. We study submanifolds by using the notion of circles in Riemannian geometry. This expository paper consists of two parts. One is to improve Nomizu and Yano's result ([11]) which characterizes extrinsic spheres in a Riemannian manifold. The other is to characterize totally geodesic immersions and some parallel immersions in terms of Frenet curves of proper order 2 (which are generalizations of circles) on submanifolds.

1. Introduction

A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M^n parametrized by its arclength s is called a *Frenet curve of proper order* 2 if there exist a field of orthonormal frames $\{V_1 = \dot\gamma, V_2\}$ along γ and a positive smooth function $\kappa(s)$ satisfying the following system of ordinary differential equations: $\nabla_{\dot\gamma}V_1(s) = \kappa(s)V_2(s)$ and $\nabla_{\dot\gamma}V_2(s) = -\kappa(s)V_1(s)$, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M^n. The function $\kappa(s)$ and the field of orthonormal frames $\{V_1, V_2\}$ are called the *curvature* and the *Frenet frame* of γ, respectively. Note that we do *not* allow the curvature $\kappa(s)$ to vanish at some point. Therefore curves with inflection points, such as $y = x^3$ on a Euclidean xy-plane, are not Frenet curves of proper order 2. A curve is called a Frenet curve *of order* 2 if it is either a geodesic or a Frenet curve of proper order 2. A Frenet curve of order 2 with nonnegative constant curvature k is called a *circle* of curvature k. Needless to say a circle of null curvature is nothing but a geodesic.

Study of circles produces new knowledge in submanifold theory. For example we here recall the following two surfaces. Let f_1 be a totally umbilic embedding of a 2-dimensional standard sphere $S^2(c)$ of curvature c into a Euclidean space

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2000 *Mathematics Subject Classification.* Primary 53B25, Secondary 53C40.

Key words and phrases. totally umbilic submanifolds, circles, constant isotropic immersions, Kähler submanifolds, Veronese embeddings, Frenet curves of proper order 2, Kähler Frenet curves.

$\mathbb{R}^5$ and $f_2 = \iota \circ f$ an isometric parallel immersion of $S^2(c)$ into $\mathbb{R}^5$. Here f is the second standard minimal immersion of $S^2(c)$ into $S^4(3c)$ and ι is a totally umbilic embedding of $S^4(3c)$ into $\mathbb{R}^5$. It is known that for each great circle γ of $S^2(c)$, both of the curves $f_1 \circ \gamma$ and $f_2 \circ \gamma$ are circles in the ambient space $\mathbb{R}^5$. This tells us that we cannot distinguish f_1 from f_2 by the extrinsic shape of *geodesics* of $S^2(c)$ in $\mathbb{R}^5$. However we emphasize that we can distinguish these two isometric immersions f_1 and f_2 by the extrinsic shape of small circles of $S^2(c)$ in $\mathbb{R}^5$. In fact, for each small circle γ on $S^2(c)$, the curve $f_1 \circ \gamma$ is also a circle in $\mathbb{R}^5$ but the curve $f_2 \circ \gamma$ is a helix of proper order 4 in the ambient space $\mathbb{R}^5$, namely in $\mathbb{R}^5$ the curve $f_2 \circ \gamma$ has constant positive three curvatures κ_1, κ_2 and κ_3 along this curve in the sense of Frenet formula. It is hence interesting to investigate the extrinsic shape of circles of the submanifold.

The first half of this paper is motivated by the follwing fact due to Nomizu and Yano [11]: Let M^n be a Riemannian submanifold of an ambient Riemannian manifold $\widetilde{M}^{n+p}$ (with Riemannian connection $\widetilde{\nabla}$) through an isometric immersion f. Then (M^n, f) is an extrinsic sphere of $\widetilde{M}^{n+p}$ (that is, M^n is a totally umbilic submanifold with parallel mean curvature vector of $\widetilde{M}^{n+p}$) if and only if, for some positive constant k and for every circle $\gamma = \gamma(s)$ of curvature k on M^n, the curve $f \circ \gamma$ is a circle in $\widetilde{M}^{n+p}$. In this fact we pay particular attention to a condition "*the curve $f \circ \gamma$ is a circle in $\widetilde{M}^{n+p}$*". It is natural to pose the following two problems.

Problem 1. *If we replace this condition by a condition that the curve $f \circ \gamma$ has constant the first curvature, that is, $\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ is constant along $f \circ \gamma$, what can we say about the isometric immersion f?*

Problem 2. *If we replace this condition by a condition that the curve $f \circ \gamma$ is a Frenet curve of order* 2*, what can we say about the isometric immersion f?*

The main purpose of the first half of this paper is to give complete answers to these problems (Theorems 1 and 2). As an application of Theorem 1 we obtain a characterization of Veronese embeddings of complex projective spaces into complex projective spaces which are typical examples of Kähler immersions (Theorem 3).

In the latter half, using the notion of Frenet curves of order 2, we establish some results. We first provide a characterization of every totally geodesic submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+p}$ by observing the extrinsic shape of a Frenet curve of proper order 2 on the submanifold M^n (Theorem 4).

We consider a Frenet curve $\gamma = \gamma(s)$ of proper order 2 in a Kähler manifold M (with complex structure J). We put $\tau_\gamma = |\langle V_1, JV_2\rangle|$, which is well-defined. Since we have

$$\frac{d}{ds}\langle V_1, JV_2\rangle = \langle \nabla_{\dot\gamma} V_1, JV_2\rangle + \langle V_1, J\nabla_{\dot\gamma} V_2\rangle = \kappa\langle V_2, JV_2\rangle - \kappa\langle V_1, JV_1\rangle = 0,$$

we see τ_γ does not depend on parameter s. A curve $\gamma = \gamma(s)$ is called a *Kähler Frenet curve* if it is either a geodesic or a Frenet curve of proper order 2 with $\tau_\gamma = 1$. In other words, a Kähler Frenet curve γ is either a geodesic or a curve satisfying $\nabla_{\dot\gamma}\dot\gamma = \kappa(s)J\dot\gamma$ or $\nabla_{\dot\gamma}\dot\gamma = -\kappa(s)J\dot\gamma$ for some positive smooth function $\kappa = \kappa(s)$. We call γ a *Kähler circle* of curvature k if the curvature function κ of a Frenet curve γ of order 2 is a nonnegative constant k.

We next give a characterization of all totally geodesic Kähler immersions of Kähler manifolds into Kähler manifolds by using the notion of Kähler Frenet curves (Theorem 5). We finally characterize all parallel isometric immersions of an n-dimensional complex space form $M_n(c)$ (which is locally congruent to either $\mathbb{C}P^n(c), \mathbb{C}^n$ or $\mathbb{C}H^n(c)$) of constant holomorphic sectional curvature c into a $(2n+p)$-dimensional real space form $\widetilde{M}^{2n+p}(\tilde c)$ (which is locally congruent to either $\mathbb{R}^{2n+p}, S^{2n+p}(\tilde c)$ or $H^{2n+p}(\tilde c)$) of constant sectional curvature $\tilde c$ by observing the extrinsic shape of Kähler Frenet curves on $M_n(c)$ (Theorem 6).

2. Fundamental equations and isotropic immersions

We review fundamental equations in submanifold theory. Let M^n, $\widetilde{M}^m$ be Riemannian manifolds and $f : M^n \to \widetilde{M}^m$ an isometric immersion. Throughout this paper we will identify a vector X of M^n with a vector $f_*(X)$ of $\widetilde{M}^m$. The Riemannian metrics on M^n, $\widetilde{M}^m$ are denoted by the same notation $\langle\ ,\ \rangle$.

We denote by ∇ and $\widetilde\nabla$ the covariant differentiations of M^n and $\widetilde{M}^m$, respectively. Then the second fundamental form σ of the immersion f is defined by

$$\sigma(X,Y) = \widetilde\nabla_X Y - \nabla_X Y, \tag{2.1}$$

where X and Y are vector fields tangent to M^n. For a vector field ξ normal to M^n, we write

$$\widetilde\nabla_X \xi = -A_\xi X + D_X\xi, \tag{2.2}$$

where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of $\widetilde\nabla_X\xi$. We define the covariant differentiation $\bar\nabla$ of the second fundamental form σ with respect to the connection in (tangent bundle)$\oplus$(normal bundle) as follows:

$$(\bar\nabla_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.3}$$

We next review the notion of isotropic immersions. An isometric immersion $f : M \to \widetilde{M}$ is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2 (= \lambda(x))$ does not depend on the choice of $X(\neq 0) \in T_xM$. If the immersion is isotropic at every point, then the immersion is said to be isotropic. When the function $\lambda = \lambda(x)$ is constant on M, we call M a constant (λ-)isotropic submanifold. Note that a totally umbilic immersion is isotropic, but not *vice versa*. The following is well known ([12]).

Lemma 1. *Let f be an isometric immersion of M into $(\widetilde{M}, \langle\ ,\ \rangle)$. Then f is isotropic at $x \in M$ if and only if the second fundamental form σ of f satisfies $\langle \sigma(u,u), \sigma(u,v) \rangle = 0$ for an arbitrary orthogonal pair $u, v \in T_xM$.*

3. Characterization of constant isotropic immersions

The following gives a geometric meaning of constant isotropic immersions in terms of circles on submanifolds.

Theorem 1 ([5]). *Let M^n be an n-dimensional connected Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. Then the following are equivalent.*

(i) *M^n is a constant (λ-) isotropic submanifold of $\widetilde{M}^{n+p}$.*

(ii) *There exists a positive constant k satisfying that for each circle γ of curvature k on the submanifold M^n the curve $f \circ \gamma$ in $\widetilde{M}^{n+p}$ has constant the first curvature κ_1 along this curve.*

Proof. (i) $\Rightarrow$ (ii): Let $f : M^n \to \widetilde{M}^{n+p}$ be a constant λ-isotropic immersion. We take a circle of curvature k, which satisfies the following equations:

$$\nabla_{\dot\gamma}\dot\gamma = kY_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -k\dot\gamma. \tag{3.1}$$

In the following, for simplicity we also denote $f \circ \gamma$ by γ. It follows from (2.1) that

$$\widetilde{\nabla}_{\dot\gamma(s)}\dot\gamma(s) = kY_s + \sigma(\dot\gamma(s), \dot\gamma(s)). \tag{3.2}$$

Then from (3.2) we can see that the first curvature $\kappa_1 = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ of the curve $f \circ \gamma$ is equal to $\sqrt{k^2 + \lambda^2}$, which is constant.

(ii) $\Rightarrow$ (i): Let $f : M^n \to \widetilde{M}^{n+p}$ be an isometric immersion satisfying the condition (ii). We take a point $x \in M$ and choose an arbitrary orthonormal pair of vectors $u, v \in T_xM$. Let $\gamma = \gamma(s)$ be a circle of curvature k on the submanifold M^n with initial condition that $\gamma(0) = x$, $\dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = kv$. By condition (ii) the first curvature $\kappa_1 = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ of the curve $f \circ \gamma$ is constant, so that equation (3.2) implies $\|\sigma(\dot\gamma, \dot\gamma)\|$ is constant. Hence we obtain

$$\begin{aligned} 0 = \frac{d}{ds}\|\sigma(\dot\gamma,\dot\gamma)\|^2 &= 2\langle D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)), \sigma(\dot\gamma,\dot\gamma)\rangle \\ &= 2\langle (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma), \sigma(\dot\gamma,\dot\gamma)\rangle \\ &= 2\langle (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma), \sigma(\dot\gamma,\dot\gamma)\rangle + 4k\langle \sigma(\dot\gamma,\dot\gamma), \sigma(\dot\gamma, Y)\rangle. \end{aligned} \tag{3.3}$$

Evaluating equation (3.3) at $s = 0$, we get

$$\langle (\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle + 2k\langle \sigma(u,u), \sigma(u,v)\rangle = 0. \tag{3.4}$$

On the other hand, for another circle $\rho = \rho(s)$ of the same curvature k on the submanifold M^n with initial condition that $\rho(0) = x, \dot\rho(0) = u$ and $\nabla_{\dot\rho}\dot\rho(0) = -kv$, we have

$$\langle(\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle - 2k\langle\sigma(u,u), \sigma(u,v)\rangle = 0 \tag{3.5}$$

which corresponds to equation (3.4). Thus, from (3.4) and (3.5) we can see that $\langle\sigma(u,u), \sigma(u,v)\rangle = 0$ for any orthonormal pair of vectors u, v at each point x of M, so that the submanifold M^n is (λ-)isotropic in $\widetilde{M}^{n+p}$ through the isometric immersion f by Lemma 1.

Next, we shall show that $\lambda : M \to \mathbb{R}$ is constant. It follows from (3.4) and (3.5) that

$$\langle(\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle = 0 \quad \text{for every unit vector } u \text{ at each point } x \text{ of } M^n.$$

Then, for every geodesic $\tau = \tau(s)$ on the submanifold M^n we see that $\lambda = \lambda(s)$ is constant along τ. Therefore we can conclude that λ is constant on M. $\square$

4. Characterization of extrinsic spheres

The following is an improvement of Nomizu and Yano's result [11].

Theorem 2 ([4]). *Let M^n be an n-dimensional connected Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. Then M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$ if and only if for some positive constant k and for every circle $\gamma = \gamma(s)$ of curvature k in M^n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$.*

Proof. The "only if" part is obvious by virtue of the well-known result of Nomizu and Yano ([11]). So it suffices to verify the "if" part.

Let x be an arbitrary point of M^n and u, v an orthonormal pair of vectors in T_xM^n. Let $\gamma = \gamma(s)$ be a circle of of positive curvature k in M^n satisfying (3.1) and the initial condition $\gamma(0) = x, \dot\gamma(0) = u$ and $Y_0 = v$. By assumption there exist a (nonnegative) smooth function $\tilde\kappa = \tilde\kappa(s)$ and a field of unit vectors $\widetilde{Y}_s$ along $f \circ \gamma$ in $\widetilde{M}^{n+p}$ satisfying that

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \tilde\kappa\widetilde{Y}_s,\ \widetilde\nabla_{\dot\gamma}\widetilde{Y}_s = -\tilde\kappa\dot\gamma. \tag{4.1}$$

It follows from (2.1) and the first equality of (4.1) that

$$\tilde\kappa\widetilde{Y}_s = kY_s + \sigma(\dot\gamma, \dot\gamma), \tag{4.2}$$

so that

$$\tilde\kappa^2 = k^2 + \|\sigma(\dot\gamma, \dot\gamma)\|^2. \tag{4.3}$$

We here note that $\tilde{\kappa}(s) > 0$ for each s.

Now we shall compute the covariant differentiation of (4.2). For the left-hand side, by using (4.1) and (4.2) we see

$$\begin{aligned}\widetilde{\nabla}_{\dot\gamma}(\tilde\kappa\widetilde{Y}_s) &= \dot{\tilde\kappa}\widetilde{Y}_s + \tilde\kappa\widetilde{\nabla}_{\dot\gamma}\widetilde{Y}_s \\ &= \frac{\dot{\tilde\kappa}}{\tilde\kappa}(kY_s + \sigma(\dot\gamma,\dot\gamma)) - \tilde\kappa^2\dot\gamma.\end{aligned} \tag{4.4}$$

For the right-hand side, using (2.1), (2.2), (2.3) and (3.1), we obtain

$$\begin{aligned}&\widetilde{\nabla}_{\dot\gamma}(kY_s + \sigma(\dot\gamma,\dot\gamma)) \\ &\quad= k\widetilde{\nabla}_{\dot\gamma}Y_s - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)) \\ &\quad= k(\nabla_{\dot\gamma}Y_s + \sigma(\dot\gamma,Y_s)) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) \\ &\quad= -k^2\dot\gamma + 3k\sigma(\dot\gamma,Y_s) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma).\end{aligned} \tag{4.5}$$

We compare the tangential components and the normal components for the submanifold M^n in (4.4) and (4.5), respectively. Then we get the following:

$$\dot{\tilde\kappa}kY_s - \tilde\kappa^3\dot\gamma = \tilde\kappa(-k^2\dot\gamma - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma), \tag{4.6}$$

$$\dot{\tilde\kappa}\sigma(\dot\gamma,\dot\gamma) = \tilde\kappa(3k\sigma(\dot\gamma,Y_s) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)). \tag{4.7}$$

Equation (4.7) gives at $s = 0$

$$\dot{\tilde\kappa}(0)\sigma(u,u) = \tilde\kappa(0)(3k\sigma(u,v) + (\bar\nabla_u\sigma)(u,u)),$$

so that

$$\dot{\tilde\kappa}(0)\tilde\kappa(0)\sigma(u,u) = \tilde\kappa(0)^2(3k\sigma(u,v) + (\bar\nabla_u\sigma)(u,u)). \tag{4.8}$$

On the other hand, we see from (4.3) and (3.3) that

$$\tilde\kappa^2(0) = k^2 + \|\sigma(u,u)\|^2, \tag{4.9}$$

and that for each s

$$2\dot{\tilde\kappa}\tilde\kappa = 2\langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle + 4\langle\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle.$$

So we have at $s = 0$

$$\dot{\tilde\kappa}(0)\tilde\kappa(0) = \langle(\bar\nabla_u\sigma)(u,u),\sigma(u,u)\rangle + 2k\langle\sigma(v,u),\sigma(u,u)\rangle. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8), we obtain

$$\begin{aligned}(4.11)\qquad &3k(k^2+\|\sigma(u,u)\|^2)\sigma(u,v)-2k\langle\sigma(u,u),\sigma(u,v)\rangle\sigma(u,u)\\ &=\langle(\bar\nabla_u\sigma)(u,u),\sigma(u,u)\rangle\sigma(u,u)-(k^2+\|\sigma(u,u)\|^2)(\bar\nabla_u\sigma)(u,u).\end{aligned}$$

Here we consider anther circle $\tau=\tau(s)$ of the same curvature k in M^n satisfying the equations $\nabla_{\dot\tau}\dot\tau = kZ_s$ and $\nabla_{\dot\tau}Z_s=-k\dot\tau$, with initial condition $\tau(0)=x$, $\dot\tau(0)=u$ and $Z_0=-v$. By assumption, the curve $f\circ\tau$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$. So we can apply the above discussion to the curve τ. Then we can see that

$$\begin{aligned}(4.11')\qquad &-3k(k^2+\|\sigma(u,u)\|^2)\sigma(u,v)+2k\langle\sigma(u,u),\sigma(u,v)\rangle\sigma(u,u)\\ &=\langle(\bar\nabla_u\sigma)(u,u),\sigma(u,u)\rangle\sigma(u,u)-(k^2+\|\sigma(u,u)\|^2)(\bar\nabla_u\sigma)(u,u).\end{aligned}$$

Hence, from (4.11) and (4.11') we obtain

$$(4.12)\qquad 3(k^2+\|\sigma(u,u)\|^2)\sigma(u,v)-2\langle\sigma(u,u),\sigma(u,v)\rangle\sigma(u,u)=0.$$

Taking the inner product of both sides of equation (4.12) with $\sigma(u,u)$, we get

$$(3k^2+\|\sigma(u,u)\|^2)\langle\sigma(u,u),\sigma(u,v)\rangle=0.$$

Hence

$$\langle\sigma(u,u),\sigma(u,v)\rangle=0.$$

Since x is arbitrary, thanks to Lemma 1, we find that our immersion f is (say, λ-)isotropic. So, again by using Lemma 1, we get $A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma=\lambda^2\dot\gamma$. Hence, from (4.6) we have

$$\dot{\tilde\kappa}kY_s-\tilde\kappa^3\dot\gamma=-\tilde\kappa(k^2+\lambda^2)\dot\gamma.$$

Taking the inner product of bith sides of this equation with unit vectors Y_s, we know that $\tilde\kappa=\tilde\kappa(s)$ is constant along the curve $f\circ\gamma$. Therefore we can see that the curve $f\circ\gamma$ is a circle in the ambient space $\widetilde{M}^{n+p}$. Thus we get the statement of Theorem 2. $\square$

5. Characterization of Veronese embeddings

As an application of Theorem 1 we shall provide a characterization of a Kähler isometric full immersion of a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c into a complex projective space $\mathbb{C}P^N(\tilde c)$ of constant holomorphic sectional curvature $\tilde c$. By virtue of the classification theorem ([1, 10]) this Kähler immersion is equivalent to a Kähler embedding $f_\nu:\mathbb{C}P^n(c/\nu)\to\mathbb{C}P^N(c)$ given by

$$[z_i]_{0\leqq i\leqq n}\mapsto\Big[\sqrt{\frac{\nu!}{\nu_0!\cdots\nu_n!}}\,z_0^{\nu_0}\cdots z_n^{\nu_n}\Big]_{\nu_0+\cdots+\nu_n=\nu},$$

where $[*]$ means the point of the projective space with homogeneous coordinates $*$ and $N=(n+\nu)!/(n!\nu!)-1$. We usually call f_ν the ν-th *Veronese embedding.* The embedding f_ν has various geometric properties. We recall the work of [1, 10] for the later use.

Theorem A. *Let $f : M_n(c) \to M_N(\tilde{c})$ be a Kähler isometric immersion of a complex space form of constant holomorphic sectional curvature c into another complex space form of constant holomorphic sectional curvature $\tilde{c}$. If $\tilde{c} > 0$ and f is full, then $\tilde{c} = \nu c$, $N = (n+\nu)!/(n!\nu!) - 1$ and f is locally equivalent to the ν-th Veronese embedding f_ν for some positive integer ν.*

We are now in a position to prove the following:

Theorem 3 ([5]). *Let $f : M_n \to M_N(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold M_n into an N-dimensional complex space form $M_N(c)$ of constant holomorphic sectional curvature $c > 0$. Then the following conditions are equivalent.*

(*i*) *For some positive integer ν, the submanifold M_n is locally congruent to $\mathbb{C}P^n(c/\nu)$, $N = (n+\nu)!/(n!\nu!) - 1$ and f is locally equivalent to the ν-th Veronese embedding f_ν.*

(*ii*) *There exists a positive constant k satisfying that for each circle γ of curvature k on the submanifold M_n the curve $f \circ \gamma$ in $M_N(c)$ has constant the first curvature κ_1 along this curve.*

Proof. $(i) \Rightarrow (ii)$: For each Veronese embedding $f_\nu : \mathbb{C}P^n(c/\nu) \to \mathbb{C}P^N(c)$ we see that $\|\sigma(X,X)\|^2 = c(\nu-1)/2\nu$ for any unit vector X at each point $x \in \mathbb{C}P^n(c/\nu)$ (see [13]). Then we find that for each circle γ of curvature k on $\mathbb{C}P^n(c/\nu)$ the curve $f_\nu \circ \gamma$ has constant the first curvature $\kappa_1 = \sqrt{k^2 + \frac{c(\nu-1)}{2\nu}}$ in the ambient manifold $\mathbb{C}P^N(c)$.

$(ii) \Rightarrow (i)$: Let $f : M_n \to M_N(c)$ be a Kähler isometric full immersion satisfying the condition (ii). Then by virtue of Theorem 1 M_n is constant (λ-)isotropic in $M_N(c)$. On the other hand we denote by R (resp. $\widetilde{R}$) the curvature tensor of M_n (resp. $M_N(c)$). We recall the Gauss equation

$$\langle \widetilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle \sigma(X,Z),\sigma(Y,W)\rangle - \langle \sigma(X,W),\sigma(Y,Z)\rangle.$$

Since M is a Kähler submanifold in $M_N(c)$, from this equation and

$$\widetilde{R}(X,Y)Z = \frac{c}{4}(\langle Y,Z\rangle X - \langle X,Z\rangle X + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ),$$

we find that the holomorphic sectional curvature $K(X,JX)$ of M_n determined by a unit vector X is given by

$$K(X,JX) = \langle R(X,JX)JX,X\rangle = c - 2\|\sigma(X,X)\|^2.$$

Thus we can see that our submanifold M_n is a complex space form. Therefore from Theorem A we obtain the statement (i). $\square$

We here make mention of other curvatures in Theorem 3. It is known that for each circle γ on $\mathbb{C}P^n(c/\nu)$ the curve $f_\nu \circ \gamma$ is an integral curve of some Killing vector field of the ambient space $\mathbb{C}P^N(c)$, so that all curvatures of $f_\nu \circ \gamma$ are constant (cf. [7]).

6. Characterization of totally geodesic immersions

In this section we consider Frenet curves of proper order 2 on submanifolds. Our aim here is to prove the following:

Theorem 4 ([17]). *Let M^n be a connected Riemannian submanifold of a Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. Asumme that there exists a nonconstant positive smooth function $\kappa = \kappa(s)$ satisfying that for every Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ in M^n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in the ambient space $\widetilde{M}^{n+p}$. Then M^n is totally geodesic in $\widetilde{M}^{n+p}$.*

Proof. Let x be an arbitrary point of M^n and u, v any orthonormal pair of vectors in T_xM^n. Without loss of generality we suppose that the function $\kappa = \kappa(s)$ is defined on some open interval $-\epsilon < s < \epsilon$. Let $\gamma = \gamma(s)$ ($|s| < \epsilon$) be a Frenet curve of proper order 2 with curvature κ in M^n satisfying the equations

$$\nabla_{\dot\gamma}\dot\gamma = \kappa(s)Y_s, \quad \nabla_{\dot\gamma}Y_s = -\kappa(s)\dot\gamma \tag{6.1}$$

and the initial condition $\gamma(0) = x$, $\dot\gamma(0) = u$ and $Y_0 = v$. By assumption the curve $f \circ \gamma$ is a Frenet curve of proper order 2 in $\widetilde{M}^{n+p}$. So there exist a positive smooth function $\tilde\kappa = \tilde\kappa(s)$ and a field of unit vectors $\widetilde{Y}_s$ along $f \circ \gamma$ in $\widetilde{M}^{n+p}$ satisfying equation (4.1).

Then by the same discussion as in the proof of Theorem 2 we have the following equations.

$$\tilde\kappa\widetilde{Y}_s = \kappa Y_s + \sigma(\dot\gamma, \dot\gamma), \tag{6.2}$$

$$\tilde\kappa^2 = \kappa^2 + \|\sigma(\dot\gamma, \dot\gamma)\|^2. \tag{6.3}$$

$$\begin{aligned}\widetilde\nabla_{\dot\gamma}(\tilde\kappa\widetilde{Y}_s) &= \dot{\tilde\kappa}\widetilde{Y}_s + \tilde\kappa\widetilde\nabla_{\dot\gamma}\widetilde{Y}_s \\ &= \frac{\dot{\tilde\kappa}}{\tilde\kappa}(\kappa Y_s + \sigma(\dot\gamma, \dot\gamma)) - \tilde\kappa^2\dot\gamma.\end{aligned} \tag{6.4}$$

$$\begin{aligned}&\widetilde\nabla_{\dot\gamma}(\kappa Y_s + \sigma(\dot\gamma, \dot\gamma)) \\ &\qquad = \dot\kappa Y_s + \kappa\widetilde\nabla_{\dot\gamma}Y_s - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma, \dot\gamma)) \\ &\qquad = \dot\kappa Y_s + \kappa(\nabla_{\dot\gamma}Y_s + \sigma(\dot\gamma, Y_s)) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma \\ &\qquad\qquad + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) \\ &\qquad = \dot\kappa Y_s - \kappa^2\dot\gamma + 3\kappa\sigma(\dot\gamma, Y_s) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma).\end{aligned} \tag{6.5}$$

We compare the tangential components and the normal components for the submanifold M^n in (6.4) and (6.5), respectively.

$$\dot{\tilde{\kappa}}\kappa Y_s - \tilde{\kappa}^3\dot{\gamma} = \tilde{\kappa}(\dot{\kappa}Y_s - \kappa^2\dot{\gamma} - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}), \tag{6.6}$$

$$\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}(3\kappa\sigma(\dot{\gamma},Y_s) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})). \tag{6.7}$$

Equation (6.7) gives at $s = 0$

$$\dot{\tilde{\kappa}}(0)\sigma(u,u) = \tilde{\kappa}(0)(3\kappa(0)\sigma(u,v) + (\bar{\nabla}_u\sigma)(u,u)),$$

so that

$$\dot{\tilde{\kappa}}(0)\tilde{\kappa}(0)\sigma(u,u) = \tilde{\kappa}(0)^2(3\kappa(0)\sigma(u,v) + (\bar{\nabla}_u\sigma)(u,u)). \tag{6.8}$$

On the other hand, we see from (6.3) and (3.3) that

$$\tilde{\kappa}^2(0) = \kappa^2(0) + \|\sigma(u,u)\|^2, \tag{6.9}$$

and that

$$\begin{aligned}\dot{\tilde{\kappa}}(0)\tilde{\kappa}(0) =& \dot{\kappa}(0)\kappa(0) + \langle(\bar{\nabla}_u\sigma)(u,u),\sigma(u,u)\rangle \\ &+ 2\kappa(0)\langle\sigma(v,u),\sigma(u,u)\rangle.\end{aligned} \tag{6.10}$$

Substituting (6.9) and (6.10) into (6.8), we obtain

$$\begin{aligned}&3\kappa(0)(\kappa(0)^2 + \|\sigma(u,u)\|^2)\sigma(u,v) - 2\kappa(0)\langle\sigma(u,u),\sigma(u,v)\rangle\sigma(u,u) \\ &\quad = (\dot{\kappa}(0)\kappa(0) + \langle(\bar{\nabla}_u\sigma)(u,u),\sigma(u,u)\rangle)\sigma(u,u) \\ &\qquad - (\kappa(0)^2 + \|\sigma(u,u)\|^2)(\bar{\nabla}_u\sigma)(u,u).\end{aligned} \tag{6.11}$$

Then by the same discussion as in the proof of Theorem 2 we find that

$$\begin{aligned}&-3\kappa(0)(\kappa(0)^2 + \|\sigma(u,u)\|^2)\sigma(u,v) + 2\kappa(0)\langle\sigma(u,u),\sigma(u,v)\rangle\sigma(u,u) \\ &\quad = (\dot{\kappa}(0)\kappa(0) + \langle(\bar{\nabla}_u\sigma)(u,u),\sigma(u,u)\rangle)\sigma(u,u) \\ &\qquad - (\kappa(0)^2 + \|\sigma(u,u)\|^2)(\bar{\nabla}_u\sigma)(u,u).\end{aligned} \tag{6.11'}$$

It follows from (6.11) and (6.11') that

$$3\kappa(0)(\kappa(0)^2 + \|\sigma(u,u)\|^2)\sigma(u,v) - 2\kappa(0)\langle\sigma(u,u),\sigma(u,v)\rangle\sigma(u,u) = 0. \tag{6.12}$$

So we have

$$(3\kappa(0)^2 + \|\sigma(u,u)\|^2)\langle\sigma(u,u),\sigma(u,v)\rangle = 0.$$

Note that $\kappa(0) \neq 0$. Hence

$$\langle \sigma(u,u), \sigma(u,v) \rangle = 0.$$

This, combined with (6.12), shows that $\sigma(u,v) = 0$ holds for each orthonormal pair of vectors u, v in T_xM^n. Since x is an arbitrary point, our immersion f is totally umbilic. Then by taking the inner product of both sides of equation (6.6) with Y_s we see that

$$\dot{\tilde{\kappa}}(s)\kappa(s) = \tilde{\kappa}(s)\dot{\kappa}(s) \quad \text{for each } s. \tag{6.13}$$

Moreover, from (6.7) we obtain

$$\dot{\tilde{\kappa}}(s)\mathfrak{h}_{\gamma(s)} = \tilde{\kappa}(s)(D_{\dot{\gamma}(s)}\mathfrak{h})_{\gamma(s)} \quad \text{for any } s, \tag{6.14}$$

where $\mathfrak{h}$ is the mean curvature vector of our immersion. Therefore equations (6.13) and (6.14) imply

$$(D_{\dot{\gamma}(s)}\mathfrak{h})_{\gamma(s)} = \frac{\dot{\tilde{\kappa}}(s)}{\tilde{\kappa}(s)}\mathfrak{h}_{\gamma(s)} = \frac{\dot{\kappa}(s)}{\kappa(s)}\mathfrak{h}_{\gamma(s)}. \tag{6.15}$$

In particular, at $s = 0$, we get

$$(D_u\mathfrak{h})_x = \frac{\dot{\kappa}(0)}{\kappa(0)}\mathfrak{h}_x.$$

This equation shows that $(D_u\mathfrak{h})_x$ is independent of the choice of a unit vector $u \in T_xM^n$. Changing u into $-u$ we see $(D_u\mathfrak{h})_x = 0$. Since x is an arbitrary point, we have shown that the mean curvature vector $\mathfrak{h}$ of f is parallel.

Finally, by assumption, as the function $\kappa(s)$ is not constant, there exists some $s_0 \in (-\epsilon, \epsilon)$ with $\dot{\kappa}(s_0) \neq 0$. This, together with equation (6.15), yields

$$\mathfrak{h}_{\gamma(s_0)} = \frac{\kappa(s_0)}{\dot{\kappa}(s_0)}(D_{\dot{\gamma}}\mathfrak{h})_{\gamma(s_0)} = 0.$$

Since $\|\mathfrak{h}\|$ is constant on M^n, we have $\mathfrak{h} = 0$ on M^n. Consequently the immersion $f : M^n \to \widetilde{M}^{n+p}$ is totally geodesic. $\square$

The discussion in the proof of Theorem 4 gives the following, which is an improvement of Theorem 2.

Theorem 4'. *Let M^n be a connected Riemannian submanifold of a Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. Then M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$ if and only if there exists a positive smooth function $\kappa = \kappa(s)$ satisfying that for every Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ in M^n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in the ambient space $\widetilde{M}^{n+p}$.*

7. Characterization of totally geodesic Kähler immersions

We consider the extrinsic shape $f \circ \gamma$ of a Kähler Frenet curve γ of a Kähler manifold M in an ambient Kähler manifold $\widetilde{M}$ through a Kähler isometric immersion f. We shall give a characterization of all totally geodesic Kähler immersions from this point of view.

Theorem 5 ([8, 15]). *Let f be a Kähler isometric immersion of a complex n-dimensional Kähler manifold M_n into an arbitrary complex m-dimensional Kähler manifold $\widetilde{M}_m$. Then the following are equivalent.*

- (*i*) *f is a totally geodesic immersion.*
- (*ii*) *There exists a positive smooth function $\kappa = \kappa(s)$ satisfying that f maps every Kähler Frenet curve $\gamma = \gamma(s)$ of curvature κ on M_n to a Frenet curve of order 2 in $\widetilde{M}_m$.*
- (*iii*) *There exists a positive constant k satisfying that f maps every Kähler circle $\gamma = \gamma(s)$ of curvature k on M_n to a Frenet curve of order 2 in $\widetilde{M}_m$.*

In order to prove Theorem 5 we prepare the following lemma:

Lemma 2. *Let M_n be a Kähler manifold with complex structure J which is immersed into an arbitrary Riemannian manifold $\widetilde{M}$ through an isometric immersion f. If for a Kähler Frenet curve $\gamma = \gamma(s)$ of positive curvature $\kappa = \kappa(s)$ on M_n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in $\widetilde{M}$, then following equalities hold*:

$$\pm\kappa\dot{\tilde{\kappa}}J\dot{\gamma} - \tilde{\kappa}^3\dot{\gamma} = \tilde{\kappa}\{\pm\dot{\kappa}J\dot{\gamma} - \kappa^2\dot{\gamma} - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}\}, \tag{7.1}$$

$$\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}\{\pm 3\kappa\sigma(\dot{\gamma}, J\dot{\gamma}) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}. \tag{7.2}$$

In equalities (7.1) *and* (7.2), *we adopt plus sign* (*resp. minus sign*) *for a Kähler Frenet curve satisfying $\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa J\dot{\gamma}$ (resp. $\nabla_{\dot{\gamma}}\dot{\gamma} = -\kappa J\dot{\gamma}$).*

Proof of Lemma 2. Since the curve $f \circ \gamma$ is a Frenet curve of order 2 in $\widetilde{M}$ by assumption, there exist a function $\tilde{\kappa} = \tilde{\kappa}(s)$ and a field of unit vectors $\widetilde{Y}_s$ along $f \circ \gamma$ in $\widetilde{M}$ satisfying (4.1). Then from (2.1) we have

$$\tilde{\kappa}\widetilde{Y}_s = \pm\kappa J\dot{\gamma} + \sigma(\dot{\gamma},\dot{\gamma}). \tag{7.3}$$

The function $\tilde{\kappa}$ is positive because $\kappa > 0$.

For the left-hand side of (7.3), by using (4.1) and (7.3) again, we see

$$\begin{aligned} \tilde{\kappa}\widetilde{\nabla}_{\dot{\gamma}}(\tilde{\kappa}\widetilde{Y}_s) &= \tilde{\kappa}\{\dot{\tilde{\kappa}}\widetilde{Y}_s + \tilde{\kappa}\widetilde{\nabla}_{\dot{\gamma}}\widetilde{Y}_s\} = \tilde{\kappa}\dot{\tilde{\kappa}}\widetilde{Y}_s - \tilde{\kappa}^3\dot{\gamma} \\ &= \dot{\tilde{\kappa}}\{\pm\kappa J\dot{\gamma} + \sigma(\dot{\gamma},\dot{\gamma})\} - \tilde{\kappa}^3\dot{\gamma}. \end{aligned} \tag{7.4}$$

On the other hand, for the right-hand side of (7.3), it follows from (2.1), (2.2) and (2.3) that

(7.5)
$$\begin{aligned}
&\tilde\kappa\widetilde\nabla_{\dot\gamma}\{\pm\kappa J\dot\gamma+\sigma(\dot\gamma,\dot\gamma)\}\\
&\quad=\tilde\kappa\Big\{\pm\dot\kappa J\dot\gamma\pm\kappa\widetilde\nabla_{\dot\gamma}(J\dot\gamma)-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma+D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma))\Big\}\\
&\quad=\tilde\kappa\Big\{\pm\dot\kappa J\dot\gamma\pm\kappa\,(\nabla_{\dot\gamma}(J\dot\gamma)+\sigma(\dot\gamma,J\dot\gamma))-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma+(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)+2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma)\Big\}\\
&\quad=\tilde\kappa\Big\{\pm\dot\kappa J\dot\gamma\pm\kappa\,(J(\pm\kappa J\dot\gamma)+\sigma(\dot\gamma,J\dot\gamma))-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma+(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)\pm2\kappa\sigma(J\dot\gamma,\dot\gamma)\Big\}\\
&\quad=\tilde\kappa\Big\{\pm\dot\kappa J\dot\gamma-\kappa^2\dot\gamma\pm3\kappa\sigma(\dot\gamma,J\dot\gamma)-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma+(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)\Big\}.
\end{aligned}$$

We compare the tangential components and the normal components of (7.4) and (7.5), respectively. Then we get equalities (7.1) and (7.2). □

We shall prove Theorem 5. Needless to say, condition (*i*) implies both conditions (*ii*) and (*iii*). We have only to show that condition (*ii*) implies condition (*i*) for each given positive smooth function $\kappa=\kappa(s)$. Let $v\in TM_n$ be an arbitrary unit vector tangent to M_n. We have Kähler Frenet curves γ_1,γ_2 of cuvature κ on M_n satisfying $\nabla_{\dot\gamma_1}\dot\gamma_1=\kappa J\dot\gamma_1,\ \nabla_{\dot\gamma_2}\dot\gamma_2=-\kappa J\dot\gamma_2$ with condition $\dot\gamma_1(0)=\dot\gamma_2(0)=v$. The curves γ_1,γ_2 satisfy the following equalities by Lemma 2:

$$\pm\kappa\dot{\tilde\kappa}_iJ\dot\gamma_i-\tilde\kappa_i^3\dot\gamma_i=\tilde\kappa_i\{\pm\dot\kappa J\dot\gamma_i-\kappa^2\dot\gamma_i-A_{\sigma(\dot\gamma_i,\dot\gamma_i)}\dot\gamma_i\},\tag{7.6}$$

$$\dot{\tilde\kappa}_i\sigma(\dot\gamma_i,\dot\gamma_i)=\tilde\kappa_i\{\pm3\kappa\sigma(\dot\gamma_i,J\dot\gamma_i)+(\bar\nabla_{\dot\gamma_i}\sigma)(\dot\gamma_i,\dot\gamma_i)\},\tag{7.7}$$

where $\tilde\kappa_i=\sqrt{\kappa^2+\|\sigma(\dot\gamma_i,\dot\gamma_i)\|^2}\ (>0)$. Since the immersion f is Kähler, we see

$$\langle A_{\sigma(\dot\gamma_i,\dot\gamma_i)}\dot\gamma_i,J\dot\gamma_i\rangle=\langle\sigma(\dot\gamma_i,\dot\gamma_i),\sigma(\dot\gamma_i,J\dot\gamma_i)\rangle=\langle\sigma(\dot\gamma_i,\dot\gamma_i),J\sigma(\dot\gamma_i,\dot\gamma_i)\rangle=0.$$

Hence, by taking the inner product of both sides of equation (7.6) with $J\dot\gamma_i$, we find $\kappa\dot{\tilde\kappa}_i=\tilde\kappa_i\dot\kappa$. Then equation (7.7) becomes

$$\pm3\kappa J\sigma(\dot\gamma_i,\dot\gamma_i)+(\bar\nabla_{\dot\gamma_i}\sigma)(\dot\gamma_i,\dot\gamma_i)=\frac{\dot{\tilde\kappa}_i}{\tilde\kappa_i}\sigma(\dot\gamma_i,\dot\gamma_i)=\frac{\dot\kappa}{\kappa}\sigma(\dot\gamma_i,\dot\gamma_i).$$

Evaluating this equation at $s=0$, we get

$$3\kappa(0)J\sigma(v,v)+(\bar\nabla_v\sigma)(v,v)=\frac{\dot\kappa(0)}{\kappa(0)}\sigma(v,v)\qquad\text{for } i=1,$$
$$-3\kappa(0)J\sigma(v,v)+(\bar\nabla_v\sigma)(v,v)=\frac{\dot\kappa(0)}{\kappa(0)}\sigma(v,v)\qquad\text{for } i=2.$$

It follows that $\sigma(v,v)=0$ for an arbitrary unit vector $v\in TM_n$. Therefore f is a totally geodesic immersion.

Remark. In the statement of Theorem 5, we can not omit the condition that κ is positive. For example, we consider the second Veronese embedding $f_2 : \mathbb{C}P^n(c/2) \longrightarrow \mathbb{C}P^{(n^2+3n)/2}(c)$ which is defined by

$$(z_0,\dots,z_n)\mapsto(z_0^2,\sqrt{2}\,z_0z_1,\dots,z_n^2)$$

in homogeneous coordinates. This (non totally geodesic) Kähler isometric embedding maps every geodesic on $\mathbb{C}P^n(c/2)$ to a totally real circle of positive curvature $\sqrt{c}/2$ in the ambient space $\mathbb{C}P^{(n^2+3n)/2}(c)$, that is a circle of curvature $\sqrt{c}/2$ in $\mathbb{R}P^2(c/4)$ which is a totally real totally geodesic submanifold of $\mathbb{C}P^{(n^2+3n)/2}(c)$. The following gives a characterization of the second Veronese embedding:

Proposition ([6, 9]). *Let M_n be a non totally geodesic Kähler submanifold of $\mathbb{C}P^N(c)$ through a full Kähler isometric immersion f. Suppose that for each geodesic γ on M_n, the curve $f\circ\gamma$ is a plane curve in the ambient space $\mathbb{C}P^N(c)$. Then the submanifold M_n is locally congruent to $\mathbb{C}P^n(c/2)$, the isometric immersion f is locally equivalent to the second Veronese embedding f_2 and $N=(n^2+3n)/2$.*

Here, a curve $\gamma=\gamma(s)$ on a Riemannian manifold M is called a *plane curve* if the curve γ is locally contained in some real 2-dimensional totally geodesic submanifold of M. As a matter of course, every plane curve with positive curvature function is a Frenet curve of proper order 2. But in general, the converse does not hold.

8. Characterization of some parallel isometric immersions

We shall provide a characterization of all parallel immersions of a complex space form $M_n(c)$ into a real space form $\widetilde{M}^{2n+p}(\tilde{c})$ by observing the extrinsic shape of a Kähler Frenet curve of $M_n(c)$ in the ambient space $\widetilde{M}^{2n+p}(\tilde{c})$.

Theorem 6 ([8]). *Let f be an isometric immersion of a Kähler manifold M_n into a real space form $\widetilde{M}^{2n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. If there exists a positive smooth function κ satisfying that f maps every Kähler Frenet curve γ of curvature κ on M_n to a plane curve in $\widetilde{M}^{2n+p}(\tilde{c})$, then f is a parallel immersion and locally equivalent to one of the following.*

(1) *f is a totally geodesic immersion of $M_n=\mathbb{C}^n$ into $\widetilde{M}^{2n+p}(\tilde{c})=\mathbb{R}^{2n+p}$, where $\tilde{c}=0$.*
(2) *f is a totally umbilic immersion of $M_n=\mathbb{C}^n$ into $\widetilde{M}^{2n+p}(\tilde{c})=\mathbb{R}H^{2n+p}(\tilde{c})$, where $\tilde{c}<0$.*
(3) *f is a parallel immersion defined by*

$$f=f_1\circ f_2 : M_n=\mathbb{C}P^n(c)\xrightarrow{f_1} S^{n^2+2n-1}((n+1)c/(2n))\xrightarrow{f_2}\widetilde{M}^{2n+p}(\tilde{c}),$$

where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding and $(n+1)c/(2n) \geqq \tilde{c}$.

Proof. Let x be any point of M_n and $v \in T_xM_n$ be any unit tangent vector at $x \in M_n$. Let $\gamma_1 = \gamma_1(s)$ and $\gamma_2 = \gamma_2(s)$ be Kähler Frenet curves of curvature κ on M_n satisfying that $\gamma_1(0) = \gamma_2(0) = x$, $\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = v$ and $\nabla_{\dot{\gamma}_1}\dot{\gamma}_1 = \kappa J\dot{\gamma}_1$, $\nabla_{\dot{\gamma}_2}\dot{\gamma}_2 = -\kappa J\dot{\gamma}_2$. Then, by Lemma 2 equations (7.6) and (7.7) hold. Note that $\tilde{\kappa}_i > 0$. For the sake of brevity, we abbreviate the indices in the following. It follows from (6.3) that

$$\begin{aligned}\tilde{\kappa}\dot{\tilde{\kappa}} &= \frac{1}{2}\frac{d}{ds}\tilde{\kappa}^2 \\ &= \kappa\dot{\kappa} + \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle \\ &= \kappa\dot{\kappa} + \langle(\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle \pm 2\kappa\langle\sigma(J\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle.\end{aligned} \tag{8.1}$$

On the other hand, equation (7.2) implies that

$$\tilde{\kappa}\dot{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \tilde{\kappa}^2\{\pm 3\kappa\sigma(\dot{\gamma},J\dot{\gamma}) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\}. \tag{8.2}$$

Substituting (6.3) and (8.1) into (8.2), at $s = 0$ we obtain

$$\begin{aligned}&\Big\{\kappa(0)\dot{\kappa}(0) + \langle(\bar{\nabla}_v\sigma)(v,v),\sigma(v,v)\rangle \pm 2\kappa(0)\langle\sigma(v,v),\sigma(v,Jv)\rangle\Big\}\sigma(v,v) \\ &= \Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\Big\{\pm 3\kappa(0)\sigma(v,Jv) + (\bar{\nabla}_v\sigma)(v,v)\Big\}.\end{aligned} \tag{8.3}$$

Then the same discussion as in the proof of Theorem 5 shows

$$2\kappa(0)\langle\sigma(v,v),\sigma(v,Jv)\rangle\sigma(v,v) = 3\kappa(0)\Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\sigma(v,Jv),$$

so that

$$2\langle\sigma(v,v),\sigma(v,Jv)\rangle\sigma(v,v) = 3\Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\sigma(v,Jv). \tag{8.4}$$

Taking the inner product of both sides of this with $\sigma(v,v)$, we get

$$2\langle\sigma(v,v),\sigma(v,Jv)\rangle\|\sigma(v,v)\|^2 = 3\Big\{\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\langle\sigma(v,v),\sigma(v,Jv)\rangle$$

hence

$$\Big\{3\kappa(0)^2 + \|\sigma(v,v)\|^2\Big\}\langle\sigma(v,v),\sigma(v,Jv)\rangle = 0.$$

Consequently $\langle\sigma(v,v),\sigma(v,Jv)\rangle = 0$, because $3\kappa(0)^2 + \|\sigma(v,v)\|^2 > 0$. Again from (8.4) we find

$$\sigma(v,Jv) = 0 \quad \text{for any } x \in M_n \text{ and any } v \in T_xM_n. \tag{8.5}$$

Replacing v by $Jv+u$, we have $\sigma(u,v)=\sigma(Ju,Jv)$ for each $u,v\in T_xM_n$. This, together with Codazzi's equation in a space of constant curvature, implies that the immersion f is parallel: $\bar\nabla\sigma=0$ (see [3]). Next, owing to (8.5), we see $\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma, J\dot\gamma\rangle=\langle\sigma(\dot\gamma,\dot\gamma),\sigma(\dot\gamma,J\dot\gamma)\rangle=0$. Therefore, taking the inner product of both sides of (7.1) with $J\dot\gamma$, we have

$$\kappa\dot{\tilde\kappa}=\tilde\kappa\dot\kappa, \tag{8.6}$$

so that equation (7.1) reduces to

$$A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma=(\tilde\kappa^2-\kappa^2)\dot\gamma,$$

which implies that

$$\langle\sigma(v,v),\sigma(v,u)\rangle=\langle A_{\sigma(v,v)}v,u\rangle=0$$

for any orthonormal pair of vectors $u,v\in T_xM_n$ at each point $x\in M_n$. Thus, by virtue of Lemma 1, the immersion f is isotropic. Hence our submanifold M_n is a complex space form immersed as an isotropic submanifold with parallel second fundamental form into the ambient space $\widetilde{M}^{2n+p}(\tilde c)$, so that the immersion f is rigid (see Proposition 1 in [18]). Therefore we can see that the submanifold (M_n,f) is locally congruent to one of (1), (2) and (3) (cf. [2, 16]).

Now we shall show that the examples (1), (2) and (3) satisfy the hypothesis of our theorem. For each positive smooth function κ the totally geodesic case (1) satisfies our hypothesis. Next, for the other cases (2) and (3) we take the function κ as a positive constant, say k. In the case of (2), for each circle γ of curvature $k(>0)$ on M_n the curve $f\circ\gamma$ is a circle of curvature $\sqrt{k^2-\tilde c}$ (see page 169 in [11]), so that it is a plane curve in the ambient space $\widetilde{M}^{2n+p}(\tilde c)$. Finally we explain the case (3) in detail. The isometric immersion f given by (3) is $(\lambda=)\sqrt{c-\tilde c}$-isotropic and the parallel second fundamental form σ of f satisfies $\sigma(JX,JY)=\sigma(X,Y)$ for each vector $X,Y\in T(\mathbb{C}P^n(c))$. Let $\gamma=\gamma(s)$ be a Kähler circle of curvature $k(>0)$ on $\mathbb{C}P^n(c)$. Then the curve $f\circ\gamma$ satisfies $\widetilde\nabla_{\dot\gamma}\dot\gamma=\pm kJ\dot\gamma+\sigma(\dot\gamma,\dot\gamma)$. Note that

$$\|\widetilde\nabla_{\dot\gamma}\dot\gamma\|=\sqrt{k^2+\lambda^2}=\sqrt{k^2+c-\tilde c}.$$

Here we put

$$\widetilde{Y}_s=\frac{\pm kJ\dot\gamma+\sigma(\dot\gamma,\dot\gamma)}{\sqrt{k^2+c-\tilde c}}.$$

So we have

$$
\begin{aligned}
\widetilde{\nabla}_{\dot\gamma}\widetilde{Y}_s &= \frac{1}{\sqrt{k^2+c-\tilde c}}\widetilde{\nabla}_{\dot\gamma}\Big(\pm kJ\dot\gamma + \sigma(\dot\gamma,\dot\gamma)\Big)\\
&= \frac{1}{\sqrt{k^2+c-\tilde c}}\Big(\pm k\widetilde{\nabla}_{\dot\gamma}(J\dot\gamma) + \widetilde{\nabla}_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma))\Big)\\
&= \frac{1}{\sqrt{k^2+c-\tilde c}}\Big(\pm k\nabla_{\dot\gamma}(J\dot\gamma) \pm k\cdot\sigma(\dot\gamma,J\dot\gamma) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma))\Big)\\
&= \frac{1}{\sqrt{k^2+c-\tilde c}}\Big(-k^2\dot\gamma - \lambda^2\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) \pm 2k\cdot\sigma(J\dot\gamma,\dot\gamma)\Big)\\
&= -\sqrt{k^2+c-\tilde c}\ \dot\gamma.
\end{aligned}
$$

Therefore we can see that for each Kähler circle γ of curvature $k(>0)$ on $\mathbb{C}P^n(c)$ the curve $f\circ\gamma$ is a circle (of curvature $\sqrt{k^2+c-\tilde c}$), so that it is a plane curve in $\widetilde{M}^{2n+p}(\tilde c)$. □

Remark. Theorem 6 also holds under the condition $\kappa\equiv 0$ (see [14]).

As a consequence of Theorem 6 we establish the following corollary ([8]):

Corollary. *Let f be an isometric immersion of a Kähler manifold M_n into a real space form $\widetilde{M}^{2n+p}(\tilde c)$ of constant sectional curvature $\tilde c$. If there exists a non constant positive smooth function κ satisfying that f maps every Kähler Frenet curve γ of curvature κ on M_n to a plane curve in $\widetilde{M}^{2n+p}(\tilde c)$, then f is a totally geodesic immersion of $M_n=\mathbb{C}^n$ into $\widetilde{M}^{2n+p}(\tilde c)=\mathbb{R}^{2n+p}$, where $\tilde c=0$.*

Proof. By assumption, as the curvature function κ is not constant, there exists some s_0 with $\dot\kappa(s_0)\neq 0$. Since $\kappa,\tilde\kappa>0$, we find $\dot{\tilde\kappa}(s_0)\neq 0$ from (8.6). In the following, we will use the fact that $\bar\nabla\sigma=0$ and equation (8.5). Then equation (7.2) yields $\sigma(\dot\gamma(s_0),\dot\gamma(s_0))=0$. Now, we can see that $\|\sigma(\dot\gamma,\dot\gamma)\|$ is constant along the curve γ. In fact, we have

$$\frac{d}{ds}\|\sigma(\dot\gamma,\dot\gamma)\|^2 = 2\langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle \pm 4\kappa\langle\sigma(J\dot\gamma,\dot\gamma),\sigma(\dot\gamma,\dot\gamma)\rangle = 0.$$

Hence we conclude $\sigma(v,v)=0$ for an arbitrary unit vector $v\in T_xM_n$ at each point $x\in M_n$. Thus $f: M_n\to\widetilde{M}^{2n+p}(\tilde c)$ is a totally geodesic immersion. □

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