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ISOTROPIC IMMERSIONS WITH LOW CODIMENSION OF SPACE FORMS INTO SPACE FORMS

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Abstract. It is known that all parallel immersions of space forms into space forms are isotropic in the sense of O'Neill. We characterize these parallel immersions with low codimension in terms of isotropic immersions. This is an improvement of S. Maeda's result [M].

1. Introduction

Let $f : M^n \longrightarrow \widetilde{M}^{n+p}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$. We recall the notion of isotropic immersion [O]: Let σ be the second fundamental form of f. The immersion f is said to be *isotropic* at $x \in M^n$ if $||\sigma(X,X)||/||X||^2$ is constant for all vectors $X(\neq 0)$ tangent to M^n at x. If the immersion is isotropic at every point, then we find a function λ on M^n defined by $x(\in M^n) \mapsto ||\sigma(X,X)||/||X||^2$ and the immersion f is said to be λ-isotropic or simply, isotropic. Note that totally umbilic immersions are isotropic, but not vice versa. We here remark that in the case that the codimension $p = 1$, isotropic immersions are nothing but totally umbilic immersions.

It is known that all parallel immersions of space forms into space forms are isotropic. On the other hand, there exist many isotropic immersions of space forms into space forms,which are not parallel [T].

A space form $M^n(c)$ is a Riemannian manifold of constant sectional curvature c which is locally isometric to one of the standard sphere $S^n(c)$, Euclidean space $\mathbb{R}^n$ and hyperbolic space $H^n(c)$.

In this paper, we characterize all parallel immersions of space forms $M^n(c)$ into space forms $\widetilde{M}^{n+p}(\tilde{c})$ with low codimension by using the notion of isotropic. The purpose of this paper is to prove the following:

Theorem . *Let f be a λ-isotropic immersion of an $n(\geq 2)$-dimensional space form $M^n(c)$ of constant sectional curvature c into an $(n+p)$-dimensional space*

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form $\widetilde{M}^{n+p}(\tilde{c})$ *of constant sectional curvature* $\tilde{c}$. *Suppose that*

$$p \leq \frac{1}{2}n(n+1) - 1.$$

Then f *is a parallel immersion. Moreover* f *is locally equivalent to one of the following:*

(I) f *is a totally umbilic immersion of* $M^n(c)$ *into* $\widetilde{M}^{n+p}(\tilde{c})$, *where* $c \geq \tilde{c}$ *and* $p \leq (n(n+1)/2) - 1$.

(II) f *is the second standard minimal immersion of* $M^n(c) = S^n(c)$ *into* $\widetilde{M}^{n+p}(\tilde{c}) = S^{n+p}(\tilde{c})$, *where* $\tilde{c} = 2(n+1)c/n$ *and* $p = (n(n+1)/2) - 1$.

2. Basic terminology

We recall terminology in this paper. Let $f : M^n \longrightarrow \widetilde{M}^{n+p}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ with metric $\langle , \rangle$. We denote by ∇ (resp. $\widetilde{\nabla}$) the Levi-Civita connection of the tangent bundle TM^n (resp. $T\widetilde{M}^{n+p}$). The second fundamental form σ of f is defined by $\sigma(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$ for $\forall X, Y \in \mathfrak{X}(M^n)$, where $\mathfrak{X}(M^n)$ denotes the Lie algebra of all vector fields on M^n. The curvature tensor R of M^n is defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for $\forall X, Y, Z \in \mathfrak{X}(M^n)$. For a vector field ξ normal to M^n, we write $\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi$ for $\forall X \in \mathfrak{X}(M^n)$, where $-A_\xi X$ (resp. $D_X \xi$) denotes the tangential (resp. the normal) component of $\widetilde{\nabla}_X \xi$. We define the covariant differentiation ∇' of the second fundamental form σ with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) as follows:

$$(\nabla'_X \sigma)(Y,Z) = D_X\big(\sigma(Y,Z)\big) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

where $X, Y, Z \in \mathfrak{X}(M^n)$. The immersion f is said to be *parallel* if $\nabla'\sigma = 0$. Now, we choose a local field of orthonormal frames $\{e_1, \cdots, e_n\}$ on M^n and define the mean curvature vector field $\mathfrak{h}$ as $\mathfrak{h} = \sum_{i=1}^n (1/n)\sigma(e_i, e_i)$. The immersion f is said to be *totally umbilic* if $\sigma(X,Y) = \langle X, Y \rangle \mathfrak{h}$ for $\forall X, Y \in \mathfrak{X}(M^n)$. The immersion f is said to be *minimal* if $\mathfrak{h} = 0$.

In case that $M^n = M^n(c)$ and $\widetilde{M}^{n+p} = \widetilde{M}^{n+p}(\tilde{c})$, Gauss and Codazzi equations are reduced to the following:

$$\begin{aligned} \langle \sigma(X,Y), \sigma(Z,W) \rangle &- \langle \sigma(Z,Y), \sigma(X,W) \rangle \\ &= (c - \tilde{c})\{\langle X,Y \rangle \langle Z,W \rangle - \langle Z,Y \rangle \langle X,W \rangle\}, \end{aligned} \tag{2.1}$$

$$(\nabla'_X \sigma)(Y,Z) = (\nabla'_Y \sigma)(X,Z), \tag{2.2}$$

where $X, Y, Z, W \in \mathfrak{X}(M^n(c))$.

3. Proof of Theorem

First of all, we prepare the following lemma.

Lemma ([O]). *Let f be a $\lambda(>0)$-isotropic immersion of a Riemannian manifold M into a Riemannian manifold $\widetilde{M}$. The discriminant Δ_x at $x \in M$ is defined by $\Delta_x = K(X,Y) - \widetilde{K}(X,Y)$, where $K(X,Y)$ (resp. $\widetilde{K}(X,Y)$) represents the sectional curvature of the plane spanned by $X, Y \in T_xM$ for M (resp. $\widetilde{M}$). Suppose that the discriminant Δ_x at $x \in M$ is constant. Then the following inequalities hold at x:*

$$-\frac{n+2}{2(n-1)}\lambda(x)^2 \leq \Delta_x \leq \lambda(x)^2.$$

Moreover,

(i) $\Delta_x = \lambda(x)^2 \iff f$ *is umbilic at* $x \iff \dim N_x^1 = 1$,
(ii) $\Delta_x = -\{(n+2)/2(n-1)\}\lambda(x)^2 \iff f$ *is minimal at* $x \iff \dim N_x^1 = (n(n+1)/2) - 1$,
(iii) $-\{(n+2)/2(n-1)\}\lambda(x)^2 < \Delta_x < \lambda(x)^2 \iff \dim N_x^1 = n(n+1)/2$.

Here, we denote by N_x^1 the first normal space at x, that is $N_x^1 = Span_{\mathbb{R}}\{\sigma(X,Y) : X, Y \in T_xM\}$.

Now, we shall prove our theorem.

Proof of Theorem. By the hypothesis, for $\forall X \in \mathfrak{X}(M^n(c))$ we have $\langle \sigma(X,X), \sigma(X,X)\rangle = \lambda^2\langle X,X\rangle\langle X,X\rangle$, which is equivalent to

$$\begin{aligned}&\langle \sigma(X,Y), \sigma(Z,W)\rangle + \langle \sigma(X,Z), \sigma(W,Y)\rangle + \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ &= \lambda^2\{\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\}\end{aligned} \tag{3.1}$$

for $\forall X, Y, Z$ and $W \in \mathfrak{X}(M^n(c))$.

It follows from (2.1) and (3.1) that

$$\begin{aligned}\langle \sigma(X,Y), \sigma(Z,W)\rangle = &\frac{\lambda^2}{3}\{\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\} \\ &+ \frac{c-\tilde{c}}{3}\{2\langle X,Y\rangle\langle Z,W\rangle - \langle X,Z\rangle\langle W,Y\rangle - \langle X,W\rangle\langle Y,Z\rangle\},\end{aligned} \tag{3.2}$$

where $\forall X, Y, Z, W \in \mathfrak{X}(M^n(c))$.

First, we consider the case that f is a totally geodesic immersion. Then this case is included in (I) of our theorem.

Next, we consider the case that f is not a totally geodesic immersion. Then there exists some point $x_0 \in M^n(c)$ such that $\lambda(x_0) \neq 0$. Since λ is a continuous function on $M^n(c)$, there exists a neighborhood U of x_0 such that $\lambda > 0$ on U. We shall study on the open subset U from now on. From the above lemma, the assumption of our theorem and the continuity of λ, we see that the function λ is constant on U, so that $\lambda^2 = c - \tilde{c}$ or $\lambda^2 = 2(n-1)(\tilde{c} - c)/(n+2)$.

In case of $\lambda^2 = c - \tilde{c}$, this case is included in (I) of our theorem.

In case of $\lambda^2 = 2(n-1)(\tilde{c} - c)/(n+2)$, from the above lemma we know that $\dim N_x^1 = (n(n+1)/2) - 1$ for all $x \in U$. Since λ is constant, differentiating (3.2) with respect to each $T \in \mathfrak{X}(M^n(c))$, we have the following:

$$\langle (\nabla'_T\sigma)(X,Y), \sigma(Z,W)\rangle = -\langle \sigma(X,Y), (\nabla'_T\sigma)(Z,W)\rangle. \tag{3.3}$$

By using (2.2) and (3.3) repeatedly, we get the following:

$$\begin{aligned}\langle(\nabla'_T\sigma)(X,Y),\sigma(Z,W)\rangle &= -\langle\sigma(X,Y),(\nabla'_T\sigma)(Z,W)\rangle = -\langle\sigma(X,Y),(\nabla'_Z\sigma)(T,W)\rangle\\ &= \langle(\nabla'_Z\sigma)(X,Y),\sigma(T,W)\rangle = \langle(\nabla'_Y\sigma)(X,Z),\sigma(T,W)\rangle\\ &= -\langle\sigma(X,Z),(\nabla'_Y\sigma)(T,W)\rangle = -\langle\sigma(X,Z),(\nabla'_W\sigma)(T,Y)\rangle\\ &= \langle(\nabla'_W\sigma)(X,Z),\sigma(T,Y)\rangle = \langle(\nabla'_X\sigma)(W,Z),\sigma(T,Y)\rangle\\ &= -\langle\sigma(W,Z),(\nabla'_X\sigma)(T,Y)\rangle = -\langle\sigma(W,Z),(\nabla'_T\sigma)(X,Y)\rangle.\end{aligned}$$

Thus we find that $\langle(\nabla'_T\sigma)(X,Y),\sigma(Z,W)\rangle = 0$. This, together with $\dim N^1 = \mathrm{codim} M^n(c)$, implies that f is parallel on U, so that there occurs the case (II) of our theorem (cf. [F, S]). Therefore we can get the conclusion. □

Finally we note that our theorem in this paper is a partial answer to the open problem in [B].

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