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LAGRANGIAN MINIMAL SURFACES WITH 1-PARAMETER FAMILY OF PAIR OF GREAT CIRCLES IN $S^2 \times S^2$

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Abstract. This paper gives the condition for surfaces with 1-parameter family of pair of great circles in $S^2 \times S^2$ to be Lagrangian. And we give conditions for those surfaces to be minimal in $S^2 \times S^2$.

1. Introduction

Lagrangian submanifolds in Kähler manifolds have been studied as very interested subjects in differential geometry. In particular, for Lagrangian submanifolds in a complex projective space with a constant holomorphic sectional curvature, many facts are known. For example, the existence of the Lagrangian immersion and their congruence. On the other hand, for general Hermitian symmetric spaces, it seems that there are no such results with respect to Lagrangian submanifolds. Let S^2 be a Riemannian sphere with the induced metric from $\mathbf{R}^3$. The Riemannian product $S^2 \times S^2$ of unit 2-spheres is a Hermitian symmetric space which has complex dimension 2 and rank 2. In this paper, we especially consider Lagrangian surfaces (in particular, minimal surfaces) in $S^2 \times S^2$ which is important next to complex projective planes CP^2 among compact Kähler surfaces of complex dimension 2.

Here, there are the following examples for Lagrangian surfaces in $S^2 \times S^2$: (i) A surface which consists of two curves γ_1 and γ_2 in S^2 embedded in $S^2 \times S^2$ by a product immersion. In this example, the surface is minimal if and only if both γ_i $(i = 1, 2)$ are great circles. (ii) Identify S^2 with a complex projective line CP^1 and be corresponded an element z of CP^1 to a pair of z and the complex conjugate $\bar{z}$, then we can get a totally geodesic Lagrangian surface in $S^2 \times S^2$.

In this paper, we first give a surface M of $S^2 \times S^2$ which consists of 1-parameter family of some geodesic on $S^2 \times S^2$. Then we get the condition for the surface M to be Lagrangian in $S^2 \times S^2$ (Proposition 1). Moreover we show that if such a Lagrangian surface M is minimal, then M is totally geodesic (Theorem of §4).

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2. Preliminaries

Let $\widetilde{M}$ be a Kähler manifold of complex dimension m with a complex structure J and let M be a real m dimensional submanifold of $\widetilde{M}$. Then M is called Lagrangian if for any tangent vector X of M, JX is contained in the normal space to M.

Now we consider Riemannian product $S^2 \times S^2$ of unit spheres S^2 in $\mathbf{R}^3$. For any $p \in S^2$, we define a linear transformation $\tilde{J}$ of the tangent space T_pS^2 at p as $\tilde{J}v = p \times v$ by the vector product $\times$ of $\mathbf{R}^3$, so $\tilde{J}$ is a complex structure on S^2. We can define a complex structure J on $S^2 \times S^2$ by

$$J(X_1, X_2) = (\tilde{J}X_1, \tilde{J}X_2) \tag{1}$$

for any tangent vector (X_1, X_2) to $S^2 \times S^2$. We define a Riemannian metric $\langle\ ,\ \rangle$ on $S^2 \times S^2$ by

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle = X_1 \cdot X_2 + Y_1 \cdot Y_2$$

where $\cdot$ is a Riemannian metric on $S^2 \subset \mathbf{R}^3$. Then $\langle\ ,\ \rangle$ is a Hermitian metric and $S^2 \times S^2$ is a Kähler manifold. We also denote $\cdot$ by $\langle\ ,\ \rangle$ in below.

3. Lagrangian surfaces in $S^2 \times S^2$

Let $\gamma_i (i = 1, 2)$ be great circles in S^2. Then (γ_1, γ_2) is a geodesic in $S^2 \times S^2$. We put a set of all such pairs (γ_1, γ_2) as $\mathcal{M}$. We denote that $SO(n)$ is a special orthogonal group. Because $SO(3) \times SO(3)$ acts transitively $\mathcal{M}$, $\mathcal{M}$ is a homogeneous space of $SO(3) \times SO(3)$. If K is a set of all elements of $SO(3) \times SO(3)$ which preserve $\gamma \in \mathcal{M}$, $K = \{(g, g) | g \in SO(2)\}$ and we can identify $SO(3) \times SO(3)/K$ with $\mathcal{M}$. We define the natural projection by

$$\pi : SO(3) \times SO(3) \to \mathcal{M}. \tag{2}$$

Moreover, when we give naturally the two-sided invariant Riemannian metric for $SO(3) \times SO(3)$, we can introduce the Riemannian metric into $\mathcal{M}$ such as π is a Riemannian submersion. Let φ be a curve from a open interval I into $SO(3) \times SO(3)/K$ and $\widetilde{\varphi}(s) = (g_1(s), g_2(s)) \in SO(3) \times SO(3)$ be a horizontal lift of φ with respect to π. Then, we define a map $\Phi : I \times S^1 \to S^2 \times S^2$ by $\Phi(s, t) = (\varphi(s)\gamma(t))$. We consider differentiations $g_1'(s)$ and $g_2'(s)$ of $g_1(s)$ and $g_2(s)$ by s. We can easily see that $g_1^{-1}(s)g_1'(s)$ is a skew-symmetric matrix of degree 3. Since we see also about $g_2(s)$, we put for some functions a_i, b_i and $c_i (i = 1, 2)$ with respect to s

$$g_1^{-1}(s)g_1'(s) = \begin{pmatrix} 0 & a_1(s) & b_1(s) \\ -a_1(s) & 0 & c_1(s) \\ -b_1(s) & -c_1(s) & 0 \end{pmatrix},$$

$$g_2^{-1}(s)g_2'(s) = \begin{pmatrix} 0 & a_2(s) & b_2(s) \\ -a_2(s) & 0 & c_2(s) \\ -b_2(s) & -c_2(s) & 0 \end{pmatrix}.$$

We express any elements of K by

$$\left(\begin{pmatrix}\cos s & -\sin s & 0\\ \sin s & \cos s & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}\cos s & -\sin s & 0\\ \sin s & \cos s & 0\\ 0 & 0 & 1\end{pmatrix}\right),$$

then the tangent space of K is a space of dimension 1 spanned by

$$\left(\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right). \tag{3}$$

Since $\widetilde{\varphi}(s) = (g_1(s), g_2(s)) \in SO(3) \times SO(3)$ is horizontal lift of φ to the fiber K, $(g_1^{-1}(s)g_1'(s), g_2^{-1}(s)g_2'(s)) \in \mathfrak{o}(3) \times \mathfrak{o}(3)$ is orthogonal to (3). Hence we have $a_1(s) + a_2(s) = 0$. So we put $a = a_1 = -a_2$.

Let $\Phi : I \times S^1 \to S^2 \times S^2$ be the map defined by

$$\Phi(s,t) = \left(g_1(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}, g_2(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}\right). \tag{4}$$

For

$$\Phi_t := \frac{\partial \Phi}{\partial t} = \left(g_1(s)\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}, g_2(s)\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}\right),$$

and

$$\begin{aligned}\Phi_s := \frac{\partial \Phi}{\partial s} \quad = \quad & \left(g_1(s)\begin{pmatrix}a(s)\cos t\\ -a(s)\sin t\\ -b_1(s)\cos t - c_1(s)\sin t\end{pmatrix},\right.\\ & \left. g_2(s)\begin{pmatrix}-a(s)\cos t\\ a(s)\sin t\\ -b_2(s)\cos t - c_2(s)\sin t\end{pmatrix}\right),\end{aligned} \tag{5}$$

$g_1(s), g_2(s) \in SO(3)$ preserve the Riemannian metric $\langle\ ,\ \rangle$ of $S^2 \times S^2$, so we get $\langle \Phi_t, \Phi_t\rangle = 2$, $\langle \Phi_t, \Phi_s\rangle = 0$. Hence the condition for Φ to be regular at (s,t) is $\langle \Phi_s, \Phi_s\rangle \neq 0$. We get from (5),

$$\begin{aligned}\langle \Phi_s, \Phi_s\rangle &= 2a(s)^2 + \{b_1(s)\cos t + c_1(s)\sin t\}^2 + \{b_2(s)\cos t + c_2(s)\sin t\}^2\\ &= 2a(s)^2 + \{b_1(s)^2 + b_2(s)^2\}\cos^2 t + \{c_1(s)^2 + c_2(s)^2\}\sin^2 t\\ &\quad + 2\{b_1(s)c_1(s) + b_2(s)c_2(s)\}\cos t \sin t,\end{aligned}$$

so we put the right hand of this equation as $f(t)$:

$$\begin{aligned}f(t) = 2a(s)^2 + \{b_1(s)^2 + b_2(s)^2\}\cos^2 t + \{c_1(s)^2 + c_2(s)^2\}\sin^2 t\\ + 2\{b_1(s)c_1(s) + b_2(s)c_2(s)\}\cos t \sin t.\end{aligned} \tag{6}$$

Now, we consider the conditions for the immersion Φ to be Lagrangian with respect to the complex structure J defined by (1), i.e., the conditions of $\langle J\Phi_t, \Phi_s\rangle =$

$\langle J\Phi_s, \Phi_t\rangle = 0$. We can express $J\Phi_t$ by the vector product $\times$ of $\mathbf{R}^3$ as

(7)
$$J\Phi_t = \left(g_1(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix} \times g_1(s)\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}, g_2(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix} \times g_2(s)\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix}\right).$$

So we get

$$\begin{aligned}\langle J\Phi_t, \Phi_s\rangle &= \det g_1(s)\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}a(s)\sin t\\ -a(s)\cos t\\ -b_1(s)\cos t - c_1(s)\sin t\end{pmatrix}\right\rangle \\ &\quad + \det g_2(s)\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-a(s)\sin t\\ a(s)\cos t\\ -b_2(s)\cos t - c_2(s)\sin t\end{pmatrix}\right\rangle \\ &= -\{b_1(s)+b_2(s)\}\cos t - \{c_1(s)+c_2(s)\}\sin t.\end{aligned}$$

Hence, the conditions for Φ to be a Lagrangian immersion are $b_1(s)+b_2(s)=0$ and $c_1(s)+c_2(s)=0$. So we put $b=b_1=-b_2$ and $c=c_1=-c_2$. Moreover, we take a parameter s as $a^2+b^2+c^2=1$. Then, (6) is

$$f(t) = 1 + a(s)^2 + \{b(s)^2 - c(s)^2\}\cos 2t + 2b(s)c(s)\sin 2t. \tag{8}$$

If we suppose $b(s)=c(s)=0$ for $f(t)$ (then we get $a=1$), $f(t)=2$ for any $t\in S^1$. If we suppose $b(s)\neq 0$ or $c(s)\neq 0$, at t satisfying $(df/dt)(t)=0$, we have

$$\cos 2t = \pm\frac{b(s)^2 - c(s)^2}{b(s)^2+c(s)^2}, \qquad \sin 2t = \pm\frac{2b(s)c(s)}{b(s)^2+c(s)^2}.$$

Then, at the extremal value t, we get $f(t)=2$ or $2a(s)^2$. Since $2 > 2a^2 \geq 0$, if $a(s)\neq 0$ for arbitrary $s\in I$, $f(s,t)>0$ for all $t\in S^1$.

Hence, we have

Proposition 1. $\mathcal{M} = \{\gamma = (\gamma_1,\gamma_2) | \gamma_i (i=1,2)$ *are great circles on* S^2 $\}$. *Let* $\varphi : I \to \mathcal{M}$ *be a curve and* $\widetilde{\varphi}(s) = (g_1(s), g_2(s)) \in SO(3)\times SO(3)$ *be a horizontal lift of* φ *about the projection* $\pi : SO(3)\times SO(3) \to \mathcal{M}$ *of* (2). *Define* $\Phi : I\times S^1 \to S^2\times S^2$ *by* $\Phi(s,t) = \varphi(s)\gamma(t)$. *If there exist functions* $a>0$ *and* b, c *on* I *which satisfy*

$$g_1^{-1}(s)g_1'(s) = -g_2^{-1}(s)g_2'(s) = \begin{pmatrix}0 & a(s) & b(s)\\ -a(s) & 0 & c(s)\\ -b(s) & -c(s) & 0\end{pmatrix},$$

Φ *is a Lagrangian immersion.*

Now we review the almost product structure of $S^2\times S^2$(cf. [2]). The almost product structure $\overline{P}$ of $S^2\times S^2$ is defined by

$$\overline{P}(X_1,X_2) = (X_1,-X_2) \qquad \text{for} \quad (X_1,X_2)\in S^2\times S^2.$$

If M is a Lagrangian surface of $S^2\times S^2$, for the almost product structure $\overline{P}$ we have the following ([1]):

Lemma . *Let $x : M \to S^2 \times S^2$ be a Lagrangian immersion and $\overline{P}$ be an almost product structure of $S^2 \times S^2$. If the vector $\overline{P}X$ is othogonal to the tangent space to M for any tangent vector X of M, then the immersion x is totally geodesic and the Gauss curvature K on M satisfies $K \equiv 1/2$.*

Here, we suppose $a(s) = 0$ for the curve φ in M. Since

$$\overline{P}\Phi_s = \left(g_1(s) \begin{pmatrix} 0 \\ 0 \\ -b(s)\cos t - c(s)\sin t \end{pmatrix}, g_2(s) \begin{pmatrix} 0 \\ 0 \\ -b(s)\cos t - c(s)\sin t \end{pmatrix} \right),$$

we get

$$\langle \overline{P}\Phi_s, \Phi_t \rangle = 0 \quad \text{and} \quad \langle \overline{P}\Phi_s, \Phi_s \rangle = 0.$$

So $\overline{P}\Phi_s$ is a normal vector to $\mathcal{M}$. Hence we get from the Lemma above

Proposition 2. *Define a Lagrangian immersion $\Phi : I \times S^1 \to S^2 \times S^2$ as*

$$\Phi(s,t) = \left(g_1(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, g_2(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right)$$

where $g_1(s), g_2(s) \in SO(3)$ satisfy

$$g_1^{-1}(s) g_1'(s) = -g_2^{-1}(s) g_2'(s) = \begin{pmatrix} 0 & 0 & b(s) \\ 0 & 0 & c(s) \\ -b(s) & -c(s) & 0 \end{pmatrix}$$

for functions b and c on I. Then the immersion Φ is totally geodesic.

4. LAGRANGIAN MINIMAL SURFACES IN $S^2 \times S^2$

Firstly, we consider the case that functions a, b and c are constant where a, b and c satisfy

$$g_1^{-1}(s) g_1'(s) = -g_2^{-1}(s) g_2'(s) = \begin{pmatrix} 0 & a(s) & b(s) \\ -a(s) & 0 & c(s) \\ -b(s) & -c(s) & 0 \end{pmatrix} =: A.$$

Note that solutions of $g_1'(s) = g_1(s)A$ and $g_2'(s) = g_2(s)(-A)$ are

$$g_1(s) = g_1(0)\exp(sA) \qquad \text{and} \qquad g_2(s) = g_2(0)\exp(-sA).$$

So we suppose that $g_1(0)$ and $g_2(0)$ are a unit matrix of degree 3. Then

$$g_1(s) = \exp(sA) \qquad \text{and} \qquad g_2(s) = \exp(-sA). \tag{9}$$

Example 1. *A case of $a = 1$ and $b = c = 0$.*

In this case, we have

$$\exp(sA) = \exp \begin{pmatrix} 0 & s & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We calculate similarly for $\exp(-sA)$. *So we get*

$$(g_1(s), g_2(s)) = \left(\begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

Then (4) *is*

$$\Phi(s,t) = \left(\begin{pmatrix} \cos(t-s) \\ \sin(t-s) \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(t+s) \\ \sin(t+s) \\ 0 \end{pmatrix} \right).$$

Since we can regard that $t-s$ *and* $t+s$ *is independent variable,* Φ *is a product immersion and* Φ *is a totally geodesic immersion. In fact, we put*

$$\Phi_1 := \left(\begin{pmatrix} \cos(t-s) \\ \sin(t-s) \\ 0 \end{pmatrix}, 0 \right), \ \Phi_2 := \left(0, \begin{pmatrix} \cos(t+s) \\ \sin(t+s) \\ 0 \end{pmatrix} \right),$$

then they are unit normal vectors to $S^2 \times S^2 \subset \mathbf{R}^3 \times \mathbf{R}^3$. *Hence for* $X \in \mathbf{R}^3 \times \mathbf{R}^3$, *if we denote* $X^\perp$ *as the normal component of* X *to* $S^1 \times S^1 \subset S^2 \times S^2$,

$$\begin{aligned} \sigma(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= (\Phi_{tt})^\perp = (-\Phi)^\perp = 0, \\ \sigma(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}) &= (\Phi_{ts})^\perp = (\Phi_1 - \Phi_2)^\perp = 0, \\ \sigma(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) &= (\Phi_{ss})^\perp = (-\Phi)^\perp = 0 \end{aligned}$$

where σ *is the second fundamental form of* Φ. *Therefore we have* $\sigma \equiv 0$, *i.e.,* Φ *is a totally geodesic immersion.*

Example 2. *A case of* $a = c = 0$ *and* $b = 1$.

By calculating also Example1, *we get*

$$(g_1(s), g_2(s)) = \left(\begin{pmatrix} \cos s & 0 & \sin s \\ 0 & 1 & 0 \\ -\sin s & 0 & \cos s \end{pmatrix}, \begin{pmatrix} \cos s & 0 & -\sin s \\ 0 & 1 & 0 \\ \sin s & 0 & \cos s \end{pmatrix} \right).$$

So we have

$$\Phi(s,t) = \left(\begin{pmatrix} \cos s \cos t \\ \sin t \\ -\sin s \cos t \end{pmatrix}, \begin{pmatrix} \cos s \cos t \\ \sin t \\ \sin s \cos t \end{pmatrix} \right).$$

Hence $\Phi : S^2 \to S^2 \times S^2$ *satisfies*

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\begin{pmatrix} x \\ y \\ -z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right).$$

Then we have

$$\begin{aligned} \Phi_t &= \left(\begin{pmatrix} -\cos s \sin t \\ \cos t \\ \sin s \sin t \end{pmatrix}, \begin{pmatrix} -\cos s \sin t \\ \cos t \\ -\sin s \sin t \end{pmatrix} \right), \\ \Phi_s &= \left(\begin{pmatrix} -\sin s \cos t \\ 0 \\ -\cos s \cos t \end{pmatrix}, \begin{pmatrix} -\sin s \cos t \\ 0 \\ \cos s \cos t \end{pmatrix} \right), \end{aligned}$$

so we get

$$\langle \Phi_t, \Phi_t \rangle = 2, \quad \langle \Phi_t, \Phi_s \rangle = 0 \quad \langle \Phi_s, \Phi_s \rangle = 2\cos^2 t.$$

Hence, the condition for Φ *to be a regular is* $t \not\equiv \pi/2(\mod \pi)$. *Then, we have for the second fundamental form of* Φ

$$\begin{aligned} \sigma(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= (\Phi_{tt})^\perp = (-\Phi)^\perp = 0, \\ \sigma(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}) &= (\Phi_{ts})^\perp = (-\tan t \ \Phi_s)^\perp = 0, \\ \sigma(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) &= (\Phi_{ss})^\perp = (\Phi_{ss} + \cos^2 t \ \Phi - \cos t \sin t \ \Phi_t)^\perp = 0, \end{aligned}$$

and so Φ *is totally geodesic.*

Example 3. *A case of* a, b *and* c *are general constant real numbers satisfying* $a^2 + b^2 + c^2 = 1$.

We have from (9)

$$\Phi(s,t) = \left(\exp(sA) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \exp(-sA) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right),$$

so we get

$$\begin{aligned} \Phi_t &= \left(\exp(sA) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}, \exp(-sA) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \right), \\ \Phi_s &= \left(\exp(sA)A \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, -\exp(-sA)A \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \right) \\ &= \left(\exp(sA) \begin{pmatrix} a \sin t \\ -a \cos t \\ -b \cos t - c \sin t \end{pmatrix}, \exp(-sA) \begin{pmatrix} -a \sin t \\ a \cos t \\ b \cos t + c \sin t \end{pmatrix} \right). \end{aligned}$$

Since A *and* $-A$ *are skew-symmetric matrices,* $\exp(sA)$ *and* $\exp(-sA)$ *are orthogonal matrices. Then these matrices preserve the Riemannian metric* $\langle \ , \ \rangle$. *Therefore, we get*

$$\begin{gathered} \langle \Phi_t, \Phi_t \rangle = 2, \qquad \langle \Phi_t, \Phi_s \rangle = 0, \\ \langle \Phi_s, \Phi_s \rangle = 1 + a^2 + (b^2 - c^2) \cos 2t + 2bc \sin 2t. \end{gathered}$$

We put also Example1, 2 *that*

$$\Phi_1 := \left(\exp(sA)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix},\ 0\right),\ \Phi_2 := \left(0\ ,\exp(-sA)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}\right),$$

and so Φ_1 *and* Φ_2 *are unit normal vectors to* $S^2\times S^2\subset \mathbf{R}^3\times\mathbf{R}^3$. *Then we consider the condition for* Φ *to be a minimal immersion. Since*

$$\sigma(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = (\Phi_{tt})^{\perp} = (-\Phi)^{\perp} = 0,$$

the condition for Φ *to be minimal is* $\sigma(\partial/\partial s,\partial/\partial s)=0$. *Now, for*

$$\Phi_{ss} = \left(\exp(sA)\begin{pmatrix}(c^2-1)\cos t - bc\sin t\\ (b^2-1)\sin t - bc\cos t\\ -ab\sin t + ac\cos t\end{pmatrix},\exp(-sA)\begin{pmatrix}(c^2-1)\cos t - bc\sin t\\ (b^2-1)\sin t - bc\cos t\\ -ab\sin t + ac\cos t\end{pmatrix}\right),$$

we have

$$\begin{aligned}\langle\Phi_{ss},\Phi_1\rangle &= -\frac{1}{2}\{1+a^2+(b^2-c^2)\cos 2t + 2bc\sin 2t\},\\ \langle\Phi_{ss},\Phi_2\rangle &= -\frac{1}{2}\{1+a^2+(b^2-c^2)\cos 2t + 2bc\sin 2t\},\\ \langle\Phi_{ss},\Phi_t\rangle &= (b^2-c^2)\sin 2t - 2bc\cos 2t,\\ \langle\Phi_{ss},\Phi_s\rangle &= 0.\end{aligned}$$

Hence, we get

$$\begin{aligned}\sigma(\frac{\partial}{\partial s},\frac{\partial}{\partial s}) &= (\Phi_{ss})^{\perp}\\ &= \Phi_{ss} + \frac{1}{2}\{1+a^2+(b^2-c^2)\cos 2t + 2bc\sin 2t\}\Phi\\ &\quad -\frac{1}{2}\{(b^2-c^2)\sin 2t - 2bc\cos 2t\}\Phi_t\\ &= \left(\exp(sA)\begin{pmatrix}0\\ 0\\ a(c\cos t - b\sin t)\end{pmatrix},\exp(-sA)\begin{pmatrix}0\\ 0\\ a(c\cos t - b\sin t)\end{pmatrix}\right).\end{aligned}$$

So the immersion Φ *is minimal if and only if* $a(c\cos t - b\sin t)=0$, *i.e.,* $a=0$ *or* $a\neq 0$ *and* $b=c=0$.

Therefore, we have from Proposition 2 and Example 1

Proposition 3. *A Lagrangian immersion* $\Phi : I\times S^1\to S^2\times S^2$ *is defined by*

$$\Phi(s,t) = \left(g_1(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix},g_2(s)\begin{pmatrix}\cos t\\ \sin t\\ 0\end{pmatrix}\right)$$

where $g_1(s), g_2(s) \in SO(3)$ *satisfy*

$$g_1^{-1}(s)g_1'(s) = -g_2^{-1}(s)g_2'(s) = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

for constant real numbers a, b *and* c. *The immersion* Φ *is minimal if and only if* $a = 0$ *or* $a \neq 0$ *and* $b = c = 0$. *And then the immersion* Φ *is totally geodesic.*

Next, we consider the condition for the Lagrangian immersion $\Phi(s,t)$ defined by functions $a(s), b(s)$ and $c(s)$, which are not necessarily constant, to be minimal. We get $\sigma(\partial/\partial t, \partial/\partial t) = 0$ such as the case of a, b and c are constant. So the condition for Φ to be minimal immersion is $\sigma(\partial/\partial s, \partial/\partial s) = 0$, i.e.,

$$\langle \Phi_{ss}, J\Phi_t \rangle = \langle \Phi_{ss}, J\Phi_s \rangle = 0.$$

By straightforward computation, we get

$$\begin{aligned} \Phi_{ss} \;=\; & \left(g_1(s) \begin{pmatrix} -\{a(s)^2 + b(s)^2\} \cos t + \{-b(s)c(s) + a'(s)\} \sin t \\ -\{b(s)c(s) + a'(s)\} \cos t - \{(a(s)^2 + c(s)^2\} \sin t \\ \{a(s)c(s) - b'(s)\} \cos t - \{a(s)b(s) + c'(s)\} \sin t \end{pmatrix}, \right. \\ & \left. g_2(s) \begin{pmatrix} -\{a(s)^2 + b(s)^2\} \cos t - \{b(s)c(s) + a'(s)\} \sin t \\ \{-b(s)c(s) + a'(s)\} \cos t - \{a(s)^2 + c(s)^2\} \sin t \\ \{a(s)c(s) + b'(s)\} \cos t + \{-a(s)b(s) + c'(s)\} \sin t \end{pmatrix} \right). \end{aligned}$$

So we have from (7)

$$\langle \Phi_{ss}, J\Phi_t \rangle = 2a(s)\{-b(s) \sin t + c(s) \cos t\}. \tag{10}$$

Hence $\langle \Phi_{ss}, J\Phi_t \rangle = 0$ if and only if $a(s) = 0$ or $a(s) \neq 0$ and $b(s) = c(s) = 0$. And for

$$\begin{aligned} J\Phi_s \;=\; & \left(g_1(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \times g_1(s) \begin{pmatrix} a(s) \sin t \\ -a(s) \cos t \\ -b(s) \cos t - c(s) \sin t \end{pmatrix}, \right. \\ & \left. g_2(s) \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \times g_2(s) \begin{pmatrix} -a(s) \sin t \\ a(s) \cos t \\ b(s) \cos t + c(s) \sin t \end{pmatrix} \right), \end{aligned}$$

we get

$$\langle \Phi_{ss}, J\Phi_s \rangle = -2a'(s)\{b(s) \cos t + c(t) \sin t\} + 2a(s)\{b'(s) \cos t + c'(s) \sin t\}. \tag{11}$$

If $a(s) = 0$ or $b(s) = c(s) = 0$,

$$\langle \Phi_{ss}, J\Phi_s \rangle = 0.$$

Hence, we get the following result for the Lagrangian minimal immersion.

Theorem . *Lagrangian minimal surfaces in* $S^2 \times S^2$ *which consist of* 1*- parameter family of pair* (γ_1, γ_2) *where* γ_1 *and* γ_2 *are great circles in* S^2 *are totally geodesic and they are locally congruent to either* (*a*) $S^1 \times S^1 \subset S^2 \times S^2$ *or* (*b*) $S^2 \subset S^2 \times S^2$.

References

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