Mem. Fac. Sci., Shimane Univ.,
18, pp. 29–35 Dec. 20, 1984

A Feature of the Diffraction Scattering Based on the K-matrix Formalization and a Uniquely Determined π-π Amplitude

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(Received September 3, 1984)

It is shown that when the imaginary part of the f-Regge contribution calculated from a uniquely determined π-π amplitude[1] is taken as input for the damping equation, the obtained differaction amplitude well reproduces the t dependence of phenomenological Pomeron in small $|t|$ region.

§ 1. Introduction

Lately, Munakata, Sakamoto and the author[1] have obtained a uniquely determined π-π scattering amplitude in the narrow-width approximation. Starting with the most general Veneziano-type amplitude having just the degree of freedom to provide an arbitrary residue at each of parents and their daughters, we have restricted it by the following two conditions: (i) The duality constraints proposed by the author[2] which involve the $SU(6)\otimes O(3)_L$ hadron spectrum of the quark model. (ii) The convergence condition at $s\to\infty$. This amplitude possesses good properties; absence of negative-norm states, compatibility with available experimental data of 2π decay of the relevant resonances, and a consistency with the PCAC hypothesis. Moreover, the amplitude reproduces[3] the inverse Mellin transform of the imaginary part of the phenomenological scattering amplitude[4]. The transform is what prospsed by Froggatt, Nielsen and Petersen[5] to examine that can the π-π scattering amplitude be represented by any Veneziano model. Therefore, we suppose that the amplitude is promising as the Born amplitude of the strong interactions.

In this paper, we discuss the diffaction scattering on the basis of the amplitude and the multi-channel K-matrix formalization for the unitarity requirements. In §2, first the partial elastic amplitude in the N-channel K-matrix formalization is presented. Next, the integral equation for the full amplitude is constructed. The solution of the equation is also presented, with some parametolized Born amplitude as input. In §3, the parameters in the input amplitude are fixed so as to express the amplitude in Ref. 1). The resultant diffraction amplitude is compared with the phenomenological Pomeron, taking the viewpoint that the Pomeron is the diffraction. The section 4 is devoted to the discussion of results.

§2. The diffraction amplitude in the K-matrix formalization

2.1 *The partial-wave amplitude for the diffraction*

In the present discussion, the spin effects are neglected.

The S-matrix for the l-wave, S^l, in the N-channel K-matrix formalization, which satisfies the unitarity requirements, is written as

$$S^l = \frac{1+iK^l}{1-iK^l}, \quad K^l = (\alpha^l_{ij}), \quad \alpha^l_{ij} = \alpha^l_{ji}, \quad i, j = 1, 2, \ldots, N. \tag{2.1}$$

Here, α^l_{ij} is the real K-matrix element, that is, the Born amplitude for the transition from the state i to the state j.

We define the partial-wave scattering amplitudes for elastic and inelastic processes by

$$f^l_{ij} = (S^l_{ij} - 1)/(2i). \tag{2.2}$$

Then, from Eq. (2.1), we have the amplitude f^l_{1i} corresponding to the transition from '1' to 'i' ($i=1, 2, \ldots, N$) as

$$f^l_{1i} = b^l_{1i}(1 + if^l_{11}), \tag{2.3}$$

where

$$\begin{aligned} b^l_{1i} = \alpha^l_{1i} &+ i \sum_{j=2}^{N} \alpha^l_{1j}\alpha^l_{ji} - \sum_{j=2}^{N}\sum_{k=2}^{N} \alpha^l_{1j}\alpha^l_{jk}\alpha^l_{ki} \\ &- i \sum_{j=2}^{N}\sum_{k=2}^{N}\sum_{m=2}^{N} \alpha^l_{1j}\alpha^l_{jk}\alpha^l_{km}\alpha^l_{mi} + \cdots . \end{aligned} \tag{2.4}$$

Now, we pay our attention only to the elastic amplitude f^l_{11} hereafter;

$$f^l_{11} = b^l_{11}(1 + if^l_{11}), \tag{2.5}$$

where

$$\begin{aligned} b^l_{11} = \alpha^l_{11} &+ i \sum_{j=2}^{N} \alpha^l_{1j}\alpha^l_{j1} - \sum_{j=2}^{N}\sum_{k=2}^{N} \alpha^l_{1j}\alpha^l_{jk}\alpha^l_{k1} \\ &- i \sum_{j=2}^{N}\sum_{k=2}^{N}\sum_{m=2}^{N} \alpha^l_{1j}\alpha^l_{jk}\alpha^l_{km}\alpha^l_{m1} + \cdots . \end{aligned} \tag{2.6}$$

The elastic amplitude f^l_{11} is expressed by the 'irreducible' term b^l_{11} and the self-damping of it, as found from Eq. (2.5),

$$f^l_{11} = \frac{b^l_{11}}{1 - ib^l_{11}} . \tag{2.5'}$$

The diffraction scattering, that is, the imaginary part of f_{11}^l at high energies is obtained from Eq. (2.5) as

$$\mathrm{Im}\, f_{11}^l = \mathrm{Im}\, b_{11}^l + \{(\mathrm{Re}\, b_{11}^l)(\mathrm{Re}\, f_{11}^l) - (\mathrm{Im}\, b_{11}^l)(\mathrm{Im}\, f_{11}^l)\}. \tag{2.7}$$

The domination of the diffraction scattering observed at high energies is qualitatively understandable by Eqs. (2.6) and (2.7). The second term of Eq. (2.6) will be very large at high energies, because the contributions from all j's, $\alpha_{1j}^l \alpha_{j1}^l = (\alpha_{1j})^2$, are constructive. And, the largeness of the term will lead to the dominance of the diffraction in the elastic scattering. Therefore, from both of the phenomena and the present model, one finds inequalities

$$|\mathrm{Im}\, f_{11}^l| \gg |\mathrm{Re}\, f_{11}^l|, \qquad |\mathrm{Im}\, b_{11}^l| \gg |\mathrm{Re}\, b_{11}^l|. \tag{2.8}$$

Finally, from Eqs. (2.7) and (2.8), we have the expression for the diffraction amplitude as

$$\mathrm{Im}\, f_{11}^l = \mathrm{Im}\, b_{11}^l - (\mathrm{Im}\, b_{11}^l)(\mathrm{Im}\, f_{11}^l), \tag{2.9}$$

or,

$$\mathrm{Im}\, f_{11}^l = \frac{\mathrm{Im}\, b_{11}^l}{1 + \mathrm{Im}\, b_{11}^l}. \tag{2.9'}$$

2.2 *The full diffraction amplitude*

We define the full amplitude $\mathrm{Im}\, B_{11}(s, t)$ corresponding to the partial-wave amplitude $\mathrm{Im}\, b_{11}^l = \mathrm{Im}\, b_{11}^l(s)$ and its parametolization as

$$\begin{aligned}\mathrm{Im}\, B_{11}(s, t) &= \frac{s}{2p^2} \sum_{l=0}^{\infty} (2l+1)\{\mathrm{Im}\, b_{11}^l(s)\} P_l\left(1 + \frac{t}{2p^2}\right) \\ &= s\gamma_o \exp(\beta_o t), \quad \gamma_o = \gamma_o(s), \quad \beta_o = \beta_o(s). \end{aligned} \tag{2.10}$$

Here, s and t are the Mandelstam variables and p is the c.m. momentum in the s-channel. Then, from Eq. (2.9), it is found that the full amplitude $\mathrm{Im}\, F_{11}(s, t)$ corresponding to $\mathrm{Im}\, f_{11}^l$ satisfies the following integral equation:

$$\begin{aligned}\mathrm{Im}\, F_{11}(s, t) &= \mathrm{Im}\, B_{11}(s, t) \\ &\quad - \left(\frac{1}{2\pi s}\right) \int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2 \{\mathrm{Im}\, F_{11}(s, t_1)\} \{\mathrm{Im}\, B_{11}(s, t_2)\} \tau(t, t_1, t_2), \end{aligned} \tag{2.11}$$

where

$$\tau(t, t_1, t_2) = \frac{\theta(-t^2 - t_1^2 - t_2^2 + 2tt_1 + 2tt_2 + 2t_1t_2)}{\sqrt{-t^2 - t_1^2 - t_2^2 + 2tt_1 + 2tt_2 + 2t_1t_2}}. \tag{2.12}$$

Here, it is assumed that $s \gg |t|$. And the solution of eq. (2.11) is

$$\operatorname{Im} F_{11}(s,t)=2s\beta_o\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\gamma_o}{2\beta_o}\right)^n\exp(\beta_o t/n). \tag{2.13}$$

For observable quantities, the amplitude predicts

$$\sigma_{tot}=16\pi\beta_o(s)\ln\left\{1+\frac{\gamma_o(s)}{2\beta_o(s)}\right\}, \tag{2.14}$$

$$\sigma_{\text{diff}}=16\pi\beta_o(s)\left[\ln\left\{1+\frac{\gamma_o(s)}{2\beta_o(s)}\right\}-\frac{\gamma_o(s)}{2\beta_o(s)+\gamma_o(s)}\right], \tag{2.15}$$

$$b(s)=\left[\frac{\partial}{\partial t}\left\{\ln\left(\left(\frac{d\sigma_{\text{diff}}}{dt}\right)\Big/\left(\frac{d\sigma_{\text{diff}}}{dt}\right)_{t=0}\right)\right\}\right]_{t=0}$$

$$=2\beta_o(s)\frac{\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\frac{\gamma_o(s)}{2\beta_o(s)}\right)^n}{\ln\left\{1+\dfrac{\gamma_o(s)}{2\beta_o(s)}\right\}}. \tag{2.16}$$

§3. A uniquely determined $\pi-\pi$ amplitude and a feature of the diffraction scattering based on it

A uniquely determined $\pi^- - \pi^+$ scattering amplitude in the narrow-width approximation, which is obtained in Ref. 1), is

$$f(s,t)=-\lambda(1-w^2)\sum_{n=1}^{\infty}\frac{1}{(n-1)!(2n-1+w)(2n-3+w)}$$
$$\times\left[\frac{\Gamma(n-\alpha(s))\,\Gamma(n-\alpha(t))}{\Gamma(n-\alpha(s)-\alpha(t))}+\frac{(1-w)\Gamma(n-\alpha(s))\,\Gamma(n-\alpha(t))}{2\Gamma(n+1-\alpha(s)-\alpha(t))}\right]. \tag{3.1}$$

Here, $\alpha(s)=\alpha' s+\alpha_0$ is the exchange-degenerate $\rho-f$ Regge trajectory, and w denotes

$$w\equiv 2-3\alpha_o-4\alpha' m_\pi^2. \tag{3.2}$$

We know a candidate of the Born amplitude for the $\pi^-\pi^+\to\pi^-\pi^+$ process, that is, the $1\to 1$ process.

We do not know the Born amplitudes for the $1\to i$ processes explicitly. One may, however, expect that at very high energies where some mass differences and spin effects will be able to be neglected, almost all of the inelastic processes $1\to i$ dominated by the f Regge exchange will have amplitudes with the t-dependence as in the $1\to 1$ amplitude. So, it might be possible to predict some feature of the t dedendence of the diffraction amplitude by the amplitude (3.1).

Now, in order to prepare the explicit input for the equation (2.11), we examine the f-Regge contribution to the amplitude (3.1). The asymptotic form of Eq. (3.1) is

$$f(s,t)\xrightarrow[t:\text{fixed}]{s\to\infty}\lambda\Gamma(1-\alpha(t))\left[\exp\{-i\pi\alpha(t)\}\right]\alpha(s)^{\alpha(t)}I(t), \tag{3.3}$$

where

$$I(t)=\frac{\Gamma\left(\frac{3+w}{2}\right)\Gamma(1+\alpha(t))}{\Gamma\left(\frac{1+w}{2}+\alpha(t)\right)}, \quad \mathrm{Re}\,\{1+\alpha(t)\}>0. \tag{3.4}$$

The separation of the asymptotic form (3.3) into the f and ρ Regge contributions is

$$f(s, t)\xrightarrow[t:\mathrm{fixed}]{s\to\infty} A_f(s, t)+A_\rho(s, t), \tag{3.5}$$

$$\begin{aligned} A_f(s, t)&=\lambda\Gamma(1-\alpha(t))\frac{1+\exp\{-i\pi\alpha(t)\}}{2}\alpha(s)^{\alpha(t)}I(t) \\ &\equiv\gamma_f(t)\left(\frac{s}{s_o}\right)^{\alpha(t)}\exp\left\{-\frac{i\pi\alpha(t)}{2}\right\}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} A_\rho(s, t)&=\lambda\Gamma(1-\alpha(t))\frac{-1+\exp\{-i\pi\alpha(t)\}}{2}\alpha(s)^{\alpha(t)}I(t) \\ &\equiv i\gamma_\rho(t)\left(\frac{s}{s_o}\right)^{\alpha(t)}\exp\left\{-\frac{i\pi\alpha(t)}{2}\right\}, \end{aligned} \tag{3.7}$$

where

$$\gamma_f(t)\equiv\lambda\Gamma(1-\alpha(t))\,(\alpha' s_o)^{\alpha(t)}I(t)\cos\frac{\pi\alpha(t)}{2}, \tag{3.8}$$

$$\gamma_\rho(t)\equiv-\lambda\Gamma(1-\alpha(t))\,(\alpha' s_o)^{\alpha(t)}I(t)\sin\frac{\pi\alpha(t)}{2}, \tag{3.9}$$

and s_o is introduced to make the residue functions $\gamma_f(t)$ and $\gamma_\rho(t)$ dimensionless and the unity in GeV^2. The numerical values of the f residue function in small $|t|$ region is

$$\gamma_f(t)\simeq-1.55\ \exp(0.586t), \qquad \text{for} \quad |t|\leqq 0.5. \tag{3.10}$$

Here λ is determined by observed decay width of $\rho\to 2\pi$,[6] and $\alpha(t)$ is taken to be

$$\alpha(t)=0.881t+0.483. \tag{3.11}$$

We can make two models for the fiffraction scattering on the basis of the $A_f(s, t)$ amplitude (3.6). One is the model where this amplitude is regarded as the Born-like amplitude and the Regge cut for it is taken as the input for the equation (2.11) after being multiplied by the number of open channels. The other is the one in which the imaginary part of the amplitude $A_f(s, t)$ is taken as the input for Eq. (2.11). Some remarks are made about the latter model.

(a) The imaginary part of $A_f(s, t)$ is

$$\mathrm{Im}\,A_f(s,t) = -\gamma_f(t)\left(\frac{s}{s_o}\right)^{\alpha(t)}\sin\frac{\pi\alpha(t)}{2}$$

$$\simeq 1.07\left(\frac{s}{s_o}\right)^{\alpha(t)}\exp(1.97t),\quad (|t|\leqq 0.2). \tag{3.12}$$

And amplitudes for inelastic processes will have imaginary parts as Eq. (3.12) except for the normalization.

(b) As for the s dependence, the model will have to be modified. There is the idea of the identity of Pomeron and f.[7]

(c) The ratios $(\mathrm{Re}\,A_f(s,t)/\mathrm{Im}\,A_f(s,t))$ and $(\mathrm{Re}\,A_\rho(s,t)/\mathrm{Im}\,A_\rho(s,t))$ reproduce experiments well at least in small $|t|$ region.

(d) The integral equation (2.11) is valid independently from the choice of the irreducible term $\mathrm{Im}\,B_{11}(s,t)$.

We start with the latter model. The input, that is, the amplitude (3.12) with some modification for s dependence is parametolized as

$$\mathrm{Im}\,B_{11}(s,t) = s\{2\beta_o(s)\}C(s)\exp\{\beta_o(s)t\}, \tag{3.13}$$

where

$$\beta_o(s) = 1.97 + \alpha'\ln\left(\frac{s}{s_o}\right). \tag{3.14}$$

From Eqs. (2.14)$\sim$(2.16), the predictions of the solution of the equation (2.11) are found to be

$$\sigma_{\mathrm{tot}} = 16\pi\beta_o(s)\ln\{1+C(s)\}, \tag{3.15}$$

$$\sigma_{\mathrm{diff}} = 16\pi\beta_o(s)\left[\ln\{1+C(s)\} - \frac{C(s)}{1+C(s)}\right], \tag{3.16}$$

$$\left[\frac{\partial}{\partial t}\{\ln(\mathrm{Im}\,F_{11}(s,t)/\mathrm{Im}\,F_{11}(s,0))\}\right]_{t=0}$$

$$= \frac{b(s)}{2}$$

$$= \beta_o(s)\frac{\int_0^{C(s)}\frac{\ln(1+t)}{t}dt}{\ln\{1+C(s)\}},\quad (|C(s)|\leqq 1). \tag{3.17}$$

When the unknown quantity $C(s)$ is estimated from Eq. (3.15) and experimental σ_{tot},[4] the quantity $b(s)/2$ in Eq. (3.17) can be explicitly predicted. In Table I, the predictions are compared with experimental $b(s)/2$.[4]

In Table I, a consistency with experiments is seen.

It is clear that the model with the Regge cut of $A_f(s,t)$ as input fails.

Table I. The prediction for slope parameter of $\pi^- - \pi^+$ diffraction scattering and phenomenological values[4] for comparison.

s(GeV^2)	$\beta_o(s)$ (GeV^{-2})	σ_{tot}^{exp}(mb)	$C(s)$	$b(s)/2$(GeV^{-2})	
				prediction	exp.
3.0	2.94	15	0.298	3.14	3.3
4.0	3.19	15	0.271	3.39	3.4
5.0	3.39	15	0.254	3.60	3.5
6.0	3.55	15	0.241	3.75	3.5
7.0	3.68	15	0.231	3.88	3.6

§ 4. Discussion of results

When the imaginary part of the f-Regge contribution from the asymptotic form of the uniquely determined amplitude is taken as input for the damping equation, the obtained diffraction amplitude well reproduces slopes of phenomenological Pomeron in small $|t|$ region. It is noted that the $I(t)$ factor in the asymptotic form ($I(t) \simeq 0.89 \exp(0.32t)$, ($|t| \leqq 0.5$)) plays a role not negligible. The model can not, however, explain the s dependence of the phenomenological Pomeron.

References

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