Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
36 (2003), pp.1–9

CIRCLES AND HYPERSURFACES IN SPACE FORMS

SADAHIRO MAEDA AND TOSHIAKI ADACHI

(Received: December 20, 2002)

Abstract. In this expository paper, we study hypersurfaces of space forms by investigating circles on their hypersurfaces.

1. Introduction

In this paper we discuss hypersurfaces in a real space form, and real and complex hypersurfaces in a complex space form by observing the extrinsic shape of circles of these hypersurfaces. An n-dimensional real space form $M^n(c)$ is a Riemannian manifold of constant curvature c, which is locally isometric to either a standard sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$ or a real hyperbolic space $H^n(c)$, according as c is positive, zero or negative. A complex n-dimensional complex space form $M_n(c)$ is a Kähler manifold of constant holomorphic sectional curvature c, which is locally complex analytically isometric to either a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative. In this paper we mean by space form either a real space form or a complex space form.

A smooth curve γ on a Riemannian manifold M parametrized by its arclength is called a *circle* if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma$ with some nonnegative constant κ, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ on M. This condition is equivalent to the condition that there exist a nonnegative constant κ and a field of unit vectors Y along this curve

---

2000 *Mathematics Subject Classification.* Primary 53B25, Secondary 53C40.

Key words and phrases. hypersurfaces, circles, first curvature, space forms, real space forms, complex space forms, Kähler submanifolds, totally geodesic, Veronese embeddings.

The first author partially supported by Grant-in-Aid for Scientific Research (C) (No. 14540080), Ministry of Education, Science, Sports and Culture.

The second author partially supported by Grant-in-Aid for Scientific Research (C) (No. 14540075), Ministry of Education, Science, Sports and Culture .

The first author talked a part of this paper in the conference of Analysis, Manifolds and Mechanics during February 5-7, 2003 to celebrate the 90th Birth Anniversary of Professor Manindra Chandra Chaki.

which satisfy the following differential equations: $\nabla_{\dot\gamma}\dot\gamma = \kappa Y$ and $\nabla_{\dot\gamma} Y = -\kappa\dot\gamma$. We call the constant κ *curvature* of γ. As we have $\kappa = \|\nabla_{\dot\gamma}\dot\gamma\|$, we treat geodesics as circles of null curvature. For given a point $x \in M$, an orthonormal pair of tangent vectors $u, v \in T_xM$ and a positive constant κ, by the existence and uniqueness theorem on solutions for ordinary differential equations we have locally a unique circle $\gamma = \gamma(s)$ with initial condition that $\gamma(0) = x, \dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = \kappa v$. It is well-known that in Euclidean space a circle of positive curvature κ is nothing but a circle of radius $1/\kappa$ in the sense of Euclidean geometry.

Our study is motivated by the following fact on extrinsic spheres due to Nomizu and Yano [NY]. An extrinsic sphere is a totally umbilic submanifold of $\widetilde{M}$ with parallel mean curvature vector in $\widetilde{M}$. Their result states that a Riemannian submanifold M^n of $\widetilde{M}^{n+p}$ through an isometric immersion f is an extrinsic sphere of $\widetilde{M}$ if and only if there exists a positive constant κ such that for every circle γ of curvature κ on M^n the curve $f \circ \gamma$ is a circle in $\widetilde{M}$.

The key word in this paper is the first curvature of a curve in the sense of Frenet formula. We establish our result by paying attention to the first curvature $\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ of the curve $f\circ\gamma$ for a circle γ on a *hypersurface* in the ambient space form $\widetilde{M}(c)$. Here we denote by $\widetilde{\nabla}$ the Riemannian connection of $\widetilde{M}(c)$. We investigate hypersuraces M^{n-1} in a real space form $\widetilde{M}^n(c)$ and *real* hypersurfaces M^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c)$ (Theorems 1, 2, 3 and 4). We also investigate *complex* hypersurfaces M_{n-1} in a complex space form $\widetilde{M}_n(c)$ (Theorems 5 and 6). At the end of this paper, as an application of our discussion on hypersurfaces we characterize some Kähler embeddings of complex projective spaces into complex projective spaces which are called Veronese embeddings (Theorem 7).

2. Hypersurfaces of real space forms

We consider parallel hypersurfaces in a real space form. By the results due to Ferus[F] and Takeuchi[T] a hypersurface M^{n-1} with parallel second fundamental form in a real space form $\widetilde{M}^n(c)$ is either totally umbilic in $\widetilde{M}^n(c)$ or locally congruent to one of the following product spaces:

i) $M^{n-1} = \mathbb{R}^k \times S^{n-k-1}(c_1), 1 \leqq k \leqq n-2$ in $\mathbb{R}^n$,
ii) $M^{n-1} = S^k(c_1) \times S^{n-k-1}(c_2), 1 \leqq k \leqq n-2$ with $(1/c_1) + (1/c_2) = 1/c$ in $S^n(c)$,
iii) $M^{n-1} = H^k(c_1) \times S^{n-k-1}(c_2), 1 \leqq k \leqq n-2$ with $(1/c_1) + (1/c_2) = 1/c$ in $H^n(c)$.

By investigating the first curvature of *each geodesic* on these parallel hypersurfaces we obtain the following well-known theorem:

Theorem 1. *A hypersurface M^{n-1} of a real space form $\widetilde{M}^n(c)$ has parallel second fundamental form if and only if every geodesic on M being considered as a curve in the ambient space has constant first curvature.*

Proof. In a neighborhood of each point of M we choose a unit normal vector field $\mathcal{N}$ in $\widetilde{M}^n(c)$. The Riemannian connections $\widetilde{\nabla}$ in $\widetilde{M}^n(c)$ and ∇ in M are related by the following formulas for arbitrary vector fields X, Y on M:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX, \tag{2.2}$$

where A is the shape operator of M in $\widetilde{M}^n(c)$. Suppose that every geodesic $\gamma : I \to M$ on M has, considered as a curve in $\widetilde{M}^n(c)$, constant first curvature. This hypothesis means, by definition, that $\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ is constant on the interval I. But from equation (2.1) it follows $\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|^2 = \langle A\dot\gamma, \dot\gamma\rangle^2$, hence our hypothesis is equivalent to the constancy of $\langle A\dot\gamma, \dot\gamma\rangle$ on I. Therefore as $\nabla_{\dot\gamma}\dot\gamma = 0$, we have $\langle(\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma\rangle = \nabla_{\dot\gamma}\langle A\dot\gamma, \dot\gamma\rangle = 0$ on I. Thus we obtain $\langle(\nabla_X A)X, X\rangle = 0$ for every tangent vector X of M. By use of the Codazzi equation $\langle(\nabla_X A)Y, Z\rangle = \langle(\nabla_Y A)X, Z\rangle$ this property shows that M has parallel second fundamental form. By the same discussion as above we get the converse. $\square$

Next, we study the first curvature of *each circle of positive curvature* on parallel hypersurfaces M^{n-1} in a real space form(cf. [M]):

Theorem 2. *A hypersurface M^{n-1} of a real space form $\widetilde{M}^n(c)$ through an isometric immersion f is totally umbilic in $\widetilde{M}^n(c)$ if and only if there exists some $\kappa > 0$ satisfying that every circle of curvature κ on M being considered as a curve in the ambient space has constant first curvature.*

Proof. Suppose that for every circle $\gamma : I \to M$ of positive curvature κ on M the curve $f \circ \gamma$ has constant first curvature in the ambient space $\widetilde{M}^n(c)$. We take a point $x \in M$ and choose an arbitrary orthonormal pair of vectors $u, v \in T_xM$. Let $\gamma = \gamma(s), s \in I$ be a circle of curvature κ on the submanifold M^n with initial condition that $\gamma(0) = x, \dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = \kappa v$. It follows from equation (2.1) that the first curvature $\widetilde{\kappa}$ of the curve $f \circ \gamma$ is expressed as $\widetilde{\kappa} = \sqrt{\kappa^2 + \langle A\dot\gamma, \dot\gamma\rangle^2}$. This equality tells us that $\langle A\dot\gamma, \dot\gamma\rangle$ is constant on I, so that we have

$$0 = \frac{d}{ds}\langle A\dot\gamma, \dot\gamma\rangle = \langle(\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma\rangle + 2\langle A\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = \langle(\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma\rangle + 2\kappa\langle A\dot\gamma, Y\rangle. \tag{2.3}$$

Evaluating the equation (2.3) at $s = 0$, we get

$$\langle(\nabla_u A)u, u\rangle + 2\kappa\langle Au, v\rangle = 0. \tag{2.4}$$

On the other hand, for another circle $\rho = \rho(s)$ of the same curvature κ on the submanifold M^{n-1} with initial condition that $\rho(0) = x, \dot\rho(0) = u$ and $\nabla_{\dot\rho}\dot\rho(0) = -\kappa v$, we find

$$\langle(\nabla_u A)u, u\rangle - 2\kappa\langle Au, v\rangle = 0, \tag{2.5}$$

which corresponds to the equation (2.4). Thus, from (2.4) and (2.5) we can see that $\langle Au, v\rangle = 0$ for each orthonormal pair of vectors u, v at each point x of M, so that the hypersurface M is totally umbilic in $\widetilde{M}^n(c)$.

The converse is obvious from the fact that every circle on a totally umbilic submanifold M^n of $\widetilde{M}^{n+p}(c)$ is mapped to a circle in the ambient space $\widetilde{M}^{n+p}(c)$. □

3. Real hypersurfaces of nonflat complex space forms

It is well-known that there exist no real hypersurfaces with parallel second fundamental form in a nonflat complex space form $\widetilde{M}_n(c)$, which is either a complex projective space or a complex hyperbolic space. We hence consider a bit weak condition on real hypersurfaces in a complex space form.

Let M^{2n-1} be an orientable real hypersurface of $\widetilde{M}_n(c)$ and $\mathcal{N}$ a unit normal vector field on M in $\widetilde{M}_n(c)$. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of $\widetilde{M}_n(c)$ which satisfies

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \text{and} \quad \langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

where Id denotes the identity map of the tangent bundle TM of M. Then it is known that the following equalities

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi \quad \text{and} \quad \nabla_X \xi = \phi AX$$

hold. The condition that the structure vector $\xi = -J\mathcal{N}$ is principal is natural. It is well-known that this condition is equivalent to the condition that every integral curve of the vector field ξ is a geodesic on M^{2n-1}. As was shown in [NR], for a real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 2)$, if $A\xi = \alpha\xi$ holds with some function α on M then α is locally constant. In $\mathbb{C}P^n(c)$ each real hypersurface M lying on a tube of constant radius r $(0 < r < \pi/\sqrt{c})$ around a complex submanifold of $\mathbb{C}P^n(c)$ satisfies this condition on ξ. In $\mathbb{C}H^n(c)$ each real hypersurface M lying on a tube of radius r $(0 < r < \infty)$ around a complex submanifold or around a totally real submanifold of $\mathbb{C}H^n(c)$ satisfies this condition. Real hypersurfaces M^{2n-1} with the structure vector ξ as a principal curvature vector in $\widetilde{M}_n(c)$ are called *Hopf hypersurfaces.* In $\mathbb{C}P^n(c)$ we have the following typical Hopf hypersurfaces:

(A_1) A tube of radius r over hyperplane $\mathbb{C}P^{n-1}(c)$, where $0 < r < \pi/\sqrt{c}$,
(A_2) a tube of radius r over totally geodesic $\mathbb{C}P^k(c)$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$.

In $\mathbb{C}H^n(c)$ we have the following typical Hopf hypersurfaces:

(A_0) A horosphere in $\mathbb{C}H^n(c)$,
(A_1) a tube of radius r over $\mathbb{C}H^k(c)$ $(k = 0, n-1)$, where $0 < r < \infty$,
(A_2) a tube of radius r over $\mathbb{C}H^k(c)$ $(1 \leqq k \leqq n-2)$, where $0 < r < \infty$.

These Hopf hypersurfaces are usually called *hypersurfaces of type A*. The following theorem gives a characterization of such hypersurfaces([NR]):

Theorem A. *Let* M^{2n-1} *be a real hypersurface in a nonflat complex space form* $\widetilde{M}_n(c)$. *Then the following conditions are mutually equivalent*:

1) M *is locally congruent to a hypersurface of type A.*
2) *The structure tensor* ϕ *and the shape operator* A *of* M *are commutative*: $\phi A = A\phi$.
3) $\langle (\nabla_X A)X, X\rangle = 0$ *on* M *for each* $X \in TM^{2n-1}$.

It follows from Theorem A and the discussion in the proof of Theorem 1 yields the following (cf. [NR]):

Theorem 3. *A real hypersurface* M^{2n-1} *of a nonflat complex form* $\widetilde{M}^n(c)$ *is of type A if and only if every geodesic of* M *being considered as a curve in the ambient space has constant first curvature.*

Since we have no totally umbilic real hypersurfaces in a nonflat complex space form, the proof of Theorem 2 implies

Theorem 4. *There exist no real hypersurfaces* M^{2n-1} *in a nonflat complex space form* $\widetilde{M}_n(c)$ *satisfying that for some* $\kappa > 0$ *every circle of curvature* κ *on* M *being considered as a curve in the ambient space has constant first curvature.*

4. Complex hypersurfaces of complex space forms

In this section we consider complex hypersurfaces with parallel second fundamental form in a complex space form $\widetilde{M}_n(c)$. It is known that they are

i) either totally geodesic in $\widetilde{M}_n(c)$,
ii) or locally congruent to a complex quadric $Q_{n-1}(\mathbb{C})$ in the case $c > 0$([NT]).

By investigating the first curvature of *each geodesic* on these parallel hypersurfaces we obtain the following theorem:

Theorem 5 [MO]. *A complex hypersurface* M_{n-1} *of a complex space form* $\widetilde{M}_n(c)$ *has parallel second fundamental form if and only if every geodesic on* M *being considered as a curve in the ambient space has constant first curvature.*

Proof. Suppose that every geodesic γ of M_{n-1}, considered as a curve in the ambient space $\widetilde{M}_n(c)$, has constant first curvature. Then by the Gauss formula $\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$ we see that $\|\sigma(\dot\gamma, \dot\gamma)\|$ is constant along γ. This implies that $\langle (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle = 0$ and hence we have

$$\langle (\bar\nabla_X \sigma)(X, X), \sigma(X, X)\rangle = 0 \tag{4.1}$$

for every vector field X tangent to M. Here, the covariant differentiation $\bar\nabla$ of the second fundamental form σ with respect to the connection in (tangent bundle)$\oplus$(normal bundle) by

$$(\bar\nabla_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

where D is the normal connection of M_{n-1} in $\widetilde{M}_n(c)$.

Let J be the complex structure of the ambient space $\widetilde{M}_n(c)$. We denote by the same letter J the complex structure of the hypersurface M_{n-1}. Then, by replacing X by JX in (4.1), we obtain

$$\langle(\bar{\nabla}_X\sigma)(X,X), J\sigma(X,X)\rangle = 0 \tag{4.2}$$

for all X. It follows from (4.1) and (4.2) that $(\bar{\nabla}_X\sigma)(X,X) = 0$ for all X. Then, thanks to the Codazzi equation $(\bar{\nabla}_X\sigma)(Y,Z) = (\bar{\nabla}_Y\sigma)(X,Z)$, we know that M has parallel second fundamental form. The converse is obvious by the same discussion as above. □

We obtain the following by studying the first curvature of *each circle of positive curvature* on parallel complex hypersurfaces M_{n-1} in a complex space form $\widetilde{M}_n(c)$.

Theorem 6 [SMA]**.** *A complex hypersurface M_{n-1} of a complex space form $\widetilde{M}_n(c)$ is totally geodesic in $\widetilde{M}_n(c)$ if and only if there exists some $\kappa > 0$ satisfying that every circle of curvature κ on M being considered as a curve in the ambient space has constant first curvature.*

Proof. Let $f : M_{n-1} \to \widetilde{M}_n(c)$ be an isometric Kähler immersion satisfying the condition on the extrinsic shape of circles. For an arbitrary orthonormal pair (u,v) of vectors at a fixed point x of M we choose a circle $\gamma = \gamma(s), s \in I$ of curvature κ on the submanifold M with initial condition that $\gamma(0) = x, \dot{\gamma}(0) = u$ and $\nabla_{\dot{\gamma}}\dot{\gamma}(0) = \kappa v$. By the same discussion as in the proof of Theorem 2 we have

$$\begin{aligned} 0 &= \frac{d}{ds}\langle\sigma(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle = 2\langle D_{\dot{\gamma}}(\sigma(\dot{\gamma},\dot{\gamma})),\sigma(\dot{\gamma},\dot{\gamma})\rangle \\ &= 2\langle(\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle + 4\kappa\langle\sigma(\dot{\gamma},Y),\sigma(\dot{\gamma},\dot{\gamma})\rangle. \end{aligned}$$

Evaluating this equation at $s = 0$, we find

$$\langle(\bar{\nabla}_u\sigma)(u,u),\sigma(u,u)\rangle + 2\kappa\langle\sigma(u,v),\sigma(u,u)\rangle = 0. \tag{4.3}$$

Starting with the circle γ with initial condition that $\gamma(0) = x, \dot{\gamma}(0) = u$ and $\nabla_{\dot{\gamma}}\dot{\gamma}(0) = -\kappa v$, we find the equation (4.3) turns to

$$\langle(\bar{\nabla}_u\sigma)(u,u),\sigma(u,u)\rangle - 2\kappa\langle\sigma(u,v),\sigma(u,u)\rangle = 0. \tag{4.4}$$

Hence these two equations (4.3) and (4.4) guarantee that

$$\langle(\bar{\nabla}_u\sigma)(u,u),\sigma(u,u)\rangle = 0, \tag{4.5}$$

$$\langle\sigma(u,u),\sigma(u,v)\rangle = 0 \tag{4.6}$$

for each orthonormal pair of vectors u, v at every point x.

The equality (4.5) tells us that our hypersurface M has parallel second fundamental form (see the proof of Theorem 5), hence that the hypersurface M is totally geodesic or locally congruent to a complex quadric $Q_{n-1}(\mathbb{C})$. Next we shall show that M is *not* a complex quadric $Q_{n-1}(\mathbb{C})$. It is well-known that the equality (4.6) implies that our immersion is λ-isotropic, namely at each point of M the normal curvature vector $\sigma(v, v)$ determined by a unit vector v has the same length λ for every v (cf.[O]). Moreover, (4.5) shows this function $\lambda : M \to \mathbb{R}$ is locally constant. Indeed, for every geodesic $\tau = \tau(s)$ on the hypersurface M_{n-1} we see by differentiating $\|\sigma(\dot\tau(s), \dot\tau(s))\|^2$ along τ that $\lambda = \lambda(s)$ is constant along τ.

On the other hand we denote by R and $\widetilde{R}$ the curvature tensors of M and $\widetilde{M_n}(c)$, respectively. Since M is a Kähler submanifold in $\widetilde{M_n}(c)$, by substituting

$$\widetilde{R}(X,Y)Z = \frac{c}{4}(\langle Y,Z\rangle X - \langle X,Z\rangle X + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ),$$

to the Gauss equation

$$\langle \widetilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle \sigma(X,Z),\sigma(Y,W)\rangle - \langle \sigma(X,W),\sigma(Y,Z)\rangle,$$

we find that the holomorphic sectional curvature $K(X, JX)$ of M_{n-1} determined by a unit vector X is given by

$$K(X,JX) = \langle R(X,JX)JX,X\rangle = c - 2\|\sigma(X,X)\|^2 = c - 2\lambda^2$$

holds for an arbitrary unit vector X. This implies that the hypersurface M is a complex space form, so that M is not a complex quadric $Q_{n-1}(\mathbb{C})$. Therefore we get the desirable result. Needless to say our totally geodesic hypersurface is zero-isotropic. □

5. Appendix

As a generalization of Theorem 6 we shall provide a characterization of a Kähler isometric full immersion of a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c into a complex projective space $\mathbb{C}P^N(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. By virtue of the classification theorem ([C, NO]) this Kähler immersion is nothing but a Kähler embedding $f_k : \mathbb{C}P^n(c/k) \to \mathbb{C}P^N(c)$ given by

$$[z_i]_{0\leqq i\leqq n} \mapsto \Big[\sqrt{\frac{k!}{k_0!\cdots k_n!}}z_0^{k_0}\cdots z_n^{k_n}\Big]_{k_0+\cdots+k_n=k},$$

where $[*]$ means the point of the projective space with the homogeneous coordinates $*$ and $N = (n+k)!/(n!k!) - 1$. We usually call f_k the k-th *Veronese embedding.* The embedding f_k has various geometric properties.

Theorem B [C, NO]. *Let $f : M_n(c) \to \widetilde{M}_N(\tilde{c})$ be a Kähler isometric immersion of a complex space form of constant holomorphic sectional curvature c into another complex space form of constant holomorphic sectional curvature $\tilde{c}$. If $\tilde{c} > 0$ and f is full, then $\tilde{c} = kc$ and $N = (n+k)!/(n!k!) - 1$ for some positive integer k.*

We now show the following characterization of Veronese embeddings:

Theorem 7 [SMA,M]. *Let $f : M_n \to \widetilde{M}_N(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold M_n into an N-dimensional complex space form $\widetilde{M}_N(c)$ of constant holomorphic sectional curvature $c > 0$. Then the following conditions are equivalent*:

(1) *For some positive integer k, the submanifold M_n is locally congruent to $\mathbb{C}P^n(c/k)$, $N = (n+k)!/(n!k!) - 1$ and f is locally equivalent to the k-th Veronese embedding f_k.*
(2) *There exists $\kappa > 0$ satisfying that for each circle γ of curvature κ on the submanifold M_n the curve $f \circ \gamma$ in $\widetilde{M}_N(c)$ has constant first curvature along this curve.*

Proof. (1) $\Rightarrow$ (2). For each Veronese embedding $f_k : \mathbb{C}P^n(c/k) \to \mathbb{C}P^N(c)$ we see that $\|\sigma(v,v)\|^2 = c(k-1)/2k$ for any unit vector v at each point $x \in \mathbb{C}P^n(c/k)$ (see [Og]). We then find for each circle γ of curvature κ on $\mathbb{C}P^n(c/k)$ that the curve $f_k \circ \gamma$ has constant first curvature $\widetilde{\kappa} = \sqrt{\kappa^2 + \frac{c(k-1)}{2k}}$ in the ambient manifold $\mathbb{C}P^N(c)$.

(2) $\Rightarrow$ (1). Let $f : M_n \to \widetilde{M}_N(c)$ be a Kähler isometric full immersion satisfying the condition (2). By virtue of the discussion in the proof of Theorem 6 we find M_n is a complex space form. This, combined with Theorem B, yields the statement (1). $\square$

Remark 1. Theorem 7 is not true when we set $\kappa = 0$ in the statement (2). Every geodesic on a parallel Kähler submanifold being considered as a curve in the ambient space has constant first curvature.

Remark 2. We make mention of other curvatures of the curve $f \circ \gamma$ for a circle γ on submanifolds in a space form through an isometric immersion f in all theorems in this paper. All curvatures of every curve $f \circ \gamma$ are constant, since this curve is an orbit of one-parameter subgroup of the isometry group of the ambient space form.

References

[C] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. Math. **269** (1982), 481–499.
[F] D. Ferus, *Immersions with parallel second fundamental form*, Math. Z. **140** (1974), 87–92.

[M] S. Maeda, *A characterization of constant isotropic immersions by circles*, to appear in Archiv der Math..

[MO] S. Maeda and K. Ogiue, *Geometry of submanifolds in terms of behavior of geodesics*, Tokyo J. Math. **17** (1994), 347–354.

[NO] H. Nakagawa and K. Ogiue, *Complex space forms immersed in complex space forms*, Trans. Amer. Math. Soc. **219** (1976), 289–297.

[NR] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms*, Tight and Taut Submanifolds, T. E. Cecil and S. S. Chern, eds., Cambridge University Press, 1998, pp. 233-305.

[NT] H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space*, J. Math. Soc. Japan **28** (1976), 638–667.

[NY] K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. **210** (1974), 163-170.

[O] B. O'Neill, *Isotropic and Kaehler immersions*, Canadian J. Math. (1965), 905–915.

[Og] K. Ogiue, *Differential geometry of Kaehler submanifolds*, Advances in Math. **13** (1974), 73–114.

[SMA] K. Suizu, S. Maeda T. Adachi, *A characterization of Veronese imbeddings into complex projective spaces,*, Math. Rep. Acad. Sci. Royal Soc. Canada **24** (2002), 61–66.

[T] M. Takeuchi, *Parallel submanifolds of space forms, in honor of Y. Matsushima* (1981), Birkhäuser, Boston, 429–447.

Sadahiro MAEDA: Department of Mathematics Shimane University Matsue, Shimane, 690-8504, JAPAN

E-mail address: smaeda@math.shimane-u.ac.jp

Toshiaki ADACHI: Department of Mathematics Nagoya Institute of Technology Gokiso, Nagoya, 466-8555, JAPAN

E-mail address: adachi@math.kyy.nitech.ac.jp