Matrices of Clifford Semigroups, and a Generalization of Rees's Theorem\textsuperscript{1)}

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Let \( S \) be a completely regular semigroup, and \( E(S) \) the partial subgroupoid of idempotents of \( S \). Let \( \gamma \) be a relation on \( E(S) \). If \( \gamma \) is a congruence on \( E(S) \), that is, if \( \gamma \) is an equivalence relation on \( E(S) \) and if \( x\gamma y \) and \( u\gamma v \) satisfy \( xu\gamma yv \) (if both \( xu \) and \( yv \) are defined in \( E(S) \)), then \( S \) is called a CS-matrix. Firstly, several characterizations of a CS-matrix are given. Secondly, split CS-matrices are investigated. In particular, matrix representations of these semigroups are discussed.

\section*{§ 1. Preliminary}

Let \( P \) be a partial groupoid, and \( \gamma \) a relation on \( P \) as follows:

\begin{enumerate}
\item \( x\gamma y \) if and only if both \( xy \) and \( yx \) are defined in \( P \), and \( xy = yx \).
\end{enumerate}

If \( \gamma \) is a congruence on \( P \), that is,

\begin{enumerate}
\item \( x\gamma x \) for all \( x \in P \),
\item \( x\gamma y \) implies \( y\gamma x \),
\item \( x\gamma y, y\gamma z \) imply \( x\gamma z \),
\item if \( x\gamma y, u\gamma v \) and if both \( xu \) and \( yv \) are defined in \( P \), then \( xu\gamma yv \),
\end{enumerate}

then \( P \) is called \( \gamma \)-\textit{compatible}. In a completely simple semigroup \( C \), it is obvious that the partial groupoid \( E(C) \) of idempotents of \( C \) (with respect to the multiplication in \( C \)) is \( \gamma \)-\textit{compatible}. If the partial groupoid \( E(S) \) of idempotents of a regular semigroup \( S \) is \( \gamma \)-\textit{compatible}, then \( S \) is called \( \gamma \)-\textit{compatible}. If a semigroup \( A \) is a rectangular band \( \Delta \) of subsemigroups \( \{ A_\delta : \delta \in \Delta \} \) of type \( \mathcal{F} \), then we shall say that \( A \) is a \textit{matrix} \( \Delta \) of semigroups \( \{ A_\delta : \delta \in \Delta \} \) of type \( \mathcal{F} \). If \( A \) is a matrix of semigroups of type \( \mathcal{F} \), then \( A \) is said to be a \( \mathcal{F} \)-\textit{semigroup matrix}. For example, if \( A \) is a rectangular band of subgroups then \( A \) is called a \textit{group matrix}. If \( A \) is a rectangular band of Clifford subsemigroups (that is, semilattices of groups), then \( A \) is called a \textit{Clifford semigroup matrix} (abbrev.,

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a CS-matrix). Let $B$ be a $\gamma$-compatible band, and $\rho$ the least semilattice congruence on $B$. It is well-known that each $\rho$-class is a rectangular subband of $B$ (therefore, $B$ is a semilattice of rectangular bands). Now, it is also easy to see that $\rho \cap \gamma = \zeta_S$ (the identity congruence on $B$). Hence, $B$ is isomorphic to a subdirect product of $B/\rho$ and $B/\gamma$. Let $e, f \in B$. Since $efe = efe, efe \gamma e$. Hence, $B/\gamma$ is a rectangular band. Since $B/\rho$ is a semilattice, $B$ is isomorphic to a subdirect product of a semilattice and a rectangular band.

Further, it is easy to see that the converse also holds; that is, a band $B$ is $\gamma$-compatible if and only if it is isomorphic to a subdirect product of a semilattice and a rectangular band. Accordingly, in this case $B$ is a rectangular band of semilattices, that is, a semilattice matrix (abbrev., an SL-matrix). Of course, an SL-matrix is a normal band. In this paper, we shall investigate the structure of CS-matrices and that of split CS-matrices. If $S$ is a completely regular semigroup, it is well-known that $S$ is uniquely decomposed into a semilattice $A$ of completely simple semigroups $\{S_\lambda : \lambda \in A\}$ (see [1]). This decomposition is called the structure decomposition of $S$, and denoted by $S = \Sigma \{S_\lambda : \lambda \in A\}$.

Hereafter the terminology "a completely regular semigroup $S = \Sigma \{S_\lambda : \lambda \in A\}$" means "$S$ is a completely regular semigroup and has $S = \Sigma \{S_\lambda : \lambda \in A\}$ as its structure decomposition". Further, it is also well-known that the least semilattice congruence $\rho$ on a completely regular semigroup $S$ induces the structure decomposition of $S$. Throughout the whole paper, if $S$ is a completely regular semigroup, $\rho_S$ and $\gamma_S$ denote the least semilattice congruence on $S$ and the $\gamma$-congruence on $E(S)$ respectively. Every terminology and notation should be referred to [1], unless otherwise stated.

§ 2. CS-matrices

Let $S$ be a $\gamma$-compatible completely regular semigroup, and $E(S)$ the set of all idempotents of $S$. Since $\gamma$ is an equivalence relation on $E(S)$, $E(S)$ is decomposed into $\gamma$-equivalence classes $\{E_\lambda : \lambda \in A\}$ (where each $E_\lambda$ is a $\gamma$-class). Now, put $S_\lambda = \{x \in S : xx^* , x^*x \in E_\lambda$ for some $x^* \in V(x)\}$, where $V(x)$ is the set of inverses of $x$.

LEMMA 2.1. (1) Each $S_\lambda$ is a maximal Clifford subsemigroup of $S$.

(2) $\mathcal{F} = \{S_\lambda : \lambda \in A\}$ is the set of all maximal Clifford subsemigroups, and $S = \Sigma \{S_\lambda : \lambda \in A\}$ (where $\Sigma$ denotes disjoint sum).

PROOF. It is obvious that each $E_\lambda$ is a subsemilattice of $S$. Let $x \in S_\lambda$. Then, $xx^* , x^*x \in E_\lambda$ for some $x^* \in V(x)$. For any $e \in E_\lambda$, $xx^*$ is an idempotent and $xx^*xe^* = xe^* = xe^*xx^*$. Hence, $xx^*\gamma xe^*$, and accordingly $xe^* \in E_\lambda$. Therefore, $xE_\lambda x^* < $
Matrices of Clifford Semigroups, and a Generalization of Rees's Theorem

As matrices of Clifford Semigroups, and a Generalization of Rees's Theorem 3

Similarly, we have \( x^*E_\lambda x \subseteq E_\lambda \). Hence, \( S_\lambda = \{ x \in S : x^*x, x^*x \in E_\lambda, x^*E_\lambda x \subseteq E_\lambda \} \) for some \( x^* \in V(\lambda) \). Therefore, \( S_\lambda \) is a maximal regular subsemigroup having \( E_\lambda \) as the set of idempotents. Since \( E_\lambda \) is a semilattice, \( S_\lambda \) is an inverse semigroup. Now, let \( x \in S_\lambda \). Then, \( xx^*, x^*x \in E_\lambda \) for some \( x^* \in V(\lambda) \). Since \( S \) is completely regular, there exists the group inverse \( x^{-1} \) of \( x \). Now, \( xx^* \gamma x^*x \) implies \( xx^*xx^{-1} \gamma x^*xxx^{-1} = x^*xx^{-1}x \), and accordingly \( xx^{-1} \gamma x^*x \). Therefore, \( xx^{-1} \in E_\lambda \). Thus, \( x^{-1} \in S_\lambda \). That is, \( x \) has the group inverse \( x^{-1} \) in \( S_\lambda \). Consequently, \( S_\lambda \) is a union of groups, and accordingly \( S_\lambda \) is a Clifford semigroup.

(2) Let \( T \) be a maximal Clifford subsemigroup of \( S \). For any \( e, f \in E(T) \) (the set of idempotents of \( T \)), \( ef = fe \). Hence, \( e \gamma f \), and accordingly \( e, f \in E_\lambda \) for some \( \lambda \in \Lambda \). Therefore, \( E(T) \subseteq E_\lambda \), and \( T \subseteq S_\lambda \). Since \( T \) is a maximal Clifford subsemigroup, \( T = S_\lambda \). Thus, \( \mathcal{S} \) is the set of all maximal Clifford subsemigroups of \( S \). It is obvious that \( S = \cup \{ S_\lambda : \lambda \in \Lambda \} \). Assume that \( x \in S_\lambda \cap S_\delta \). Then, there exist \( x^*, x^* \in V(\lambda) \) such that \( xx^*, x^*x \in E_\lambda \) and \( xx^*, x^*x \in E_\delta \). Since \( xx^* \gamma x^*x \) and \( xx^* \gamma x^*x \), we have \( xx^* = xx^*xx^* \gamma x^*xx^*x = xx^*x \). Hence, \( xx^*x \in E_\lambda \cap E_\delta \), and \( \lambda = \delta \). Therefore, \( S_\lambda \cap S_\delta = \emptyset \) for \( \lambda \neq \delta \).

As characterizations of a CS-matrix, we have the following:

**Theorem 2.2.** For a completely regular semigroup \( S \), the following conditions (1)-(6) are equivalent:

1. \( S \) is the disjoint sum of maximal Clifford subsemigroups of \( S \).
2. \( S \) is \( \gamma \)-compatible.
3. \( S \) is a matrix of Clifford semigroups, that is, \( S \) is a CS-matrix.
4. For the least matrix congruence (that is, the least rectangular band congruence) \( \sigma_S \) on \( S \), each \( \sigma_S \)-class is a Clifford subsemigroup.
5. \( S \) is an SL-matrix cryptogroup (that is, an SL-matrix of groups).
6. The relation \( \tau \) on \( S \) defined by

\[
xy \text{ if and only if } [x][y] = [y][x]
\]

is a matrix congruence, and \([x][y] = [y][x]\) if and only if \([xy][yx] = [yx][xy]\), where \([u]\) denotes the identity of the maximal subgroup \( H_u \) containing \( u \).

**Proof.** (2)\(\Rightarrow\)(1) follows from Lemma 2.1, and (1)\(\Rightarrow\)(3) has been shown in Pastijn [2]. Further, it is easy to see that (3)\(\Rightarrow\)(2). (3)\(\Rightarrow\)(4): Since \( S \) is a CS-matrix, there exists a matrix congruence \( \eta_S \) on \( S \) such that each \( \eta_S \)-class is a Clifford subsemigroup. Let \( e, f \) be idempotents of a \( \eta_S \)-class. Then, \( ef = fe \). Hence, \( e \sigma_S e = e \sigma_S f \). Therefore, \( e \sigma_S f \). Let \( x \sigma_S \) be the \( \eta_S \)-class containing \( x \in S \). For any \( y \in x \sigma_S \), there exists a unique inverse \( y' \) of \( y \) in \( x \sigma_S \). For \( a \in S \), let \( \bar{a} \) be the \( \eta_S \)-class containing \( a \). Now, \( xx' \sigma_S x'x \) implies \( xx' = \bar{x}'x \), and hence \( \bar{x} = x' \), that is, \( x \sigma_S x' \). Hence, \( xx' \sigma_S x \).

3) Several other characterizations of a CS-matrix have been also given by [3].
Let \( y \in x_{\mathcal{G}} \). Then, \( x_{\mathcal{G}} = y_{\mathcal{G}} \), and \( y_{\mathcal{G}} \sigma_{\mathcal{G}} y \). Since \( y_{\mathcal{G}} \eta_{\mathcal{G}} x_{\mathcal{G}} \), we have \( y_{\mathcal{G}} \sigma_{\mathcal{G}} x_{\mathcal{G}} \), and \( x \sigma_{\mathcal{G}} y \). Therefore, \( y \in x \sigma_{\mathcal{G}} \). Thus, \( x_{\mathcal{G}} \sigma_{\mathcal{G}} x_{\mathcal{G}} \). Since \( \sigma_{\mathcal{G}} \) is the least matrix congruence, \( \eta_{\mathcal{G}} = \sigma_{\mathcal{G}} \). (4)\( \Rightarrow \) (3): Obvious. (4)\( \Rightarrow \) (5): It is easy to see that \( \sigma_{\mathcal{G}} \cap \rho_{\mathcal{G}} = \mathcal{G} \), where \( \mathcal{G} \) is the Rees H-relation on \( S \). Put \( S / \mathcal{G} = A \) and \( S / \mathcal{G} = Y \). Then, \( S \) is \( H \)-compatible, and \( S / \mathcal{G} \) is isomorphic to a direct product \( A \times Y \) of \( A \) and \( Y \) (where \( \times \) denotes a direct product). Since \( A \times Y \) is an \( SL \)-structure and since each \( \mathcal{G} \)-class is a subgroup of \( S \), \( S \) is an \( SL \)-structure cryptogroup. (5)\( \Rightarrow \) (3): Obvious. (6)\( \Rightarrow \) (4): It is obvious that each \( \sigma_{\mathcal{G}} \)-class is a union of groups. Let \( e \sigma_{\mathcal{G}} f \) for \( e, f \in E(S) \). Hence, \( e \sigma_{\mathcal{G}} f \). Since \( \sigma_{\mathcal{G}} \) is the least matrix congruence, \( ef \). Therefore, \( [ef][fe] = [fe][ef] \) and \( [e][f] = [f][e] \), that is, \( ef = fe \). Thus, every \( \sigma_{\mathcal{G}} \)-class is an inverse semigroup, and hence it is a Clifford subsemigroup. (5)\( \Rightarrow \) (6): Let \( S \) be an \( SL \)-structure \( A \times Y \) of groups \( \{ H_{\xi} : (\alpha, \eta) \in A \times Y \} \), where \( A \), \( Y \) are a rectangular band and a semilattice respectively, and \( A \times Y \) is a direct product of \( A \) and \( Y \). It is obvious that \( H_{\xi} \) is an \( H \)-class of \( S \). Suppose that \( [x][y] = [y][x] \) for \( x, y \in S \). Then, \( [x][y] = [y][x] \). where \( \mathcal{G} \) is the Rees H-relation on \( S \). Hence, \( [xy][yx] = [yx][xy] \). Conversely, suppose that \( [xy][yx] = [yx][xy] \). There exist \( H_{\xi} \) such that \( x \in H_{\xi} \) and \( y \in H_{\xi} \). Then, \( x H_{\xi} y \) and \( y H_{\xi} x \). Hence, \( [x][y] = [y][x] \). where \( \delta \) is a Clifford subsemigroup of \( S \). Since \( x \in H_{\xi} \) and \( y \in H_{\xi} \) and since \( S_{\xi} \) is a Clifford subsemigroup, \( [x][y] = [y][x] \). Next, suppose that \( x \tau y \). Then, \( [x][y] = [y][x] \), and hence \( x, y \in S_{\xi} \). for some \( \lambda \in A \). Let \( \tau \) be the congruence on \( S \) which gives the decomposition of \( S \) into the Clifford subsemigroups \( \{ S_{\xi} : \xi \in A \} \). Then, \( \tau \) is a matrix congruence and satisfies \( \tau \subset \tau \). Conversely, it is obvious that \( \tau \subset \tau \). Accordingly, \( \tau = \tau \). Thus, \( \tau \) is a matrix congruence.

From the theorem above, it is easy to see that a matrix decomposition (that is, a rectangular band decomposition) of a \( CS \)-structure \( S \) into Clifford subsemigroups \( \{ C_{\xi} : \alpha \in \Gamma \} \) is unique, and it is given by the least matrix congruence \( \sigma_{\mathcal{G}} \) on \( S \). In this case, each \( C_{\xi} \) is a maximal Clifford subsemigroup of \( S \). Further, it is also obvious that \( \sigma_{\mathcal{G}} \) is a matrix decomposition (that is, a rectangular band decomposition) of a \( CS \)-structure \( S \) into Clifford subsemigroups \( \{ C_{\xi} : \alpha \in \Gamma \} \) is unique, and it is given by the least matrix congruence \( \sigma_{\mathcal{G}} \) on \( S \). In this case, each \( C_{\xi} \) is a maximal Clifford subsemigroup of \( S \). Further, it is also obvious that \( \sigma_{\mathcal{G}} \) is a matrix congruence (the restriction of \( \sigma_{\mathcal{G}} \) to \( E(S) \)). In this case, each \( C_{\xi} \) is a maximal Clifford subsemigroup of \( S \). Further, it is also obvious that \( \sigma_{\mathcal{G}} \) is a matrix congruence (the restriction of \( \sigma_{\mathcal{G}} \) to \( E(S) \)).

**Lemma 2.3.** Let \( S \) be a \( CS \)-structure, and \( \{ E_{\xi} : \lambda \in A \} \) the \( \gamma \)-classes of \( E(S) \). Then, \( E_{\xi} SE_{\lambda} \) is a Clifford subsemigroup of \( S \), and \( S = \cup \{ E_{\xi} SE_{\lambda} : \lambda \in A \} \).

**Proof.** For each \( \lambda \in A \), let \( S_{\lambda} = \{ x \in S : xx^*x^*x \in E_{\lambda} \} \). As was shown above, \( S_{\lambda} \) is a maximal Clifford subsemigroup. Hence, it is a \( \sigma_{\mathcal{G}} \)-class. Therefore, we can consider \( A \) as a rectangular band and \( S \) as a matrix \( A \) of the maximal Clifford subsemigroups \( \{ S_{\lambda} : \lambda \in A \} \). Since \( A \) is a rectangular band, \( E_{\xi} SE_{\lambda} \subset S_{\lambda} \) for \( \lambda \in A \). Conversely, let \( x \in S_{\lambda} \). Then, there exists a group inverse \( x^{-1} \) of \( x \) in \( S_{\lambda} \). Hence, \( xx^{-1}x = xx^{-1}x \in E_{\xi} \), and hence \( x = xx^{-1}x x^{-1}x \in E_{\xi} SE_{\lambda} \). Hence, \( E_{\xi} SE_{\lambda} = S_{\lambda} \).

To consider a description of all possible \( CS \)-structures, we need only to construct all

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4) A semigroup \( S \) is said to be \( H \)-compatible if Green's \( H \)-relation is a congruence on \( S \).
possible $SL$-matrices $\Omega$ of groups $\{N_\omega: \omega \in \Omega\}$ for a given $SL$-matrix $\Omega$ and given groups $\{H_\omega: \omega \in \Omega\}$. This can be obtained as a special case of Schein's theorem of [4] which has given a construction of bands of unipotent monoids (see also [5]). However, we omit to state it here again.

§ 3. Split CS-matrices

Let $M \equiv \Sigma\{M_\lambda: \lambda \in A\}$ be a CS-matrix, and $\rho_M$ the congruence which gives the structure decomposition $M \sim \Sigma\{M_\lambda: \lambda \in A\}$. We can consider $M/\rho_M = A$ by identifying each $\rho_M$-class $M_\lambda$ with $\lambda \in A$. Let $f$ be the natural homomorphism of $M$ onto $M/\rho_M = A$; that is, the homomorphism $f$ such that $xf = \lambda$ if $x \in M_\lambda$. If there exists a homomorphism $g$ of $A = M/\rho_M$ into $M$ such that $gf = e_A$ (the identity mapping on $A$), then $M$ is called split. Let $M \equiv \{M_\lambda: \lambda \in A\}$ be a CS-matrix. Then, there exists an $SL$-matrix $(L \times R) \bowtie A$, where $L$, $R$ are a left zero semigroup, a right zero semigroup, $L \times R$ the direct product of $L$ and $R$ and $(L \times R) \bowtie A$ a subdirect product of $L \times R$ and $A$, such that $M$ is an $SL$-matrix $(L \times R) \bowtie A$ of groups $\{H_\gamma(i,j): ((i, j), \alpha) \in (L \times R) \bowtie A\}$ (where each $H_\gamma(i,j)$ is an $H$-class of $M$). In this case, it is easy to see that $M_\lambda = \Sigma\{H_\gamma(i,j): (i, j) \in I_\lambda\}$, where $I_\lambda = \{(i, j): ((i, j), \lambda) \in (L \times R) \bowtie A\}$, for each $\lambda \in A$.

**Lemma 3.1.** $M$ splits if and only if

(3.1) there exists $(i, j) \in L \times R$ such that $H_\gamma(i,j)$ exists in $M$ for all $\alpha \in A$.

**Proof.** Suppose that $M$ satisfies (3.1). There exists $(i, j) \in L \times R$ such that $H_\gamma(i,j)$ exists in $M$ for all $\alpha \in A$. Let $C(i,j) = \cup \{H_\gamma(i,j): \alpha \in A\}$. Then, $C(i,j)$ is a Clifford subsemigroup. Let $e^\gamma(i,j)$ be the idempotent of $H_\gamma(i,j)$ for all $\alpha \in A$. Now, define $f: A \rightarrow M$ by $zf = e^\gamma(i,j)f$. Since $C(i,j)$ is a Clifford semigroup, $f$ is a homomorphism. On the other hand, the mapping $h: M \rightarrow A$ defined by $M.h = \{\alpha\}$ is a homomorphism of $M$ onto $A$. The congruence induced by $h$ is $\rho_M$. Since $fh = e_A$ (the identity mapping on $A$), $M$ splits. Conversely, suppose that $M$ splits. Then, there exists a surjective homomorphism $f: M/\rho_M \rightarrow M$ such that $fh = e_M$ (the identity mapping on $A$), $M$ splits. If we identify an element $M_\lambda$ of $M/\rho_M$ with $\lambda$, then we can consider $h$ and $f$ as a surjective homomorphism of $M$ onto $A$ and a homomorphism of $A$ into $M$ such that $fh = e_A$. For every $\alpha \in A$, let $\alpha f = e_s$. Then, $e_s \in M_s$. Hence, there exists $(u, v) \in L \times R$ such that $e_s = e^u$ and $((u, v), \alpha) \in (L \times R) \bowtie A$. For $\beta \in A$, similarly there exists $(s, k) \in L \times R$ such that $((s, k), \beta) \in (L \times R) \bowtie A$ and $e_k = e^s$. Now, $e_s e_k = (\alpha f)(\beta f) = (\alpha \beta f = (\beta \alpha f = (\beta f)(\alpha f)) = e_{\beta \alpha}$. Hence, $e_s^u e_k^k = e_s^s e_k^s$. Since $e_s^u e_k^k$, $e_s^s e_k^s$ are idempotents contained in $H^{\beta \alpha}(u, v)$, $H^{\alpha \beta}(s, k)$ respectively, $u = s$ and $k = v$. Thus, $(u, v) = (s, k)$. Consequently, $H^{\beta \alpha}(u, v)$ exists for every $\lambda \in A$.

Now, let $M \sim \Sigma\{M_\lambda: \lambda \in A\}$ be the above-mentioned split CS-matrix. Then, $M$ is an $SL$-matrix $(L \times R) \bowtie A$ of $H$-classes $\{H^{\alpha}(i,j): ((i, j), \alpha) \in (L \times R) \bowtie A\}$.
Further, there exists \((i, j) \in L \times R\) such that \(H_{\alpha}^{i, j}\) exists for all \(\alpha \in A\). Denote \((i, j)\) by (1, 1), and put \(C_{(1, 1)} = \bigcup \{H_{\alpha}^{i, j} : (s, k) \in I_\alpha\}\). For every \(x \in H_{(s, k)}^{i, j}\), \(x\) is uniquely written in the form \(x = e_{s1}^{i}ue_{s1}^{j}\), \(u \in H_{(s, 1)}^{1}\), where \(e_{sk}\) is the identity of \(H_{(s, k)}^{n}\) for every \((s, k, \alpha) \in (L \times R) \times A\) (see [1]).

For \(x = e_{s1}^{i}ue_{s1}^{j} \in H_{(s, 1)}^{1, j}, \ y = e_{s1}^{n}ve_{s1}^{n} \in H_{(s, 1)}^{n, k}\), \(u \in H_{(1, 1)}^{n}\) and \(v \in H_{(1, 1)}^{n}\),

\[
(3.2) \quad xy = e_{s1}^{i}ue_{s1}^{j}e_{s1}^{n}ve_{s1}^{n} = e_{s1}^{i}ue_{s1}^{n}p_{s1}^{(s, n)}e_{s1}^{n}ve_{s1}^{n},
\]

where \(e_{s1}^{i}e_{s1}^{n} = p_{s1}^{(s, n)} \in H_{(s, 1)}^{n}\).

Now,

\[
(3.2) = e_{s1}^{i}e_{s1}^{n}ue_{s1}^{n}p_{s1}^{(s, n)}e_{s1}^{n}ve_{s1}^{n}.
\]

Next, \((e_{s1}^{i}e_{s1}^{n})^2 = e_{s1}^{i}e_{s1}^{n}e_{s1}^{n} = e_{s1}^{i}e_{s1}^{n}\). Hence, \(e_{s1}^{i}e_{s1}^{n}\) is an idempotent of \(H_{(s, 1)}^{n}\), and hence \(e_{s1}^{i}e_{s1}^{n} = e_{s1}^{i}\). Thus, \((3.2) = e_{s1}^{i}ue_{s1}^{n}p_{s1}^{(s, n)}e_{s1}^{n}ve_{s1}^{n}\). Put \(e_{s1}^{n} = p_{s1}^{(s, n)}\).

Then, \(q_{s1}^{(s, n)} \in H_{(1, 1)}^{1}\), and \(q_{s1}^{(s, n)} = e_{s1}^{i}\) for all \(\alpha, \beta \in A\). Further, \((3.2) = e_{s1}^{i}ue_{s1}^{n}p_{s1}^{(s, n)}ve_{s1}^{n}\).

It is easy to see that \(uq_{s1}^{(s, n)}v \in C_{(1, 1)}\) and the product of \(u, q_{s1}^{(s, n)}\) and \(v\) can be obtained in the semigroup \(C_{(1, 1)}\). Hence, if we rewrite \(x, y\) in the form \(x = [u]_{s1}, \ y = [v]_{s1}\), then

\[
xy = [u]_{s1}[v]_{s1} = [uq_{s1}^{(s, n)}v]_{s1}^{n},
\]

and \(M = \{[u]_{s1} : ((i, j), \delta) \in (L \times R) \times A, u \in H_{(1, 1)}^{n}\}\).

Since \([u]_{s1}[v]_{s1}\)[\(r\)\]_{mn} = \([u]_{s1}[v]_{s1}\)[\(r\)\]_{mn}, \(uq_{s1}^{(s, n)}vq_{s1}^{(s, n)}\) is an element of \(H_{(1, 1)}^{n}\). For \(\alpha, \eta, j, s\) such that \(((1, j), \delta), ((s, 1), \eta) (L \times R) \times A\), let \(q_{s1}^{(s, n)}\) be an element of \(H_{(1, 1)}^{n}\). Put \(Q = \{q_{s1}^{(s, n)} : ((1, j), \delta), ((s, 1), \eta) (L \times R) \times A\}\) and assume that \(Q\) satisfies (3.3) for \(((1, j), \delta), ((s, 1), \eta), ((1, k), \eta) (L \times R) \times A\) and the following (3.4):

\[
(3.4) \quad q_{s1}^{(s, n)} = e_{s1}^{i} \quad \text{for all } \alpha, \beta \in A, \text{where } e_{s1}^{i} \text{ is the identity of } H_{(1, 1)}^{n}.
\]

In this case, if multiplication is defined in \(M\) by

\[
[u]_{s1}[v]_{s1}^{n} = [uq_{s1}^{(s, n)}v]_{s1}^{n},
\]
then $M$ becomes a split CS-matrix. The set $Q$ above is called the sandwich matrix of $M$ over the Clifford semigroup $C(A)$, and the split CS-matrix $M$ above is denoted by $\mathcal{A}((L \times R) \rtimes A; C(A); Q)$.

Now, it follows from the results above that:

**Theorem 3.2.** $\mathcal{A}((L \times R) \rtimes A; C(A); Q)$ is a split CS-matrix. Conversely, every split CS-matrix can be obtained in this way.

**Remark.** In Theorem 3.2, consider the case where $A$ consists of a single element $\alpha$ and $C(A)$ is a group $H_{(1,1)}^\alpha$. Then, $Q = \{q_{js}^{\alpha, \alpha} : (j, s) \in R \times L\}$. Denote $q_{js}^{\alpha, \alpha}$ simply by $q_{js}$, and $[u]_{ij}$ simply by $[u]_{ij}$. Then, $\mathcal{A}((L \times R) \rtimes \{\alpha\}; H_{(1,1)}^\alpha; Q) = \{[u]_{ij} : (i, j) \in L \times R\}$ and

$$[u]_{ij}[v]_{kl} = [uq_{jk}v]_{ls}.$$  
That is, it is the regular Rees $L \times R$-matrix semigroup with sandwich matrix $Q$ over the group $H_{(1,1)}^\alpha$. Hence, $\mathcal{A}((L \times R) \rtimes A; C(A); Q)$ in Theorem 3.2 is a generalization of the concept of a regular Rees matrix semigroup.

**References**