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Note on Bundles over Spheres with Group Actions

Dedicated to Professor Hiroshi Toda on his 60th birthday

Hiromichi MATSUNAGA
Department of Mathematics, Shimane University, Matsue, Japan
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In this note we shall consider G-vector bundles which admit in a sense simple representatives of characteristic maps, and for these bundles we shall call that the actions on the total spaces are of diagonal type. Some examples of this type have been given in [3], [4].

In section 1 we shall prepare some generalities to apply the work in [2]. The author has obtained Theorems 1 and 2 in this note by an idea due to Professor H. Toda, and would like to thank him for a private communication.

In section 2 we shall obtain more examples of actions of diagonal type.

§ 1. Some general results

Let B be a compact Hausdorff space and G, Γ be compact connected Lie groups. Let $p: E \to B$ be a fibre bundle with fibre V and structure group Γ. We assume that the spaces E, B are G-spaces and the projection p is an equivariant map.

DEFINITION. If the action $g: E \to E$ is a bundle map for each $g \in G$, then the G-action on the total space E is called a *bundle action*. A bundle action $G \times E \to E$ is called *of diagonal type* when there exist a covering $B = \cup U_i$ by G-stable open sets, homomorphisms $\alpha_i: G \to \Gamma$ and G-local trivialities $\phi_i: U_i \times V \to p^{-1}(U_i)$ for each i such that there holds the relation

$$\phi_i(gx, \alpha_i(g)(v)) = g \cdot \phi_i(x, v) \qquad \text{for} \quad g \in G,\ (x, v) \in U_i \times V.$$

If a bundle space admits a bundle action of diagonal type, then the transition functions $\chi_{ji}: U_i \times U_j \to \Gamma$ satisfies the relation $\alpha_j(g)\chi_{ji}(x) = \chi_{ji}(gx)\alpha_i(g)$, and vice versa.

Now we consider the case where $G = S^1$, the circle group, $B = S^0 * X$ the unreduced suspension of a G-space X. Then the space B admits the obvious action. We denote by $\Omega\Gamma$ the space of loops on the Lie group Γ, which can be identified with the space of maps $S^1 \to \Gamma$ preserving base points. We define an S^1-action on $\Omega\Gamma$ by

$$(g\omega)(g_1) = \omega(g_1 g)\omega(g)^{-1} \qquad \text{for} \quad g, g_1 \in G,\ \omega \in \Omega\Gamma.$$

Let $V \to E \to B$ be a fibre bundle with a characteristic map $\chi: X \to \Gamma$ and $\alpha: S^1 \to \Gamma$ be a homomorphism. We define a map $f: X \to \Omega\Gamma$ by $f(x)(g) = \chi(gx)\alpha(g)\chi(x)^{-1}$. Then we have

THEOREM 1. *Suppose that the map f is G-homotopic to the constant map onto α in $\Omega\Gamma$. Then there is a bundle action of S^1 on E.*

PROOF. The total space E is obtained from the union $((+1)*X)\times V\cup((-1)*X)\times V$ by the identification

$$((+1)*X)\times V\supset X\times V\ni(x, v)\equiv(x, \chi(x)(v))\in X\times V\subset((-1)*X)\times V.$$

We define an S^1-action on $(+1)*X\times V$ by

$$g(b, v)=(gb, \alpha(g)v) \qquad \text{for} \quad g\in G,\ b\in(+1)*X,\ v\in V.$$

By assumption, the map f can be extended to a map $\bar{f}: ((-1)*X)\to\Omega\Gamma$ such that

$$\bar{f}(\bar{x})(gg_1)=\bar{f}(g_1\bar{x})(g)\cdot(\bar{f}(\bar{x})(g_1)) \quad \text{for} \quad \bar{x}\in(-1)*X,\ g, g_1\in S^1,$$

and $\bar{f}((-1))=\alpha$. Then we can define an S^1-action on $((-1)*X)\times V$ by

$$g(b, v)=(gb, \bar{f}(b)(g)(v)) \qquad \text{for} \quad g\in S^1,\ v\in V.$$

Thus we obtain an S^1-action on the total space E which is a bundle action.

COROLLARY. *The S^1-action given by Theorem 1 is of diagonal type.*

PROOF. The sets $(+1)*X$ and $(-1)*X$ are G-contractible to $(+1)$ and (-1) respectively, and by the proof of Theorem 1, S^1-actions on the fibres over $(+1)$, (-1) coincide with α. Then we have the corollary.

Next we consider the case $X=S^1*Y$, where the S^1-action on X is given by

$$g(g', y, t)=(gg', y, t) \qquad \text{for} \quad g, g'\in G,\ y\in Y,\ 0\leqq t\leqq 1.$$

We have isomorphisms of groups consisting of homotopy classes,

$$(1) \qquad [X, \Gamma]\cong[(e*Y, Y), (\Omega'\Gamma, e_0)]\cong[S^0*Y, \Omega'\Gamma],$$

where e is the unit element, $\Omega'\Gamma$ the component of the constant loop e_0. Here we refer the map g^α given in [2], which is obtained as follows. Let Γ_α be the centralizer of the group $\alpha(S^1)$ in Γ and Γ^α be the space Γ/Γ_α. Then the map of Γ into $\Omega'\Gamma$ given by $\gamma\to\gamma\alpha(g)\gamma^{-1}\alpha(g)^{-1}$ induces the injective map $g^\alpha: \Gamma^\alpha\to\Omega'\Gamma$. Thus we have a homomorphism $i_*: [S^0*Y, g^\alpha(\Gamma^\alpha)]\to[S^0*Y, \Omega'\Gamma]$ induced by the inclusion map $i: g^\alpha(\Gamma^\alpha)\subset\Omega'\Gamma$. Then we have

THEOREM 2. *If an element $[\chi']\in[S^0*Y, \Omega'\Gamma]$ belongs to the i_*-image, then the map $\chi: S^1*Y\to\Gamma$ which corresponds to the map χ', satisfies the assumption in Theorem 1.*

PROOF. For the map $\chi': (e*Y, Y)\to(g^\alpha(\Gamma^\alpha), e_0)$, the map $\chi: S^1*Y\to\Gamma$ is given by

$$\chi(gx)=\chi'(x)(g) \quad \text{for} \quad g\in S^1,\ x\in e*Y,$$
$$=\gamma\alpha(g)\gamma^{-1}\alpha(g)^{-1} \quad \text{for some} \quad \gamma\in\Gamma.$$

Then

$$f(gx)(g')=\chi(g'gx)\alpha(g')\chi(gx)^{-1}$$
$$=\gamma\alpha(g'g)\gamma^{-1}\alpha(g'g)^{-1}\alpha(g')(\gamma\alpha(g)\gamma^{-1}\alpha(g)^{-1})^{-1}$$
$$=\gamma\alpha(g')\gamma^{-1}=f(x)(g'),$$

therefore the map $f\colon S^1*Y\to\Omega\Gamma$ is the composite map

$S^1*Y\xrightarrow{\pi} e*Y\xrightarrow{f|} f^{\alpha}(\Gamma^{\alpha})\subset\Omega\Gamma$, where π is the orbit map, $f|$ the restriction of f and f^{α} is the map given by $f^{\alpha}(\gamma\Gamma_{\alpha})(g)=\gamma\alpha(g)\gamma^{-1}$ for $\gamma\in\Gamma$. Hence the map f is G-homotopic to the constant map $S^1*Y\to\alpha\in\Omega\Gamma$.

§ 2. Applications

Here we consider two cases of $\Gamma=SU(2n-2)$ and $\Gamma=SO(2n+2)$. We have the following homomorphisms.

$\alpha_1(t)=(\exp \pi\ it)^{n-1}\times(\exp -\pi\ it)^{n-1}$ in Ad $SU(2n-2)$, the adjoint group,

$\alpha_2(t)=(\exp 2\pi\ it)^{n-1}\times(\exp -2\pi\ it)^{n-1}$ in $SU(2n-2)$,

and

$\alpha_1(t)=r(\exp \pi\ it)^{n+1}$ in Ad $SO(2n+2)$, the adjoint group,

$\alpha_2(t)=r(\exp 2\pi\ it)^{n+1}$ in $SO(2n+2)$,

where r is the realification map.

Let denote by $\Delta\colon\Omega'\Gamma\to\Omega'\Gamma\times\Omega'\Gamma$ the diagonal map, $\mu\colon\Omega'\Gamma\times\Omega'\Gamma\to\Omega'\Gamma$ the product in the loop space, $\nu\colon S\Gamma^{\alpha}\to S\Gamma^{\alpha}\vee S\Gamma^{\alpha}$ the comultiplication, $d\colon S\Gamma^{\alpha}\vee S\Gamma^{\alpha}\to S\Gamma^{\alpha}$ the map given by $d|S\Gamma^{\alpha}\times(x_0)=d|(x_0)\times S\Gamma^{\alpha}=$ the identity map of $S\Gamma^{\alpha}$, where $S\Gamma^{\alpha}$ is the reduced suspension of the space Γ^{α} and $\alpha=\alpha_1,\alpha_2$. Denote by Map (Z, W) the set of continuous maps preserving base points. We have the known equivalence

$$\mathrm{Map}(\Gamma^{\alpha},\Omega'\Gamma)\approx\mathrm{Map}(S\Gamma^{\alpha},\Gamma).$$

For a map $a\colon S^m\to\Gamma^{\alpha}$, let $Sa\colon S^{m+1}\to S\Gamma^{\alpha}$ be its suspension and $[a]$, $[Sa]$ be the homotopy classes. Then we have

PROPOSITION 3. $[g^{\alpha_2}a]=2[g^{\alpha_1}a]$.

PROOF. For maps $h\colon S\Gamma^{\alpha}\to\Gamma$ and $k\colon\Gamma^{\alpha}\to\Omega'\Gamma$, let $h'\colon\Gamma^{\alpha}\to\Omega'\Gamma$ and $\hat{k}\colon S\Gamma^{\alpha}\to\Gamma$ be the corresponding maps. Then we have $\widehat{g^{\alpha_1}}\circ d\circ\nu=\widehat{g^{\alpha_2}}$, $(\mu\circ\Delta)g^{\alpha_1}=g^{\alpha_2}$, and $\widehat{g^{\alpha_2}_*}([Sa])=2\widehat{g^{\alpha_1}_*}([Sa])$. Since $(\widehat{g^{\alpha_2}}\circ Sa)'=g^{\alpha_2}a$ and $[(\widehat{g^{\alpha_2}}\circ Sa)']=[g^{\alpha_2}\circ a]=2[(\widehat{g^{\alpha_1}}\circ Sa)']$

$=2[g^{\alpha_1}a]$ we have the proposition.

Now we have

PROPOSITION 4. *Let $\Gamma=SU(2n-2)$ or $SO(2n+2)$ and $n\geqq 2$. Then any element of the group $2\pi_{2n-1}(\Gamma)$ gives a G-vector bundle of diagonal type over the base space S^1*S^{2n-2}.*

PROOF. By the proposition 8.3 in [2], $g_*^{\alpha_1}$: $\pi_i(U(2n-2)/U(n-1)\times U(n-1))\to\pi_i(\Omega' SU(2n-2))$ is an isomorphism for $0\leqq i\leqq 2n-1$. Then by Corollary to the proof of Theorem 1 and Proposition 3, we have the proposition in the case of $\Gamma=SU(2n-2)$. For the case of $\Gamma=SO(2n+2)$, by the section 5 in [2], α_1 is a generating circle. By the proposition 31.2 in [1], $H_*(SO(2n+2)/U(n+1))\approx \mathrm{Hom}(H^*(S^2\times\cdots\times S^{2n}), Z)$. Then by Theorem 1 and Proposition 10.1 in [2], $g_*^{\alpha_1}$: $H_i(SO(2n+2)/U(n+1))\to H_i(\Omega' SO(2n+2))$ is an isomorphism for $0\leqq i\leqq 2n-2$. Therefore by the theorem of J. H. C. Whitehead $g_*^{\alpha_1}$: $\pi_i(SO(2n+2)/U(n+1))\to\pi_i(\Omega' SO(2n+2))$ is an isomorphism for $0\leqq i\leqq 2n-2$. Thus again by Corollary and Proposition 3 we have the proposition.

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