A Structure Theory
of Freudenthal-Kantor Triple Systems III

Dedicated to Professor Nathan Jacobson on his 80th birthday

By

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In this paper, we give a construction of balanced Freudenthal-Kantor triple systems
and investigate a structure of the Jordan triple systems associated with reduced balanced
Freudenthal-Kantor triple systems.

Introduction

The triple systems studied here are a specialization of the class of Freudenthal-Kantor
triple systems given in [21, 22, 13], which is called balanced by ourselves. This triple system is a variation of Freudenthal triple systems [7, 18],
symplectic ternary algebras [6] and symplectic triple systems [23]. This paper is a
continuation of the previous articles [13, 14]. The main purpose of this article is to
give followings:
(i) A construction of Jordan triple systems from a vector space equipped with
relations of a cross product and a bilinear form.
(ii) A construction of balanced Freudenthal-Kantor triple systems from a class of
vector matrices as follows:
\[
\begin{bmatrix}
\alpha & a \\
b & \beta
\end{bmatrix}, \quad \alpha, \beta \in \Phi, \quad a, b \in V
\]
where \( \Phi \) is a base field, \( V \) is the Jordan triple system defined by (i).
(iii) If a simple balanced Freudenthal-Kantor triple system \( \mathcal{M} \) is reduced, then
\( \mathcal{M} \cong \mathcal{M}(V) \), where \( \mathcal{M}(V) \) is the balanced Freudenthal-Kantor triple system defined
by (ii).

We shall be concerned with algebras and triple systems which are finite
dimensional over a field \( \Phi \) of characteristic different from 2 or 3, unless otherwise
specified. We shall mainly employ the notation and terminology in [13, 14].
In this section, we shall give a construction of Jordan triple systems and consider the norm similarity.

**Theorem 1.1.** Let $V$ be a vector space over an arbitrary field $\mathcal{F}$ equipped with a bilinear form $B(a, b)$ and a cross product $a \times b$ satisfying the following conditions:

1. $a \times b = b \times a$
2. $B(a, b) = B(b, a)$
3. $B(a, b \times d) = B(a \times b, d)$
4. $((a \times b) \times e) \times d + ((b \times d) \times e) \times a + ((d \times a) \times e) \times b$

$$= B(a \times b, d)e + B(a, e)b \times d + B(b, e)d \times a + B(d, e)a \times b$$

for all $a, b, d, e \in V$.

Then $V$ becomes a Jordan triple system with respect to the triple product

$$\{xyz\} = B(x, y)z + B(z, y)x - (x \times z) \times y.$$

**Proof.** By the definition of the triple product, it is clear that

$$\{xyz\} = \{zyx\}.$$

We compute as follows;

$$\{uv\{xyz\}\} - \{\{uvw\}yz\} + \{x\{uvy\}z\} - \{xy\{uvw\}\}$$

$$= B(u, v)(B(x, y)z + B(z, y)x - (x \times z) \times y)$$
$$+ B(B(x, y)z + B(z, y)x - (x \times z) \times y, v)u$$
$$- (u \times (B(x, y)z + B(z, y)x - (x \times z) \times y)) \times v$$
$$- B(B(u, v)x + B(x, v)u - (u \times x) \times v, y)z$$
$$- B(z, y)(B(u, v)x + B(x, v)u - (u \times x) \times v)$$
$$+ ((B(u, v)x + B(x, v)u - (u \times x) \times v) \times z) \times y$$
$$+ B(x, B(v, u)y + B(y, u)v - (v \times y) \times u)z$$
$$+ B(z, B(v, u)y + B(y, u)v - (v \times y) \times u)x$$
$$- (x \times z) \times (B(v, u)y + B(y, u)v - (v \times y) \times u)$$
$$- B(B(u, v)x + B(x, v)u - (u \times z) \times v)$$
$$- B(B(u, v)x + B(x, v)u - (u \times z) \times v, y)x$$
$$+ (x \times (B(u, v)x + B(z, v)u - (u \times z) \times v)) \times y$$

$$= - B((x \times z) \times y, v)u + (u \times ((x \times z) \times y)) \times v$$
$$+ B(x, v)(u \times z) \times y - (x \times z) \times B(y, u)v$$
$$- ((u \times x) \times v) \times z \times y + (x \times z) \times ((v \times y) \times u)$$
$$+ B(z, v)(x \times u) \times y - (x \times ((u \times z) \times v)) \times y$$
Freudenthal-Kantor Triple Systems

\begin{align*}
= (B(x, v)(u \times z) - ((u \times x) \times v) \times z \\
+ B(z, v)(x \times u) - x \times ((u \times z) \times v)) \times y
\end{align*}

(by the relation (4) of the assumption, that is, \(B((x \times z) \times y, v)u - (u \times ((x \times z) \times y)) \times v + (x \times z) \times B(y, u)v - (x \times z) \times ((v \times y) \times u) = ((v \times (x \times z)) \times u) \times y - B(x \times z, u)v \times y - B(v, u)y \times (x \times u))

= 0.

(by the relation (4))

This completes the proof.

If \(N\) is a cubic form on a vector space \(V\) and \(c \in V\) a basepoint where \(N(c) = 1\), then we can form the trace form

\[ T(x, y) = - \partial_x \partial_y \log N|_c = (\partial_x N|_c)(\partial_y N|_c) - \partial_x \partial_y N|_c \]

of \(N\) at \(c\). We say \(N\) is nondegenerate at \(c\) if its trace form is nondegenerate. For nondegenerate forms we have a unique quadratic mapping \(x \mapsto x^g\) in \(V\) defined by \(T(x^g, y) = \partial_y N|_x\). We say a nondegenerate cubic form \(N\) and basepoint \(c\) are admissible if the adjoint identity \(x^{gg} = N(x)x\) holds under all scalar extensions (see [17]). We denote this vector space \(V\) by \(\mathfrak{J}(N, c)\). For \(ch \neq 2, 3\), to apply the case of our construction, we put \(2x^g = x \times x\), \(T(x, y) = B(x, y)\) and \(N(x) = 1/3 T(x^g, x)\). We can easily show that if \(N(x)x = x^{gg}\), then \(4/3 B(x \times x, x) x = (x \times x) \times (x \times x)\). Also these identities yield the relation \(x \times (x^g \times y) = N(x) y + T(x, y)x^g\) (by the argument of density of \(V\)). Hence this result implies that

\[ ((x \times x) \times y) \times x = 1/3 B(x \times x, x) y + B(x, y)x \times x, \]

which reduce the relation (4) of the assumption in Theorem 1. Thus we obtain the following corollary.

Corollary [17]. If the cubic form \(N\) and basepoint \(c\) are admissible then \(\mathfrak{J}(N, c)\) is a Jordan triple system with respect to the triple product

\[ \{xyz\} = T(x, y)z + T(z, y)x - (x \times z) \times y. \]

Theorem 1.2. Let \(V\) be a vector space over a field \(\Phi\) of characteristic \(\neq 2\) or 3 equipped with a bilinear form \(B(a, b)\) and a cross product \(a \times b\) satisfying the relations (1) \sim (4) of Theorem 1.1 and the following conditions;

\[ (a \times b) \times (e \times d) + (b \times e) \times (d \times a) + (e \times a) \times (b \times d) = B(a \times b, e)d + B(a \times e, d)b + B(a \times d, b)e + B(b \times e, d)a, \]

(6) there exists an element \(c \in V\) such that

\[ x \times c = B(x, c)c - x \text{ for all } x \in V. \]

Then it holds
\[ x^3 - T(x)x^2 + S(x)x - N(x)c = 0 \]

and \( x \times x = 2x^2 - 2T(x)x + 2S(x)c \) for all \( x \in V \),

where \( x^3 = 1/2\{xxx\}, \ x^2 = 1/2\{xcx\}, \ T(x) = B(x, c), \ S(x) = 1/2B(x \times x, c) \) and \( N(x) = 1/6B(x \times x, x) \).

**Proof.** From \( x^3 = 1/2\{xxx\} \) and \( x^2 = 1/2\{xcx\} \), we have

\[
\begin{align*}
    x^3 - B(x, c)x^2 &= 1/2\{xxx\} - 1/2B(x, c)\{xcx\} \\
    &= B(x, x)x - 1/2(x \times x) \times x - 1/2B(x, c)(2B(x, c)x - (x \times x) \times c) \\
    &= (B(x, x) - B(x, c)^2)x - 1/2(x - B(x, c)c) \times (x \times x). \quad (1-1)
\end{align*}
\]

On the other hand, we have

\[ B(x \times y, c) = B(x, y \times c) \]

\[ = B(y, c)B(x, c) - B(x, y) \quad \text{(by the relation (6) of the assumption)}. \]

If we put \( y = x \), then this implies that

\[ B(x \times x, c) = B(x, c)^2 - B(x, x). \]

Combining this with the identity (1-1), we get

\[
\begin{align*}
    x^3 - B(x, c)x^2 &= -B(x \times x, c)x + (x \times c) \times (x \times x). \quad (1-2)
\end{align*}
\]

By the relation (5) of the assumption, we have

\[
\begin{align*}
    (x \times c) \times (x \times x) &= 1/3B(x \times x, x)c + B(x \times c, x)x. \quad (1-3)
\end{align*}
\]

From (1-2) and (1-3), it follows that

\[ x^3 - B(x, c)x^2 + 1/2B(x \times x, c)x - 1/6B(x \times x, x)c = 0. \]

Hence this yields that

\[ x^3 - T(x)x^2 + S(x)x - N(x)c = 0, \]

where \( T(x) = B(x, c), \ S(x) = 1/2B(x \times x, c) \) and \( N(x) = 1/6B(x \times x, x) \). Also, it follows from \( x \times c = B(x, c)c - x \times x \) that

\[ (x \times x) \times c = B(x \times x, c)c - x \times x. \]

From this identity and the identity \( x^2 - T(x)x = -1/2(x \times x) \times c \), we obtain

\[ x^2 - T(x)x = -1/2(B(x \times x, c)c - x \times x), \]

which implies \( x \times x = 2x^2 - 2T(x)x + 2S(x)c \). This completes the proof. \( \blacksquare \)

**Theorem 1.3.** Let \( V \) (resp. \( V' \)) be a vector space over an infinite field \( \Phi \) of characteristic \( \neq 2 \) or 3 equipped with a nondegenerate bilinear form \( B(a, b) \) (resp. \( B(a, b') \)) and a cross product \( a \times b \) (resp. \( a' \times b' \)) satisfying the relations (1), (2),
(3) and (5) of Theorem 1.2 (resp. (1)', (2)', (3)' and (5)'). If a mapping $g$ is invertible (linear and bijective) from $V$ onto $V'$, then the followings are equivalent:

(i) $B(ga \times ga, ga') = \lambda B(a \times a, a)$, $\lambda \in \Phi^*$, for all $a \in V$

(ii) $g$ is an isotopy of the Jordan triple system with respect to the triple product

$$\{xyz\} = B(x, y)z + B(z, y)x - (x \times z) \times y.$$ 

Furthermore, in the case of (ii), we have

$$g(x \times y) = \lambda \hat{g}x \times \hat{g}y, \quad \hat{g}(a \times b) = \lambda^{-1} ga \times gb$$

and $B(\hat{g}a, gb)' = B(a, b)$, where $B(g^* a', b) = B(a', gb)'$ and $\hat{g} = g^{-1}$. 

**Proof.** (i) $\Rightarrow$ (ii) If $g$ is a bijective linear mapping, one may define a bijective linear mapping $g^*$ of $V'$ onto $V$ by

$$B(g^* a', b) = B(a', gb').$$

Hence we have

$$B(ga \times ga, ga') = B(g^* (ga \times ga), a). \quad (1-4)$$

From the assumption that $B(ga \times ga, ga') = \lambda B(a \times a, a)$ and $B(.)$ is nondegenerate, we get

$$g^* (ga \times ga) = \lambda a \times a$$

and so $\hat{g}(a \times a) = \lambda^{-1} ga \times ga$, where $\hat{g} = g^{-1}$. \quad (1-5)

Using (1-5), we obtain

$$(ga \times ga) \times (ga \times ga) = \lambda^2 \hat{g}(a \times a) \times \hat{g}(a \times a). \quad (1-6)$$

By relation (5)' of the assumption, we have

$$4B(ga \times ga, ga') ga = 3 (ga \times ga) \times (ga \times ga).$$

The left-hand side of equation (1-6) is equal to

$$4/3 B(ga \times ga, ga') ga$$
$$= 4/3 \lambda B(a \times a, a) ga.$$ 

Consequently, we get

$$4/3 B(a \times a, a) ga = \lambda \hat{g}(a \times a) \times \hat{g}(a \times a).$$

Replacing $a$ by $a \times a$, we have

$$4/3 B((a \times a) \times (a \times a), a \times a) g(a \times a)$$
$$= \lambda \hat{g}((a \times a) \times (a \times a)) \times \hat{g}((a \times a) \times (a \times a)).$$

Using the relation $(a \times a) \times (a \times a) = 4/3 B(a \times a, a)a$, we get
\[(4/3)^2 \left( B(a \times a, a) \right)^2 g(a \times a) = \lambda (4/3 \left( B(a \times a, a) \right)^2 \hat{g}a \times \hat{g}a. \]

By using a density argument, that is, \(B(a \times a, a) \neq 0\) for all \(a \neq 0\) in \(V\), we obtain
\[g(a \times a) = \lambda \hat{g}a \times \hat{g}a.\]

In \(B(g^* a', b) = B(a', gb)'\), putting \(a = g^* a'\), we have
\[B(a, b) = B(\hat{g}a, gb)'.\]

From the definition of the triple product
\[\{xyz\} = B(x, y)z + B(z, y)x - (x \times z) \times y,\]
we can see that
\[g\{xyz\} = B(x, y)gz + B(z, y)gx - g((x \times z) \times y)\]
\[= B(gx, \hat{g}y)gz + B(gz, \hat{g}y)gx - \lambda(\hat{g}(x \times z) \times \hat{g}y)\]
\[= B(gx, \hat{g}y)gz + B(gz, \hat{g}y)gx - (gx \times gz) \times \hat{g}y\]
\[= \{gx\hat{g}ygz\}'.\]

Similarly we have
\[\hat{g}\{xyz\} = \{\hat{g}xgygz\}'.\]

(ii) \(\Rightarrow\) (i). Let \(g\) be an isotopy satisfying \(\hat{g}(x \times y) = \lambda^{-1}(gx \times gy)\) and \(g(x \times y) = \lambda \hat{g}x \times \hat{g}y\). From \(g\{xyz\} = \{gx\hat{g}ygz\}'\) and the definition of the triple product, we have
\[B(x, y)gz + B(z, y)gx - (gx \times gz) \times \hat{g}y (1-7)\]
\[= B(gx, \hat{g}y)gz + B(gz, \hat{g}y)gx - (gx \times gz) \times \hat{g}y.\]

Putting \(x = z\) in the identity (1–7), we get
\[B(x, y) = B(gx, \hat{g}y)'.\]

Replacing \(y\) by \(x \times x\) in the equation (1–8), we have
\[B(x \times x, x) = \lambda^{-1}B(gx \times gx, gx)'\]
(by \(\hat{g}(x \times x) = \lambda^{-1}(gx \times gx)\)).

This completes the proof. \(\blacksquare\)

Theorem 1.3 can be regarded as a generalization of the following proposition for a Jordan triple system.

**Proposition 1.4.** [12] Let \(V\) and \(V'\) be reduced simple exceptional Jordan algebras. Then the following conditions are equivalent:

1. \(V\) and \(V'\) are isotopic,
2. \(V\) and \(V'\) are norm similar.
If $A$ is a linear mapping of a vector space $V$ equipped with a bilinear form $B(x, y)$ and a cross product $x \times y$ into itself satisfying

$$B(Ax, x \times x) = \rho B(x \times x, x) \quad \text{for all } x \in V \quad (1-9)$$

where $\rho \in \Phi^*$ is fixed and satisfying (1–9) for all field extensions of $\Phi$, then $A$ is said to be a Lie similarity of $V$. Then we have the following;

**Theorem 1.5.** Let $V$ be as in Theorem 1.3. If $A$ is a linear mapping of $V$ into itself, then the followings are equivalent;

(i) $A$ is a Lie similarity of $V$.
(ii) There exists a linear mapping $A^*$ of $V$ into itself satisfying

$$A\{xyz\} = \{Axyz\} + \{xA^*yz\} + \{xyAz\}$$

where $A^*$ is the linear mapping of $V$ into itself defined by $B(A^*x, y) = -B(x, Ay)$.

**Remark.** The above theorem implies that the notion of structure algebra of the Jordan triple system $V$ coincides that of Lie similarity. (for the definition of structure algebra, see [13]). In particular, if the cross product is zero, then an arbitrary linear mapping $A$ of $V$ is a Lie similarity, hence if $V$ has a nondegenerate bilinear form, the mapping $A$ is a structure algebra of $V$.

In this section, we shall study a construction of the prototype of a balanced Freudenthal-Kantor triple system with $\varepsilon = 1$.

For $\varepsilon = \pm 1$, a triple system $U(\varepsilon)$ with the triple product $\langle - , - , - \rangle$ is called a Freudenthal-Kantor triple system if

(U1) $[L(a, b), L(c, d)] = L(\langle abc \rangle, d) + \varepsilon L(c, \langle bad \rangle)$ \hspace{1cm} (2–1)

(U2) $K(K(a, b)c, d) - L(d, c)K(a, b) + \varepsilon K(a, b)K(c, d) = 0$, \hspace{1cm} (2–2)

where $L(a, b)c = \langle abc \rangle$ and $K(a, b)c = \langle acb \rangle - \langle bca \rangle$.

**Definition.** A Freudenthal-Kantor triple system is balanced if there exists an anti-symmetric bilinear form $\langle , \rangle$ such that $K(x, y) = \langle x, y \rangle \text{ Id, } \langle x, y \rangle \in \Phi^*$.

**Remark.** From results in [14], we note the following:

(i) The case of $\varepsilon = -1$ does not occur in a balanced Freudenthal-Kantor triple system.

(ii) A balanced Freudenthal-Kantor triple system is simple if and only if the anti-symmetric bilinear form $\langle , \rangle$ is nondegenerate.

(iii) The derivation of semisimple Freudenthal-Kantor triple systems over a field of characteristic 0 is a finite sum of inner derivations of $L(a, b) + \varepsilon L(b, a)$ (denoted by $S(a, b)$).
Let $V$ be a vector space over an arbitrary field $\Phi$ equipped with a bilinear form $B(a, b)$ and a cross product $a \times b$ satisfying the following conditions:

(1) $a \times b = b \times a$

(2) $B(a, b) = B(b, a)$

(3) $B(a, b \times d) = B(a \times b, d)$

(4) $((a \times b) \times e) \times d + ((b \times d) \times e) \times a + ((d \times a) \times e) \times b$

$= B(a \times b, d)e + B(a, e)b \times d + B(b, e)d \times a + B(d, e)a \times b$

(5) $(a \times b) \times (e \times d) + (b \times e) \times (d \times a) + (e \times a) \times (b \times d)$

$= B(a \times b, e)d + B(a \times e, d)b + B(a \times d, b)e + B(b \times e, d)a$

for all $a, b, d, e \in V$

In particular, for $\Phi \neq 2, 3$, $3(a \times a) \times (a \times a) = 4B(a, a \times a)a$ holds under two conditions that "$a \times b = 0 \Rightarrow a = 0$ or $b = 0"$ (division property) and $(a \times a) \times b \times a = 1/3B(a \times a, a)b + B(a, b)a \times a$.

**EXAMPLE.** \(\mathcal{Z}(N, c)\) satisfies the conditions (1) ~ (5).

We can consider the set of vector matrices with coefficients in the vector space $V$ as follows:

$$\mathfrak{M}(V) = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta \in \Phi, a, b \in V \right\}.$$ 

In \(\mathfrak{M}(V)\), we shall introduce an operation $\circ$, that is,

$$\begin{pmatrix} \alpha_1 & a_1 \\ b_1 & \beta_1 \end{pmatrix} \circ \begin{pmatrix} \alpha_2 & a_2 \\ b_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \alpha_2 + B(a_1, b_2) & \alpha_1 a_2 + \beta_2 a_1 + b_1 \times b_2 \\ \alpha_2 b_1 + \beta_1 b_2 + a_1 \times a_2 & \beta_2 \beta_1 + B(a_2, b_1) \end{pmatrix}.$$ 

Next we shall use the following mapping to consider a triple product

$$P: \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \rightarrow \begin{pmatrix} -\alpha & a \\ -b & \beta \end{pmatrix}$$

and

$$- : \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \beta & a \\ b & \alpha \end{pmatrix}.$$ 

Thereby we can define a triple product on \(\mathfrak{M}(V)\) as follows:

$$\langle x_1 x_2 x_3 \rangle = x_1 \circ (\overline{P} x_2 \circ x_3) + x_3 \circ (\overline{P} x_2 \circ x_1) - P x_2 \circ (\overline{x_1} \circ x_3)$$

(2-3)

where $x_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \mathfrak{M}(V)$. 

We have the following result on this vector matrix $\mathcal{M}(V)$.

**Theorem 2.1.** Let $\mathcal{M}(V)$ be the set of vector matrices of the above. Then $(\mathcal{M}(V), \langle - , - , - \rangle)$ is a balanced Freudenthal-Kantor triple system with respect to the above triple product $(2-3)$.

**Proof.** From the assumptions (1), (2), (3), (4) and (5) of vector space $V$, we can obtain this theorem by straightforward but very long calculations and we omit it. 

We call $\mathcal{M}(V)$ the balanced Freudenthal-Kantor triple system induced from the Jordan triple system $V$ satisfying the conditions (1) – (5).

**Remark.** For $\text{ch } \Phi \neq 2$, we note that

$$p_{\text{loc}2} - o_{\text{loc}2} + B(a_1, b_2) - B(b_1, a_2) \quad (2\text{-}4)$$

and

$$y(x_1, x_2) = \frac{1}{4}[\text{tr}^2(R(x_1, x_2) - R(x_2, x_1)) + L(x_1, x_2) - L(x_2, x_1)] \quad (2\text{-}5)$$

where $y(x_1, x_2) = 1/2[\text{tr}^2(R(x_1, x_2) - R(x_2, x_1)) + L(x_1, x_2) - L(x_2, x_1)]$.

**Definition [7, 18].** A Freudenthal triple system is a vector space $\mathcal{M}$ with trilinear product $(x, y, z) \rightarrow [xyz]$ and anti-symmetric bilinear form $(x, y) \rightarrow \langle x, y \rangle_F$ such that

(A1) $[xyz]$ is symmetric in all arguments;

(A2) $q_F(x, y, z, w) = \langle x, [yzw] \rangle_F$ is a nonzero symmetric 4-linear form;

(A3) $[[xxx]xy] = \langle y, x \rangle_F [xxx] + \langle y, [xxx] \rangle_F x$

for $x, y, z, w \in \mathcal{M}$.

**Proposition 2.2.** If $(\mathcal{M}, \langle - , - , - \rangle)$ is a balanced Freudenthal-Kantor triple system equipped with $K(x, y) = \langle x, y \rangle \text{Id}$ over a field $\text{ch } \Phi \neq 2$, then $(\mathcal{M}, [ - , - , - ])$ is a Freudenthal triple system satisfying $\langle x, y \rangle_F = 1/2 \langle x, y \rangle$ with respect to the triple product

$$[xyz] = 1/2(\langle xyz \rangle + \langle yxz \rangle).$$

**Proof.** (i) By the balanced condition, we have

$$\langle xyz \rangle - \langle yxz \rangle = -\langle xzy \rangle + \langle yzx \rangle.$$ 

Hence we have

$$[xyz] = 1/2(\langle xzy \rangle + \langle yzx \rangle)$$

$$= 1/2(\langle yzx \rangle + \langle yxz \rangle)$$

$$= [yzx].$$

From the definition of triple product, we have
\[ [xyz] = [xzy]. \]

(ii) Since \( L(x, y) - L(y, x) = \langle y, x \rangle \text{Id} \), we have
\[
[L(x, y) - L(y, x), L(z, w)] = 0.
\]

Similarly, \([L(y, x), L(z, w) - L(w, z)] = 0\) holds. Hence we get \([L(x, y), L(z, w)] = [L(y, x), L(w, z)]\). From (U1) with \( \varepsilon = 1 \), it follows that
\[
L(\langle xyz \rangle, w) + L(z, \langle yxw \rangle) - L(\langle yxw \rangle, z) - L(w, \langle xyz \rangle) = 0.
\]

Therefore we obtain
\[
\langle \langle xyz \rangle, w \rangle + \langle z, \langle yxw \rangle \rangle = 0.
\]

Similarly, \(\langle \langle xyz \rangle, w \rangle + \langle x, \langle wzy \rangle \rangle = 0\) holds.

On the other hand, we have
\[
\langle z, [ywz] \rangle_F = 1/4(\langle z, \langle ywz \rangle \rangle + \langle z, \langle yxz \rangle \rangle)
\]
where \(\langle a, b \rangle_F = 1/2\langle a, b \rangle\) (i.e., \(\langle \cdot, \cdot \rangle_F\): the anti-symmetric bilinear form induced from an anti-symmetric form \(\langle \cdot, \cdot \rangle\) of balanced Freudenthal-Kantor triple system). Combining this with (2-7), we get
\[
\langle z, \langle ywx \rangle \rangle_F = \langle y, \langle zwz \rangle \rangle_F.
\]

(iii) From (U1) with \( \varepsilon = 1 \), we have
\[
\langle x \langle xxx \rangle y \rangle = - \langle \langle xxx \rangle xy \rangle.
\]

Putting \( z = x \) in \( K(y, z)x = \langle zyx \rangle - \langle yzx \rangle \), we have
\[
2\langle yxx \rangle = \langle xyx \rangle + \langle xxy \rangle.
\]

Linearizing this relation, we get
\[
\langle yxz \rangle + \langle yzx \rangle = 1/2(\langle yxz \rangle + \langle xzy \rangle + \langle zyx \rangle + \langle zxy \rangle).
\]

Replacing \( z = \langle xxx \rangle \), we have
\[
\langle y \langle xxx \rangle x \rangle + \langle y \langle xxx \rangle y \rangle
\]
\[
= 1/2(\langle xy \langle xxx \rangle \rangle + \langle x \langle xxx \rangle y \rangle + \langle xxx \rangle yx + \langle xxx \rangle xy).\]

Combining this with (2-9), we have
\[
\langle y \langle xxx \rangle x \rangle + \langle y \langle xxx \rangle y \rangle
\]
\[
= 1/2(\langle xy \langle xxx \rangle \rangle + \langle xxx \rangle yx). \quad (2-12)
\]

From \( K(x, y) \langle xxx \rangle = - L(x, y) \langle xxx \rangle + L(y, x) \langle xxx \rangle \), we have
Freudenthal-Kantor Triple Systems

\[ \langle xy \langle xxx \rangle \rangle - \langle yx \langle xxx \rangle \rangle = - \langle x, y \rangle \langle xxx \rangle. \]  
(2-13)

From \( L(\langle xxx \rangle, y)x - L(y, \langle xxx \rangle)x = - K(\langle xxx \rangle, y)x \), we have

\[ \langle \langle xxx \rangle yx \rangle - \langle y \langle xxx \rangle x \rangle = - \langle \langle xxx \rangle, y \rangle x. \]  
(2-14)

Therefore by (2-13) and (2-14), we have

\[ \langle xy \langle xxx \rangle \rangle + \langle \langle xxx \rangle yx \rangle - \langle yx \langle xxx \rangle \rangle - \langle y \langle xxx \rangle x \rangle \]
\[ = - \langle x, y \rangle \langle xxx \rangle - \langle \langle xxx \rangle, y \rangle x. \]  
(2-15)

From (2-12) and (2-15), we obtain

\[ \langle yx \langle xxx \rangle \rangle + \langle y \langle xxx \rangle x \rangle \]
\[ = - \langle x, y \rangle \langle xxx \rangle - \langle \langle xxx \rangle, y \rangle x. \]

Consequently, by means of \[ [xyz] = 1/2(\langle xyz \rangle + \langle xyz \rangle) \] and \[ \langle x, y \rangle^F = 1/2 \langle x, y \rangle \], we have

\[ [yx [xxx]] = \langle y, x \rangle^F [xxx] + \langle y, [xxx] \rangle^F x. \]

This completes the proof.

PROPOSITION 2.3. If \((\mathcal{W}, [\quad, \quad, \quad, \quad], \langle \quad, \quad \rangle_F)\) is a Freudenthal triple system over a field of char \( \Phi \neq 2 \), then \((\mathcal{W}, \langle \quad, - \quad, - \quad \rangle)\) is a balanced Freudenthal-Kantor triple system with respect to the triple product

\[ \langle xyz \rangle := [xyz] + \langle y, z \rangle^F x + \langle x, z \rangle^F y + \langle y, x \rangle^F z. \]

In this case, it holds \( K(x, y) = 2 \langle x, y \rangle^F Id. \)

PROOF. From the definition of triple system, we have

\[ \langle xyz \rangle - \langle yxz \rangle = 2 \langle y, x \rangle^F z \]

and

\[ \langle xzy \rangle - \langle yzx \rangle = 2 \langle x, y \rangle^F z. \]

Hence we get

\[ K(x, y) = - L(x, y) + L(y, x) = 2 \langle x, y \rangle^F Id \] (balanced property). Consequently, this yields that

\[ K(K(x, y)a, b) - L(b, a)K(x, y) + K(x, y)L(a, b) = 0. \]

We shall next show that the following equality holds,

\[ \langle xy \langle abz \rangle \rangle = \langle \langle xyz \rangle bz \rangle + \langle a \langle yxb \rangle z \rangle + \langle ab \langle xyz \rangle \rangle. \]

This is verified by using (A2) and the following relation, which can be obtained from linearizations of (A3):
\[
= - \langle b, [xaz] \rangle_F y - \langle y, [xaz] \rangle_F u + \langle y, a \rangle_F [xbz] + \langle y, z \rangle_F [xba]
+ \langle y, x \rangle_F [baz] + \langle b, z \rangle_F [yax] + \langle b, a \rangle_F [xyz] + \langle b, x \rangle_F [yza].
\]
This completes the proof.

The Freudenthal triple system \((\mathfrak{M}, [-, -, -])\) defined above is called the Freudenthal triple system associated with a balanced Freudenthal-Kantor triple system.

Let \(V = \mathfrak{Z}(N, c)\), and let the base field be characteristic zero. Then combining the above propositions with Satz 8, 4 in [18], we have dimensional formulas as follows;

**Theorem 2.4.** Under the assumption of above, let \(T(\mathfrak{M}(V))\) be the Lie triple system associated with \(\mathfrak{M}(V)\) and \(L(\mathfrak{M}(V))\) be the standard imbedding Lie algebra. If \(\dim \mathfrak{M}(V) = n\), then we have

\[
\begin{align*}
\dim \text{Der } \mathfrak{M}(V) &= 3n(n + 1)/(n + 16), \\
\dim \text{Anti-Der } \mathfrak{M}(V) &= 1, \\
\dim T(\mathfrak{M}(V)) &= 2n \text{ and} \\
\dim L(\mathfrak{M}(V)) &= (5n^2 + 38n + 48)/(n + 16).
\end{align*}
\]

**Proof.** Since the correspondence between the inner derivation \(S(x, y)\) of a simple balanced Freudenthal-Kantor triple system and the derivation \(D(x, y)\) of the Freudenthal triple system associated with it is given by

\[
S(x, y)z = 2D(x, y)z := 2[xyz] - 2\langle z, y \rangle_F x - 2\langle z, x \rangle_F y,
\]
the theorem is verified.

On the other hand, we have

<table>
<thead>
<tr>
<th>(\mathfrak{M}(V)) (V)</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\dim \mathfrak{M}(V))</td>
<td>14</td>
<td>20</td>
<td>32</td>
<td>56</td>
</tr>
<tr>
<td>(\dim \mathfrak{M}(V))</td>
<td>6</td>
<td>9</td>
<td>15</td>
<td>27</td>
</tr>
</tbody>
</table>

where \(V = \begin{pmatrix} \xi_1 & c & b \\ \xi_2 & a & \xi_3 \end{pmatrix} \Rightarrow \xi_i \in \Phi, a, b, c \in \mathfrak{A}. \) (a composition algebra over a field \(\Phi\))

(For composition algebras, see [12, 19]).

Therefore, for simple balanced Freudenthal-Kantor triple system over an algebrai-
cally closed field of characteristic 0, from the fact that $\mathfrak{M}(V)$ is simple if and only if $L(\mathfrak{M}(V))$ is simple [14], we can obtain simple Lie algebras.

$$\begin{array}{|c|c|c|c|c|}
\hline
\dim \mathfrak{A} & 1 & 2 & 4 & 8 \\
\hline
\text{Lie alg } L(\mathfrak{M}(V)) & F_4 & E_6 & E_7 & E_8 \\
\hline
\end{array}$$

For $E_6$, we note the followings: From dimension 78 of simple Lie algebras, it follows that there exist the type $B_6$, $C_6$ and $E_6$. In our case, since the dimension of the simple Lie triple system is 40's, we can obtain the type of $E_6$.

**Remark.** If $\dim \mathfrak{A} = 0$, then we have

$$\mathfrak{M}(V) := \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} : a = (\xi_1, \xi_2, \xi_3), b = (\eta_1, \eta_2, \eta_3), \xi_i, \eta_i \in \Phi \right\},$$

and $B(a, b) = \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3, a \times a = 2(\xi_2 \xi_3, \xi_1 \xi_3, \xi_1 \xi_2)$. Hence by straightforward calculations it is shown that the Lie algebra $L(\mathfrak{M}(V))$ is a simple Lie algebra of type $D_4$.

**Remark.** If an element $a$ is rank one in the vector space $V$ equipped with the conditions (1) - (5) in Section 2 (that is, $a \times a = 0$ and $a \neq 0$), then the element $c(a$ is rank one in $\mathfrak{M}(V)$, where $c$ is an arbitrary element in $\Phi$.

**Lemma 3.1.** Let $(\mathfrak{M}, \langle -, -,-, - \rangle)$ be a balanced Freudenthal-Kantor triple systems.
system and \((\mathbb{M}, [\cdot, \cdot, \cdot])\) be the Freudenthal triple system associated with it. Then an element \(u\) is rank one in \((\mathbb{M}, \langle \cdot, \cdot, \cdot \rangle)\) if and only if \(u\) is strictly regular in \((\mathbb{M}, [\cdot, \cdot, \cdot])\). (for the definition of strictly regular element, for example [7])

PROOF. “only if”: Let \(\langle uux \rangle = 0\) for all \(x \in \mathbb{M}\). Since \(\langle u, x \rangle u = \langle uux \rangle - \langle xuu \rangle\), we get

\[
\langle xuu \rangle = -\langle u, x \rangle u. \tag{3-2}
\]

On the other hand, by the balanced property, we have

\[
\langle u, x \rangle u = -\langle uxu \rangle + \langle xuu \rangle. \tag{3-3}
\]

From (3-2) and (3-3), it follows that

\[
\langle uxu \rangle = -2\langle u, x \rangle u. \tag{3-4}
\]

By the definition of the triple product

\[
[xyz] = 1/2(\langle xyz \rangle + \langle xzy \rangle),
\]

we obtain

\[
[uxu] = -\langle u, x \rangle u,
\]

which implies that \(u\) is strictly regular in \((\mathbb{M}, [\cdot, \cdot, \cdot])\).

“if”: Let \(u\) be strictly regular. From the equation (5) in [7, p317], we have \([uuy] = 2\langle y, u \rangle_F u\), where \(\langle \cdot, \cdot \rangle_F\) is the anti-symmetric bilinear form of Freudenthal triple system. From Proposition 2.3, we get

\[
\langle uuy \rangle = [uuy] + 2\langle u, y \rangle_F u.
\]

Therefore we obtain \(\langle uuy \rangle = 0\) for all \(y \in \mathbb{M}\). This completes the proof.

DEFINITION. A balanced Freudenthal-Kantor triple system \(\mathbb{M}\) is said to be reduced if \(\mathbb{M}\) contains a rank one element \(u\).

DEFINITION. A pair of rank one element \((u, v)\) is said to be supplementary if

\[
K(u, v) = 2\lambda d. \tag{3-5}
\]

PROPOSITION 3.2. Let \(\mathbb{M}\) be a simple balanced Freudenthal-Kantor triple system. Then \(\mathbb{M}\) is reduced if and only if \(\mathbb{M}\) contains a pair of supplementary rank one elements.

PROOF. Combining the above lemma 3.1 with Theorem 3.3 in [7], we can easily show the proposition.

COROLLARY. Let \(\mathbb{M}\) be a simple balanced Freudenthal-Kantor triple system and \(q(x) = \langle xxx \rangle, x \rangle\) be a nonzero 4-linear form of \(\mathbb{M}\). Then \(\mathbb{M}\) is reduced if and only if \(\mathbb{M}\) contains an element \(x\) with \(q(x) = -24\beta^2, \beta \in \Phi^*\).
REMARK. Let \( \mathfrak{M} \) be a balanced Freudenthal-Kantor triple system. Then for the 4-linear form \( q(x, y, z, w) = \langle \langle xyz \rangle, w \rangle \) in \( x, y, z, w \in \mathfrak{M} \), we have the following identities by straightforward calculations:

\[
q(x, y, z, w) = q(w, z, y, x) = q(y, x, w, z) = q(z, w, x, y).
\]

In particular,

\[
q(x, x, x, y) = q(x, x, y, x) = q(x, y, x, x) = q(y, x, x, x).
\]

Furthermore, we have

\[
q(S(x, y)z, z, z, z) = 0 \quad \text{for all } x, y, z \in \mathfrak{M},
\]

where \( S(x, y) = L(x, y) + L(y, x) \).

**PROPOSITION 3.3.** Let \( \mathfrak{M} \) be a simple balanced Freudenthal-Kantor triple system. If the 4-linear form \( q(x) \) is identically zero, then it holds

\[
\langle\langle xyz \rangle, x \rangle = 1/2(\langle y, x \rangle z + \langle y, z \rangle x + \langle x, z \rangle y),
\]

for all \( x, y, z \in \mathfrak{M} \).

**PROOF.** By the fact that \( \langle \cdot, \cdot \rangle \) is nondegenerate if and only if \( \mathfrak{M} \) is simple, and from linearizing of \( \langle\langle xxx \rangle, x \rangle = 0 \) and the above remark it follows that

\[
\langle\langle xxx \rangle \rangle = 0 \quad \text{for all } x \in \mathfrak{M}.
\]

Linearizing the identity \( \langle\langle xxx \rangle \rangle = 0 \), we have

\[
\langle\langle xxy \rangle \rangle + \langle\langle xyx \rangle \rangle + \langle\langle yxx \rangle \rangle = 0. \tag{3-6}
\]

From the assumption to be balanced, we have

\[
\langle\langle xxy \rangle \rangle = 2\langle\langle yxx \rangle \rangle - \langle\langle xyx \rangle \rangle. \tag{3-7}
\]

Combining (3-6) with (3-7), we get from \( \text{ch } \Phi \neq 3 \)

\[
\langle\langle yxx \rangle \rangle = 0.
\]

Hence we have \( \langle\langle xyx \rangle \rangle = -\langle x, y \rangle x \). Linearizing this identity, we have

\[
\langle\langle xyz \rangle \rangle + \langle\langle zyx \rangle \rangle = -\langle x, y \rangle z - \langle z, y \rangle x. \tag{3-8}
\]

On the other hand, we have

\[
\langle\langle yzx \rangle \rangle - \langle\langle zyx \rangle \rangle = \langle x, z \rangle y. \tag{3-9}
\]

From (3-8) and (3-9), we obtain

\[
\langle\langle xzy \rangle \rangle = 1/2(\langle y, x \rangle z + \langle y, z \rangle x + \langle x, z \rangle y). \tag*{\blacksquare}
\]

**LEMMA 3.4.** Let \( (u, v) \) be a pair of supplementary rank one elements of simple balanced Freudenthal-Kantor triple system \( \mathfrak{M} \). Then it holds

\[
1/4(R(u, v) + R(v, u))^2 x = x + 3/2\langle u, x \rangle v - 3/2\langle v, x \rangle u
\]
for all $x \in \mathfrak{M}$.

PROOF. From (U1) with $\varepsilon = 1$, we obtain the following relation by straightforward calculations:

$$R(c, d) R(a, b) x = R(a, \langle bcd \rangle) x - L(b, c) R(a, d) x$$
$$- M(b, d) M(a, c) x,$$

(3–10)

where $R(a, b) x = \langle xab \rangle$ and $M(a, c) x = \langle acx \rangle$. By making use of the relation (3–10), we have

$$R(u, v) R(u, v) x = R(u, \langle uvv \rangle) x - L(v, u) R(u, v) x$$
$$- M(v, v) M(u, u) x$$
$$= R(u, -2 \langle v, u \rangle v) x - L(v, u) R(u, v) x$$
$$- 4 \langle v, \langle u, x \rangle u \rangle v$$

(by (3–4))

$$= 4R(u, v) x - L(v, u) R(u, v) x + 8 \langle u, x \rangle v.$$  

(by $\langle u, v \rangle = 2$)

Similarly, we have

$$R(v, u) R(v, u) x = -4R(v, u) x - L(u, v) R(v, u) x - 8 \langle v, x \rangle u.$$  

Hence we get

$$(R(u, v) + R(v, u))^2 x = (4R(u, v) - L(v, u) R(u, v) + R(v, u) R(v, u)$$
$$+ R(v, u) R(u, v) - 4R(v, u) - L(u, v) R(v, u)) x$$
$$+ 8 \langle u, x \rangle v - 8 \langle v, x \rangle u.$$  

(3–11)

We compute

$$(4R(u, v) - L(v, u) R(u, v) + R(u, v) R(v, u)$$
$$+ R(v, u) R(u, v) - 4R(v, u) - L(u, v) R(v, u)) x$$
$$= (4R(u, v) x - 2R(u, v) x + \langle R(u, v) x, u \rangle v$$
$$+ 2R(v, u) x + \langle R(v, u) x, v \rangle u - 4R(v, u) x$$

(by means of the relations;

$- L(v, u) y + R(v, u) y = -2y + \langle y, u \rangle v$  for all $y \in \mathfrak{M}$

$- L(u, v) z + R(u, v) z = 2z + \langle z, v \rangle u$  for all $z \in \mathfrak{M}$)

$$= 2R(u, v) x - 2R(v, u) x + \langle R(u, v) x, u \rangle v + \langle R(v, u) x, v \rangle u$$

$$= 4x + \langle R(u, v) x - 2x, u \rangle v + \langle R(v, u) x + 2x, v \rangle u$$

(by means of the relation;

$$R(u, v) x - R(v, u) x = 2x + \langle x, v \rangle u + \langle u, x \rangle v$$

$$= 4x + 2 \langle x, u \rangle v - 2 \langle x, v \rangle u.$$  

(3–12)
(by means of the relations;
\[ \langle \langle xuv \rangle, u \rangle = - \langle x, \langle uvu \rangle \rangle = 4 \langle x, u \rangle \]
\[ \langle \langle xuv \rangle, v \rangle = - \langle x, \langle vwu \rangle \rangle = - 4 \langle x, v \rangle \].

Combining (3-12) with (3-11), we obtain
\[ (R(u, v) + R(v, u))^2 = 4x + 6 \langle u, x \rangle v - 6 \langle v, x \rangle u. \]

This completes the proof.

We denote \(1/2(R(u, v) + R(v, u))\) by \(J(u, v)\). Thus on \((\Phi u \oplus \Phi v)^\perp\), we have
\[ J(u, v)^2 = \text{Id}, \]
so \((\Phi u \oplus \Phi v)^\perp = \mathcal{M}_1 \oplus \mathcal{M}_{-1}\), where \(\mathcal{M}_\epsilon\) is the eigenspace for the eigenvalue \(\epsilon\) of \(J(u, v)\) for \(\epsilon = \pm 1\). Moreover, since \(\langle -, - \rangle\) is nondegenerate, and its restriction to \((\Phi u \oplus \Phi v)^\perp\) is nondegenerate, we have
\[ \mathcal{M} = \Phi u \oplus \Phi v \oplus \mathcal{M}_1 \oplus \mathcal{M}_{-1}. \]

Since \(J(u, v) u = - 2u\) (resp. \(J(u, v) v = 2v\)), these imply \(u\) (resp. \(v\)) is the eigenspace for \(J(u, v)\) with eigenvalue \(- 2\) (resp. \(2\)). Consequently we have the following decomposition of \(\mathcal{M}\);
\[ \mathcal{M} = \mathcal{M}_{-2} \oplus \mathcal{M}_{-1} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2, \]
where \(\mathcal{M}_i\) is the eigenspace for the eigenvalue \(i\) of \(J(u, v)(i = \pm 1, \pm 2)\). We call this decomposition the Peirce decomposition of a simple reduced balanced Freudenthal-Kantor triple system. We remark that all Peirce spaces \(\mathcal{M}_i\) are totally isotopic (that is, \(\langle \mathcal{M}_i, \mathcal{M}_{-j} \rangle \neq 0\) if \(i = j\) and \(\langle \mathcal{M}_i, \mathcal{M}_{-j} \rangle = 0\) otherwise). Using results of the coordinatization of simple reduced Freudenthal triple system, we can prove the following lemma.

\[ \mathcal{M} = \mathcal{M}_{-2} \oplus \mathcal{M}_{-1} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2, \]

where \(\mathcal{M}_i\) is the eigenspace for the eigenvalue \(i\) of \(J(u, v)(i = \pm 1, \pm 2)\). We call this decomposition the Peirce decomposition of a simple reduced balanced Freudenthal-Kantor triple system. We remark that all Peirce spaces \(\mathcal{M}_i\) are totally isotopic (that is, \(\langle \mathcal{M}_i, \mathcal{M}_{-j} \rangle \neq 0\) if \(i = j\) and \(\langle \mathcal{M}_i, \mathcal{M}_{-j} \rangle = 0\) otherwise). Using results of the coordinatization of simple reduced Freudenthal triple system, we can prove the following lemma.

\[ \mathcal{M} = \mathcal{M}_{-2} \oplus \mathcal{M}_{-1} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2, \]

where \(\mathcal{M}_i\) is the eigenspace for the eigenvalue \(i\) of \(J(u, v)(i = \pm 1, \pm 2)\). We call this decomposition the Peirce decomposition of a simple reduced balanced Freudenthal-Kantor triple system. We remark that all Peirce spaces \(\mathcal{M}_i\) are totally isotopic (that is, \(\langle \mathcal{M}_i, \mathcal{M}_{-j} \rangle \neq 0\) if \(i = j\) and \(\langle \mathcal{M}_i, \mathcal{M}_{-j} \rangle = 0\) otherwise). Using results of the coordinatization of simple reduced Freudenthal triple system, we can prove the following lemma.

**Lemma 3.5.** For \(t\) as above,
(i) \(\langle a, tb \rangle = - \langle ta, b \rangle\)
(ii) \(\langle v, tatata \rangle = \lambda/12 \langle u, aaa \rangle\)
(iii) \(t\) is nonsingular
(iv) \(t \langle vtab \rangle = - \lambda/12 \langle uab \rangle\)
for all \(a, b \in \mathcal{M}_1\).

We can next define a bilinear form \(B(.)\) and a cross product on \(\mathcal{M}_1\) as follows:
\[ B(a, b) = \lambda^{-1}/6 \langle a, tb \rangle \]
and \(a \times b = - \lambda^{-1}/2(\langle vtab \rangle + \langle vtbta \rangle)\) if \(\lambda \neq 0\).
\( B(a, b) = 1/6 \langle a, tb \rangle \) and \( a \times b = 0 \), if \( \lambda = 0 \).

**Proposition 3.6.** Under the above definition, we have the following identities on \( \mathfrak{M}_1 \):

1. \( a \times b = b \times a \)
2. \( B(a, b) = B(b, a) \)
3. \( B(a \times b, d) = B(a, b \times d) \)
4. \( (a \times a) \times b = 1/3B(a, a \times a)b + B(b, a)a \times a \)
5. \( (a \times a) \times (a \times a) = 4/3B(a \times a, a)a \)

**Proof.** By the definition of the above bilinear form and cross product, the relations obtained from Lemma 3.5 yield the proof.

**Theorem 3.7.** Let \( \mathfrak{M} \) be a reduced simple balanced Freudenthal-Kantor triple system over \( \Phi \). Then it holds \( \mathfrak{M} \cong \mathfrak{M}(V) \), where \( V \) is a vector space equipped with the bilinear form \( B(a, b) \) and the cross product \( \times \) satisfying the relations (1) - (5) of Proposition 3.6.

**Proof.** We can show that if \( \lambda \neq 0 \), then the map \( f: \mathfrak{M}(V) \to \mathfrak{M} \) defined as follows is an isomorphism of balanced Freudenthal-Kantor triple systems:

\[
\left( \begin{array}{c}
\alpha_1 \\
b_1
\end{array} \right) \mapsto 36\lambda\alpha_1v + 1/72\lambda^{-1}\beta_1u + a_1 + 1/6\lambda^{-1}tb
\]

if \( \lambda = 0 \), similarly,

\[
\left( \begin{array}{c}
\alpha_1 \\
b_1
\end{array} \right) \mapsto 36\alpha_1v + 1/72\beta_1u + a_1 + 1/6tb_1.
\]

As the proof of this isomorphism is very long and of straightforward calculations, we omit it.

Finally, from results of this paper, Theorem 6.8 and Theorem 7.4 in [7], we can obtain the following.

**Theorem 3.8.** Let \( V(\text{resp. } V') \) be a Jordan triple system induced from an admissible cubic form \( N(\text{resp. } N') \) with basepoint \( c(\text{resp. } c') \). Then the followings are equivalent

(i) \( V \) and \( V' \) are isotopic.
(ii) \( \mathfrak{M}(V) \) and \( \mathfrak{M}(V') \) are isomorphic.

**References:**

Freudenthal-Kantor Triple Systems