Mem. Fac. Sci. Shimane Univ.,
23, pp. 11–15 Dec. 23, 1989

Some Remarks on Subdirect Products of Completely Regular Semigroups

Miyuki YAMADA

Department of Mathematics, Shimane University, Matsue, Japan
(Received September 6, 1989)

In this paper, the concept of a subspined product of completely regular semigroups are introduced. Firstly, we give a necessary and sufficient condition for a subspined product $A \divideontimes B$ of cryptogroups A and B to be also a cryptogroup. Secondly, it is shown that a subspined product $A \divideontimes B$ is necessarily a cryptogroup if one of A and B is a band and the other is a cryptogroup. It is also shown that any subspined product $A \divideontimes B$ coincides with the spined product $A \bowtie B$ if one of A and B is a Clifford semigroup and the other is a band.

Finally, the concept of a subspined product is extended to the concept of a $\mathscr{P}$-subspined product for the class of completely $\mathscr{P}$-regular semigroups, and some considerations are given for $\mathscr{P}$-subspined products of $\mathscr{P}$-cryptogroups.

Let S be a completely regular semigroup. Then, S is uniquely decomposed into a semilattice Y of completely simple semigroups $\{S_i: i \in Y\}$ (see [1]).

This decomposition is called *the structure decomposition* of S (see [3]), and denoted by $S \sim \Sigma\{S_i: i \in Y\}$. In this case, Y is also uniquely determined, up to isomorphism, and it is called *the structure semilattice* of S. Hereafter, "a completely regular semigroup $S \equiv \Sigma\{S_i: i \in Y\}$" means "$S$ is a completely regular semigroup and has the structure decomposition $S \sim \Sigma\{S_i: i \in Y\}$".

Let $A \equiv \Sigma\{A_i: i \in Y\}$ and $B \equiv \Sigma\{B_i: i \in Y\}$ be completely regular semigroups having the same structure semilattice Y. Put $A \bowtie B = \{(x, y): x \in A_i, y \in B_i, i \in Y\}$. Define multiplication in $A \bowtie B$ as follows: $(x, y)(u, v) = (xu, yv)$ for (x, y), $(u, v) \in A \bowtie B$. Then, $A \bowtie B$ is a regular subsemigroup of the direct product $A \times B$ of A and B. This $A \bowtie B$ is called *the spined product of A and B* (*with respect to the structure decompositions of A and B*). Hereafter, we shall omit the term "with respect to the structure decompositions of A and B". Now, let S be a regular subsemigroup of the spined product $A \bowtie B$ of completely regular semigroups A and B such that

(C.1) the first and the second projections ϕ, Ψ (that is, the mappings ϕ: $S \to A$ and Ψ: $S \to B$ defined by $(x, y)\phi = x$ and $(x, y)\Psi = y$, $(x, y) \in S$) are surjective homomorphisms.

Then, S is also a subdirect product of A and B. Such an S is called *a subspined product of A and B* (*with respect to the structure decompositions of A and B*). Of course, a subspined product of A and B is not necessarily unique in general.

LEMMA 1. *If A and B are cryptogroups (that is, bands of groups; see [2]), then a subspined product $A \divideontimes B$ of A and B is also a cryptogroup if and only if $A \divideontimes B$ satisfies the following*:

(C.2) $A \divideontimes B \ni (a, b)$ *implies* $(a^{-1}, b^{-1}) \in A \divideontimes B$, *where* x^{-1} *is the group inverse of* x.

PROOF. The "if" part: Let A be a band Γ_1 of groups $\{A'_\gamma : \gamma \in \Gamma_1\}$ and B a band Γ_2 of groups $\{B'_\tau : \tau \in \Gamma_2\}$. Let $e \in E_A$ (the set of idempotents of A) and $f \in E_B$. Put $C_{(e,f)} = \{(a, b) \in A \divideontimes B : aa^{-1} = e \text{ and } bb^{-1} = f\}$. It is obvious that $(e, f) \in C_{(c,f)}$. If $(a, b) \in C_{(e,f)}$, then $(a^{-1}, b^{-1}) \in C_{(e,f)}$. Further, (a, b), $(c, d) \in C_{(e,f)}$ implies that $ac(c^{-1}a^{-1}) = aea^{-1} = aa^{-1} = e$ and similarly $(c^{-1}a^{-1})ac = e$. Further, $bd(d^{-1}b^{-1}) = (d^{-1}b^{-1})bd = f$. Therefore, $c^{-1}a^{-1} = (ac)^{-1}$, and $d^{-1}b^{-1} = (bd)^{-1}$. Thus, $(ac, bd) \in C_{(e,f)}$, that is, $(a, b)(c, d) \in C_{(e,f)}$. Therefore, $C_{(e,f)}$ is a group. Let e_γ be the identity of A'_γ for $\gamma \in \Gamma_1$, and f_τ the identity of B'_τ for $\tau \in \Gamma_2$. Let $x \in C_{(e_\gamma, f_\lambda)}$ and $y \in C_{(e_\mu, f_\nu)}$. Put $x = (a, b)$, $y = (c, d)$. Then, $aa^{-1} = e_\gamma$, $bb^{-1} = f_\lambda$, $cc^{-1} = e_\mu$ and $dd^{-1} = f_\nu$. Since $a \in A'_\gamma$, $c \in B'_\mu$, it follows that $ac \in A'_{\gamma\mu}$. Similarly, $bd \in B'_{\lambda\nu}$. Further, $(ac, bd) \in A \divideontimes B$. Now, $ac(ac)^{-1} = e_{\gamma\mu}$ and $bd(bd)^{-1} = f_{\lambda\nu}$. Hence, $xy \in C_{(e_{\gamma\mu}, f_{\lambda\nu})}$. Thus, $C_{(e_\gamma, f_\lambda)} C_{(e_\mu, f_\nu)} \subset C_{(e_{\gamma\mu}, f_{\lambda\nu})}$. This implies that $A \divideontimes B$ is a cryptogroup. The "only if" part: Suppose that a subspined product $A \divideontimes B$ of A and B is a cryptogroup. Let $(x, y) \in A \divideontimes B$. Then, (x, y) is contained in a subgroup of $A \divideontimes B$, and accordingly there exists an group-inverse (u, v) of (x, y) in $A \divideontimes B$. Since $(u, v)(x, y) = (x, y)(u, v) = (e, f) \in E_{A \divideontimes B}$, $ux = xu = e$ and $vy = yv = f$. Further, $((u, v), (x, y))$ is a regular pair. Then, $ex = xe = x$, $ue = eu = u$, $vf = fv = v$ and $yf = fy = y$. Therefore, u and v are group-inverses of x and y respectively. Hence, $(x^{-1}, y^{-1}) = (u, v) \in A \divideontimes B$.

Let $X(I)$ be a $\mathscr{P}$-regular semigroup (for the definition, see [4]). If X is completely regular, then $X(I)$ is called *a completely $\mathscr{P}$-regular semigroup*. Let $A(P)$ and $B(Q)$ be completely $\mathscr{P}$-regular semigroups. Let $A \sim \Sigma\{A_\lambda : \lambda \in \Lambda\}$ and $B \sim \Sigma\{B_\lambda : \lambda \in \Lambda\}$ be the structure decompositions of A and B respectively. Let $P_\lambda = A_\lambda \cap P$ and $Q_\lambda = B_\lambda \cap Q$, and let $U = \Sigma\{P_\lambda \times Q_\lambda \text{ (direct product)}: \lambda \in \Lambda\}$. Let $C = A \bowtie B$. Then, $C(U)$ is also a completely $\mathscr{P}$-regular semigroup, which is denoted by $C(U) = A(P) \underset{\mathscr{P}}{\bowtie} B(Q)$. Let $D(V)$ be a $\mathscr{P}$-regular subsemigroup (see [4]) of $A(P) \underset{\mathscr{P}}{\bowtie} B(Q)$. If $D(V)$ satisfies the following (C.3) then $D(V)$ is called *a $\mathscr{P}$-subspined product* of $A(P)$ and $B(Q)$, and some times denoted by $A(P) \underset{\mathscr{P}}{\divideontimes} B(Q)$, etc.:

(C.3) The first and the second projections $\phi: D(V) \to A(P)$ and $\Psi: D(V) \to B(Q)$ are surjective $\mathscr{P}$-homomorphisms.

Of course, in this case D is a subspined product of A and B. Especially, if A and B are cryptogroups and if D satisfies (C.2), then $D(V)$ is also a $\mathscr{P}$-regular cryptogroup (abbrev., a $\mathscr{P}$-cryptogroup). Since the converse is also satisfied, we have the following:

LEMMA 2. *Let $A(P)$ and $B(Q)$ be $\mathscr{P}$-cryptogroups, and assume that A and B has the same structure semilattice. Then, every $\mathscr{P}$-subspined product $D(V)$ of $A(P)$ and $B(Q)$ is a $\mathscr{P}$-cryptogroup if and only if D satisfies* (C.2).

THEOREM 3. *Let A be a completely regular semigroup, and B a band. Assume that A and B have the same structure semilattice. Then, every subspined product of A and B satisfies* (C.2). *Accordingly, if A is a cryptogroup, every subspined product of A and B is also a cryptogroup.*

PROOF. Let $A \sim \Sigma\{A_\lambda: \lambda \in \Lambda\}$ and $B \sim \Sigma\{B_\lambda: \lambda \in \Lambda\}$ be the structure decompositons of A and B. Hence, each B_λ is a rectangular band. Suppose that C is a subspined product of A and B. Let $(b, e) \in C$. Then, there exists f such that $(b^{-1}, f) \in C$. Of course, if $(b, e) \in A_\lambda \times B_\lambda$ then $(b^{-1}, f) \in A_\lambda \times B_\lambda$. Now, (bb^{-1}, ef), $(b^{-1}b, fe) \in C$, and $(bb^{-1}, ef)(b^{-1}, f)(b^{-1}b, fe) = (b^{-1}, e) \in C$ since B_λ is a rectangular band. Hence, (C.2) is satisfied.

COROLLARY. *Let $A(P)$ be a completely $\mathscr{P}$-regular semigroup, and $B(Q)$ a $\mathscr{P}$-band. Assume that A and B have the same structure semilattice. Then, every $\mathscr{P}$-subspined product of $A(P)$ and $B(Q)$ satisfies* (C.2). *Hence, if $A(P)$ is a $\mathscr{P}$-cryptogroup then every $\mathscr{P}$-subspined product of $A(P)$ and $B(Q)$ is also a $\mathscr{P}$-cryptogroup.*

THEOREM 4. *Let $A \equiv \Sigma\{A_\lambda: \lambda \in \Lambda\}$ be a Clifford semigroup (that is, a semilattice Λ of groups $\{A_\lambda: \lambda \in \Lambda\}$), and $B \equiv \Sigma\{B_\lambda: \lambda \in \Lambda\}$ a band. Then, a subspined product of A and B coincides with $A \bowtie B$.*

PROOF. Let C be a subsined product of A and B. Now, $A \bowtie B \equiv \Sigma\{A_\lambda \times B_\lambda: \lambda \in \Lambda\}$. Let $C_\lambda = C \cap (A_\lambda \times B_\lambda)$. Then, $C \equiv \Sigma\{C_\lambda: \lambda \in \Lambda\}$. If $(a, e) \in C_\lambda$, it follows from Theorem 3 that $(a^{-1}, e) \in C_\lambda$. Hence, $(aa^{-1}, e) \in C_\lambda$. Now, for any $b \in A_\lambda$, there exists f such that $(b, f) \in C_\lambda$. Therefore, $(aa^{-1}, e)(b, f)(aa^{-1}, e) \in C_\lambda$. That is, $(b, e) \in C_\lambda$. Next, let $u \in B_\lambda$. Then, there exists $c \in A_\lambda$ such that $(c, u) \in C_\lambda$. Since $cc^{-1} = aa^{-1}$, $(aa^{-1}, u) \in C_\lambda$. Therefore, $(aa^{-1}, u)(b, f)(aa^{-1}, u) = (b, u) \in C_\lambda$. Thus, $C_\lambda \supset A_\lambda \times B_\lambda$, and accordingly $C = A \bowtie B$.

COROLLARY. *Let $A(P)$ be a $\mathscr{P}$-Clifford semigroup (a Clifford semigroup which is $\mathscr{P}$-regular), and $B(Q)$ a $\mathscr{P}$-band. Assume that A and B have the same structure semilattice. Then, every $\mathscr{P}$-subspined product of $A(P)$ and $B(Q)$ coincides with $A(P) \underset{\mathscr{P}}{\bowtie} B(Q)$.*

REMARK. Let A be a Clifford semigroup, and E_A the semilattice of idempotents of A. Let $P \subset E_A$. If $A(P)$ is $\mathscr{P}$-regular, then every $\mathscr{L}$-class and $\mathscr{R}$-class (where $\mathscr{L}$ and $\mathscr{R}$ are Green's L and R relations respectively) contain an element of P. Hence, we have $P = E_A$. Conversely, it is obvious that $A(E_A)$ is $\mathscr{P}$-regular. Therefore, $A(P)$ is $\mathscr{P}$-regular if and only if $P = E_A$.

LEMMA 5. *Let A be a rectangular band, and B a completely simple semigroup. Then, a subdirect product C of A and B is the direct product $A \times B$ of A and B if and only if C contains all idempotents of $A \times B$.*

PROOF. Let B be a rectangular band Γ of groups $\{B_\gamma: \gamma \in \Gamma\}$. Let C be a subdirect product of A and B. If $C = A \times B$, then it is obvious that C contains all idempotents of $A \times B$. Conversely, suppose that C contains all idempotents of $A \times B$. Let $(e, b) \in C$. For $f \in A$, $(f, h) \in C$, where $h = bb^{-1}$. Then, $(f, h)(e, b)(f, h) = (f, b) \in C$. Hence, $C = A \times B$.

This is extended to the following result:

THEOREM 6. *Let $A \equiv \Sigma\{A_\lambda: \lambda \in \Lambda\}$ be a band, and $B \equiv \Sigma\{B_\lambda: \lambda \in \Lambda\}$ a completely regular semigroup. Then, a subspined product of A and B is the spined product $A \bowtie B$ of A and B if and only if it contains all idempotents of $A \bowtie B$.*

PROOF. Now, $A \bowtie B = \Sigma\{A_\lambda \times B_\lambda: \lambda \in \Lambda\}$. Let C be a subspined product of A and B. Let $C_\lambda = C \cap (A_\lambda \times B_\lambda)$. Then, it is obvious that C_λ is a subdirect product of A_λ and B_λ. The "only if" part is obvious. Suppose that C contains all idempotents of $A \bowtie B$. Then, it follows that C_λ contains all idempotents of $A_\lambda \times B_\lambda$. Hence, it follows from Lemma 6 that $C_\lambda = C_\lambda = A_\lambda \times B_\lambda$. Therefore, $C = A \bowtie B$.

LEMMA 7. *Let $A(P)$ and $B(Q)$ be a completely simple $\mathscr{P}$-regular semigroup and a rectangular $\mathscr{P}$-band. Let $C(U)$ be a $\mathscr{P}$-subdirect product of $A(P)$ and $B(Q)$ (that is, C is a subdirect product of A and B, $U = C \cap (P \times Q)$, and $C(U)$ is $\mathscr{P}$-regular). Then, $C(U) = A(P) \underset{\mathscr{P}}{\times} B(Q)$ if and only if $C \supset P \times Q$.*

PROOF. The "only if" part is obvious. The "if" part: It is obvious that $U^2 = E_A \times E_B = E_{A \times B}$ Since $U = P \times Q$. Therefore, $C \supset E_{A \times B}$. It follows from Lemma 5 that $C(U) = A(P) \underset{\mathscr{P}}{\times} B(Q)$.

By using Lemma 7, we obtain the following result:

THEOREM 8. *Let $A(P)$ and $B(Q)$ be a completely $\mathscr{P}$-regular semigroup and a $\mathscr{P}$-band respectively. Let $A \equiv \Sigma\{A_\lambda: \lambda \in \Lambda\}$ and $B \equiv \Sigma\{B_\lambda: \lambda \in \Lambda\}$. Then, a $\mathscr{P}$-subspined product of $A(P)$ and $B(Q)$ is the $\mathscr{P}$-spined product $A(P) \underset{\mathscr{P}}{\bowtie} B(Q)$ if and only if it contains $\Sigma\{P_\lambda \times Q_\lambda: \lambda \in \Lambda\}$, where $P_\lambda = P \cap A_\lambda$ and $Q_\lambda = Q \cap B_\lambda$.*

PROOF. Let $C(U)$ be a $\mathscr{P}$-subspined product of $A(P)$ and $B(Q)$. Let $C_\lambda = C \cap (A_\lambda \times B_\lambda)$ and $U_\lambda = C_\lambda \cap U$. Then, $C_\lambda(U_\lambda)$ is a $\mathscr{P}$-subdirect product of $A_\lambda(P_\lambda)$ and $B_\lambda(Q_\lambda)$. Hence, if C_λ contains $P_\lambda \times Q_\lambda$ then $C_\lambda = A_\lambda \times Q_\lambda$, and $U_\lambda = P_\lambda \times Q_\lambda$, and accordingly $C_\lambda(U_\lambda) = A_\lambda(P_\lambda) \underset{\mathscr{P}}{\times} B_\lambda(Q_\lambda)$. Hence, $C(U) = A(P) \underset{\mathscr{P}}{\bowtie} B(Q)$. The converse is obvious.

References

[1] Clifford, A. H. and Preston, G. B.: Algrbraic theory of semigroups, Vol. 1, Amer. Math. Soc., Providence, R. I., 1961.

[2] Pastijn, F. J. and Petrich, M.: Regular semigroups as extensions, Research Notes in Mathematics 136, Pitman Advanced Publishing Program, Boston, 1985.

[3] Yamada, M.: The structure of quasi-orthodox semigroups, Mem. Fac. Sci., Shimane Univ. **14** (1980), 1–18.

[4] Yamada, M. and Sen, M. K.: $\mathscr{P}$-regular semigroups, Semigroup Forum **39** (1989), 157–178.