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GENERATING FUNCTIONS OF EULERIAN AND SEPARATING EULERIAN SUBGRAPHS

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Abstract. Throughout this paper, all graphs are assumed to be embedded into an orientable surface. A graph is Eulerian if the degree of every vertex is even. An Eulerian graph is separating if the regions into which the surface is divided by the graph are 2-colorable. Let G be a graph and G^* its dual. We show an identity which relates the generating function of Eulerian subgraphs of G and the generating function of separating Eulerian subgraphs of G^*.

1. Eulerian subgraphs

We define a *graph* G as a triple $(V(G), E(G), \phi_G)$, where $V(G)$ and $E(G)$ are finite sets and $\phi_G : E(G) \to V(G) \times V(G)$. The elements of $V(G)$ and $E(G)$ are called the *vertices* and *edges*, respectively, of G. $i(e)$ and $t(e)$ are defined by $\phi_G(e) = (i(e), t(e))$ for $e \in E(G)$; we say that the vertices $i(e)$ and $t(e)$ are *on* the edge e, and the edge e are *on* the vertices $i(e)$ and $t(e)$. The *degree*, $\deg_G(\alpha)$, in G of a vertex $\alpha \in V(G)$ is

$$\#\{e \in E(G) \mid i(e) = \alpha\} + \#\{e \in E(G) \mid t(e) = \alpha\}.$$

A graph is *Eulerian* if the degree of every vertex is even. A *subgraph* H of G is a graph containing a subset of the edges of G and those vertices of G which on these edges. A subgraph H of G is *Eulerian* if the graph H is Eulerian. $\mathcal{E}(G)$ denotes the collection of all Eulerian subgraphs of G. Let $\mathbf{u} = \{u(e) \mid e \in E(G)\}$ be a set of commutative indeterminates. A polynomial

$$S(G, \mathbf{u}) = \sum_{H \in \mathcal{E}(G)} \prod_{e \in E(H)} u(e). \tag{1}$$

is the *generating function of Eulerian subgraphs* of G. Define the *Ising partition function*, $Z(G, K)$, of G by

$$Z(G, K) = \sum_{\sigma \in \mathcal{C}(G)} \exp\left(\sum_{e \in E(G)} K(e)\sigma(i(e))\sigma(t(e)) \right) \tag{2}$$

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where $\mathcal{C}(G)$ denotes $\mathrm{Map}(V(G), \{-1,1\})$, and K is a complex-valued function on $E(G)$. Expanding the exponential in (2) and noting, since σ takes values in $\{-1,1\}$, that

$$(\sigma(i(e))\sigma(t(e)))^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ \sigma(i(e))\sigma(t(e)) & \text{if } n \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} \exp\left(K(e)\sigma(i(e))\sigma(t(e))\right) &= \cosh K(e) + \sigma(i(e))\sigma(t(e)) \sinh K(e) \\ &= \cosh K(e) \left(1 + z(e)\sigma(i(e))\sigma(t(e))\right) \end{aligned}$$

where

$$z(e) = \tanh K(e). \tag{3}$$

We can then write the partition function (2) in the form

$$Z(G,K) = \left(\prod_{e \in E(G)} \cosh K(e)\right) \sum_{\sigma \in \mathcal{C}(G)} \prod_{e \in E(G)} \left(1 + z(e)\sigma(i(e))\sigma(t(e))\right).$$

We now expand the product. Since

$$\sum_{\sigma \in \mathcal{C}(G)} \prod_{\alpha \in V(G)} \sigma(\alpha)^{n_\alpha} = \begin{cases} 2^{\#V(G)} & \text{if all } n_\alpha \text{ are even} \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$Z(G,K) = 2^{\#V(G)} \left(\prod_{e \in E(G)} \cosh K(e)\right) S(G, \mathbf{z}), \tag{4}$$

where $\mathbf{z} = \{z(e) \mid e \in E(G)\}$ and $S(G,\mathbf{z})$ is the generating function of Eulerian subgraphs (1).

2. Embedding into a surface, a dual graph, and separating Eulerian subgraphs

Let $\mathcal{F}_g$ be an orientable surface of genus g. Now, graphs are supposed to be drawn on the surface (*i.e.* embedded into $\mathcal{F}_g$). An Eulerian graph is *separating* if the regions into which the surface is divided by the graph are 2-colorable. An Eulerian subgraph H of G is *separating* if the Eulerian graph H is separating. Note that whether an Eulerian subgraph of G is separating or not depends on the embedding of G we assumed. $\mathcal{E}_0(G)$ denotes the collection of all separating Eulerian subgraphs of G. A polynomial

$$S_0(G, \mathbf{u}) = \sum_{H \in \mathcal{E}_0(G)} \prod_{e \in E(H)} u(e). \tag{5}$$

is the *generating function of separating Eulerian subgraphs* of G.

A graph G drawn on the surface $\mathcal{F}_g$ has a *dual* graph G^*. The edges of G^* are in 1–1 correspondence with the edges of G; so we shall identify them, $E(G^*) = E(G)$. Now we write the partition function (2) in the other form:

$$Z(G,K) = \left(\prod_{e\in E(G)} \mathrm{e}^{K(e)}\right) \sum_{\sigma\in\mathcal{C}(G)} \prod_{e\in E(G)} \left\{\left(\mathrm{e}^{-2K(e)}\right)^{\frac{1-\sigma(i(e))\sigma(t(e))}{2}}\right\}.$$

Choose arbitrarily a vertex $\alpha_0 \in V(G)$, and divide the sum over $\mathcal{C}(G)$ into two, according to

$$\mathcal{C}(G) = \mathcal{C}_+ + \mathcal{C}_- \quad \text{where} \quad \mathcal{C}_\pm = \{\sigma \in \mathcal{C}(G) \mid \sigma(\alpha_0) = \pm 1\},$$

we have

$$Z(G,K) = \left(\prod_{e\in E(G)} \mathrm{e}^{K(e)}\right) \cdot 2 \sum_{\sigma\in\mathcal{C}_+} \prod_{e\in E(G)} \left\{\left(\mathrm{e}^{-2K(e)}\right)^{\frac{1-\sigma(i(e))\sigma(t(e))}{2}}\right\}.$$

Observing that there is a 1–1 correspondence between $\mathcal{C}_+$ and $\mathcal{E}_0(G^*)$, the set of all separating Eulerian subgraphs of G^*, by

$$\sigma \mapsto \begin{array}{c}\text{a subgraph of } G^* \text{ whose edge set is} \\ \{e \in E(G) \mid \frac{1-\sigma(i(e))\sigma(t(e))}{2} = 1\},\end{array}$$

we obtain

$$Z(G,K) = \left(\prod_{e\in E(G)} \mathrm{e}^{K(e)}\right) \cdot 2S_0(G^*, \mathbf{x}), \tag{6}$$

where $\mathbf{x} = \{x(e) \mid e \in E(G)\}$,

$$x(e) = \mathrm{e}^{-2K(e)}, \tag{7}$$

and S_0 is the generating function defined in (5).

3. An identity which relates S and S_0

We continue to suppose that graphs are drawn on the surface $\mathcal{F}_g$. So far we have obtained two different expressions, (4) and (6), for the Ising partition functon. Combining the two expressions, we obtain

$$2^{\#V(G)} \left(\prod_{e\in E(G)} \cosh K(e)\right) S(G,\mathbf{z}) = \left(\prod_{e\in E(G)} \mathrm{e}^{K(e)}\right) \cdot 2S_0(G^*, \mathbf{x}),$$

where $z(e)$ and $x(e)$ are defined in (3) and (7), respectively. Eliminating $K(e)$ completely we have symmetric relations between these variables, $z(e)$ and $x(e)$,

$$z(e) = \frac{1-x(e)}{1+x(e)}, \qquad x(e) = \frac{1-z(e)}{1+z(e)}, \qquad e \in E(G), \tag{8}$$

and relations between S and S_0,

$$S(G,\mathbf{z}) = 2^{-\#V(G)+\#E(G)+1} \prod_{e\in E(G)} \left\{\frac{1}{1+x(e)}\right\} \cdot S_0(G^*,\mathbf{x}),$$

$$S_0(G^*,\mathbf{x}) = 2^{\#V(G)-1} \prod_{e\in E(G)} \left\{\frac{1}{1+z(e)}\right\} \cdot S(G,\mathbf{z}).$$

At this stage, we can forget what the variables $z(e)$ and $x(e)$ were — equations (3) and (7); instead, we can think of them as variables, related to each other by equation (8). We can write the result in more symmetric form; since $\#V(G) - \#E(G) + \#V(G^*) = 2 - 2g$ (g is the genus of the surface), $(G^*)^* = G$, and $E(G^*) = E(G)$, we have

Theorem. *Let $\mathcal{F}_g$ be an orientable surface of genus g, and G a graph embedded into the surface. Then the generating function S of Eulerian subgraphs and the generating function S_0 of separating Eulerian subgraphs are related by the identities*

$$S(G, \frac{1-\mathbf{u}}{1+\mathbf{u}}) \;=\; 2^{\#V(G^*)-1+2g} \cdot \prod_{e\in E(G^*)} \left\{\frac{1}{1+u(e)}\right\} \cdot S_0(G^*,\mathbf{u}), \tag{9}$$

$$S_0(G, \frac{1-\mathbf{u}}{1+\mathbf{u}}) \;=\; 2^{\#V(G^*)-1} \cdot \prod_{e\in E(G^*)} \left\{\frac{1}{1+u(e)}\right\} \cdot S(G^*,\mathbf{u}), \tag{10}$$

where

$$\frac{1-\mathbf{u}}{1+\mathbf{u}} = \left\{\frac{1-u(e)}{1+u(e)} \mid e \in E(G)\right\}$$

and $\mathbf{u} = \{u(e) \mid e \in E(G)\}$ is a set of indeterminates. □

4. Remarks

(i) When $g = 0$ (the surface is a sphere) Eulerian graphs on the surface are always separating; hence $\mathcal{E}_0(G) = \mathcal{E}(G)$, and $S_0 = S$; in this case equations (4) and (6) yield the well-known Kramers–Wannier duality relation for the partition function ([1]; this is described in any statistical mechanics textbook at advanced level, *e.g.* [2]).

(ii) If a graph G is self-dual, $G^* \cong G$, then the theorem gives a relationship between S and S_0 for the same graph G.

(iii) If we specialize the variables: $u(e) \mapsto u$ for all e, then the coefficient of u^p in $S(G,\mathbf{u})$ is the number of Eulerian subgraphs of G with p edges, and the coefficient of u^p in $S_0(G,\mathbf{u})$ is the number of separating Eulerian subgraphs of G with p edges.

(iv) In view of statistical mechanics, the expression (4) of the Ising partition function is a high-temperature expansion, and equation (6) is a low-temperature expansion for Z [since K is (ferromagnetic coupling constant)/(absolute temperature); and, therefore, $z = \tanh K$ is small at high temperature, and $x = \mathrm{e}^{-2K}$ is small at low temperature].

References

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