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Some Distribution-free Multiple Comparison Procedures

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§ 1. **Introduction.** We shall propose the distribution-free multiple comparison procedures for several treatments with a control which are some nonparametric extension of those discussed by Dunnett [4] and Paulson [6] respectively under the normal assumption. Nonparametric procedures for some multiple comparison problems have been discussed by Steel [8] and Dunn [3], but we shall here deal with this problems in more general type. That is, the randomized rank statistics proposed in the theory of testing the hypothesis by Bell-Doksum [1] will be available.

We assume that c treatments π_i with distributions $F_i(x) = F(x-\Delta_i)$, $i=1,\dots, c$, are given and π_0 is a control whose distribution is $F(x)$ where $F(x)$ is continuous but unknown otherwise. The "goodness" of a treatment is characterized by a large value of a translation parameter Δ.

Now let $X_{i1},\dots, X_{in_i}$ be the random sample of size n_i from π_i, $i=0, 1,\dots, c$. We consider the following two problems (A) and (B).

(**A**). Seperate the treatments which are better than the control π_0 from what are not better under a restriction that a control π_0 should be selected as best when all $\Delta_i = 0$ with a pre-assiged probability $1-\alpha$. Hence the goal is to accept one of the following 2^c decisions :

D_0 : a control π_0 is best

D_i : π_i is the only one better than π_0, $i=1,\dots,c$

D_{ij} : π_i and π_j are the only two better than π_0, $i, j=1,\dots,c\,(i<j)$

. . . .

$D_{1\dots c}$: all π_i are better than π_0

where

(1) $P[D_0$ is accepted when all $\Delta_i = 0] = 1-\alpha$.

Dunnett's procedure A_0 is given under the normal $F(x)$ as follows,

accept D_0 if $\bar{X}_i - \bar{X}_0 \leqq d\sigma\,(N_{i0}/n_0 n_i)^{\frac{1}{2}}$ for all $i \neq 0$

accept D_i if $\bar{X}_i - \bar{X}_0 > d\sigma\,(N_{i0}/n_0 n_i)^{\frac{1}{2}}$ and

$\bar{X}_j - \bar{X}_0 \leqq d\sigma\,(N_{j0}/n_0 n_i)^{\frac{1}{2}}$ for all $j \neq i, 0$

. . . .

accept $D_{1\dots c}$ if $\overline{X}_i - \overline{X}_0 > d\sigma(N_{i0}/n_0 n_i)^{\frac{1}{2}}$ for all $i \neq 0$

where σ^2 is the variance of $F(x)$ and $n_0 + n_i = N_{i0}$ and $\overline{X}_i = n_i^{-1} \sum_{\alpha=1}^{ni} X_{i\alpha}$.

(**B**). Determine the best of the $(c+1)$ treatments when comparing c treatments with a control π_0 under the restriction (1). Hence the goal is to accept one of the $(c+1)$ decisions,

D_0 : a control π_0 is best

D_i : π_i is best, $i=1,\dots,c$.

Then Paulson's procedure B_0 under the normal assumptiom may be expressed by the following,

accept D_0 if $\overline{X}_i - \overline{X}_0 \leqq d\sigma(N_{i0}/n_0 n_i)^{\frac{1}{2}}$ for all $i \neq 0$

accept D_i if $\overline{X}_i - \overline{X}_0 > d\sigma(N_{i0}/n_0 n_i)^{\frac{1}{2}}$ and $\overline{X}_i > \overline{X}_j$ for all $j \neq i$.

We propose in the section 2 distribution-free procedures for the problems (A) and (B) and compute in the section 3 the probability that each decision is correct. The section 4 is concerned with the asymptotic relative efficiency of our procedure with respect to that in the normal theory.

§2. Distribution-free procedures. Now let $X_{i1},\dots,X_{in_i}$ be the random sample of size n_i from $F_i(x)$ and $R(X_{i\alpha})$ be the rank of $X_{i\alpha}$ in the combined sample of size $N=\sum_{i=0}^{c} n_i$. Moreover let $Z_1,\dots, Z_N$ be the random sample from the standard normal distribution $\Phi(x)$ and $Z(1) < \dots < Z(N)$ be the order statistics. Bell-Doksum has proposed the new distribution free statistics such as

$$(2) \quad T_{Ni} = n_i^{-1} \sum_{\alpha=1}^{ni} Z(R(X_{i\alpha})),\ i=0,\ 1,\dots,c$$

in order to construct some nonparametric tests of any exact size where the existing tables as the normal or Chi-square distribution are available. We here apply them for our procedures. Our decision procedures may be formulated as follows.

Procedure A for the problem A :

accept D_0 if $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni} - T_{N0}) \leqq z_\alpha$ for all $i \neq 0$

accept D_i if $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni} - T_{N0}) > z_\alpha$ and

$(n_0 n_j/N_{j0})^{\frac{1}{2}}(T_{Nj} - T_{No}) \leqq z_\alpha$ for all $j \neq i,\ 0$

. . . .

accept $D_{1\dots c}$ if $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni} - T_{N0}) > z_\alpha$ for all $i \neq 0$

where z_α is determined for a pre-assigned α by the relation

(3) $P[D_0$ is accepted when all $\Delta_i=0]=1-\alpha$.

Procedure B for the problem B :

accept D_0 if $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni}-T_{N0}) \leqq z_\alpha$ for all $i \neq 0$

accept D_i if $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni}-T_{N0}) > z_\alpha$ and $T_{Ni} > T_{Nj}$ for all $j \neq i, 0$.

§ 3. **Probabilities of correct decisions.** The following lemma 1 and 2 are the fundamental properties shown by Bell-Doksum and lemma 3 and 4 are also easily shown from their discussions.

Lemma 1. Assume that all $\Delta_i=0$, then it holds that

(i) $Z(R(X_{i\alpha}))$ $i=0, 1,\dots, c$; $\alpha=1,\dots, n_i$, have the same joint distribution as the random sample Z_i, $i=1,\dots, N$.

(ii) T_{Ni} has the same distribution as $\bar{Y}_i = n_i^{-1}\sum_{\alpha=1}^{ni} Y_{i\alpha}$, $i=0, 1, \dots c$ where $Y_{i\alpha}$ is the random sample from $\Phi(x)$.

Lemma 2. When all $\Delta_i=0$ or $\Delta_i=\delta_i/\sqrt{N}$ for all i, the statistic $\sqrt{N}(T_{Ni}-T_{Nj})$ converges in probability to the statistic $\sqrt{N}(S_{Ni}-S_{Nj})$ as $N\to\infty$ where

(4) $$S_{Ni}=n_i^{-1}\sum_{\alpha=1}^{ni} E[Z(R(X_{i\alpha})) \mid \Phi].$$

When $\Delta_i=0$ for all i, we easily get that

$$\operatorname{cov}[(n_0 n_i/N_{i0})^{\frac{1}{2}}(\bar{Y}_i-\bar{Y}_0), (n_0 n_j/N_{j0})^{\frac{1}{2}}(\bar{Y}_j-\bar{Y}_0)] = \begin{cases} 1 & i=j \\ (n_i n_j/N_{i0}N_{j0})^{\frac{1}{2}} & i \neq j. \end{cases}$$

Hence we get the Lemma 3.

Lemma 3. The joint distribution of $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni}-T_{N0})$, $i=1, \dots, c$ is the c-variate normal under all $\Delta_i=0$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$ where

(5) $$\boldsymbol{\Sigma}=[\sigma_{ij}], \quad \sigma_{ij}=\begin{cases} 1 & i=j \\ (n_i n_j/N_{i0}N_{j0})^{\frac{1}{2}} & i \neq j. \end{cases}$$

Lemma 4. Under $\Delta_i=\delta_i/\sqrt{N}$ and $n_i=\lambda_i N$ for all i, the asymptotic joint distribution of $(n_0 n_i/N_{i0})^{\frac{1}{2}}(T_{Ni}-T_{N0})$ is c-variate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ if the Hodges-Lehmann assumption [5] holds where $\boldsymbol{\mu}=(\mu_1,\dots,\mu_c)$

(6) $$\mu_i=(n_0 n_i/N_{i0} N)^{\frac{1}{2}}\delta_i \int_{-\infty}^{\infty} \frac{d}{dF}\Phi^{-1}(F(x))f(x)\,dF(x).$$

Proof. $\sqrt{N}(T_{Ni}-T_{N0})$ converges in probability as $N\to\infty$ to $\sqrt{N}(S_{Ni}-S_{N0})$ from Lemma 2 and $\sqrt{N}(S_{Ni}-S_{N0})$ are jointly normally distributed from the works of Puri [7]. Moreover it holds that

$$N\ \mathrm{cov}\,(S_{Ni},\ S_{Nj})=\begin{cases}-1+\lambda_i^{-1}+O(N^{-\frac{1}{2}}) & i=j\\ -1+O(N^{-\frac{1}{2}}) & i\neq j\end{cases}$$

and

$$E(S_{Ni})=\int_{-\infty}^{\infty}\Phi^{-1}\Big(\sum_{\alpha=0}^{c}\lambda_\alpha\ F(x-\delta_\alpha/\sqrt{N})\Big)\,dF(x-\delta_i/\sqrt{N})$$

$$=\int_{-\infty}^{\infty}\Phi^{-1}(F(x))dF(x)+\frac{1}{\sqrt{N}}(\delta_i-\bar{\delta})\int_{\infty-}^{\infty}\frac{d}{dF}\Phi^{-1}(F(x))f(x)dF(x)$$

$$+O(1/\sqrt{N})\qquad \bar{\delta}=\sum_{\alpha=1}^{c}\lambda_\alpha\,\delta_\alpha$$

where the second equality is derived from the Hodges-Lehmann assumption. The discussions above lead to Lemma 4.

Theorem 1. The value of z_α is determined by the identity

$$(7)\quad 1-\alpha=\int_{-\infty}^{z_\alpha}\cdots\int_{-\infty}^{z_\alpha}n\,(\mathbf{0},\ \boldsymbol{\Sigma})\ dx_1\ldots dx_c\quad(=P_0)$$

where $n\,(\boldsymbol{\mu},\ \boldsymbol{\Sigma})$ is the c-variate normal density with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

The probabilities that each decision is correct are given by the following.

Procedure A.

(8) $P_i=P[D_i$ is correct$]$

$$\sim\int_{z_\alpha-\mu_i}^{\infty}dx_i\int_{-\infty}^{z_\alpha-\mu_1}\cdots\int_{-\infty}^{z_\alpha-\mu_c}n\,(\mathbf{0},\ \boldsymbol{\Sigma})\prod_{j\neq i}dx_j$$

(9) $P_{ij}=P[D_{ij}$ is correct$]$

$$\sim\int_{z_\alpha-\mu_i}^{\infty}\int_{z_\alpha-\mu_j}^{\infty}dx_i\,dx_j\int_{-\infty}^{z_\alpha-\mu_1}\cdots\int_{-\infty}^{z_\alpha-\mu_c}n\,(\mathbf{0},\ \boldsymbol{\Sigma})\prod_{k\neq i,j}dx_k\ \ldots\ldots$$

. . . .

(10) $P_{1\ldots c}=P[D_{1\ldots c}$ is correct$]$

$$\sim\int_{z_\alpha-\mu_1}^{\infty}\cdots\int_{z_\alpha-\mu_c}^{\infty}n\,(\mathbf{0},\ \boldsymbol{\Sigma})\ dx_1\ldots dx_c\,.$$

Procedure. B

(11) $P_i = P[D_i \text{ is correct}]$

$$\sim \int_{z_\alpha - \nu_{i_0}}^{\infty} dx_i \int_{-\nu_{i_1}}^{\infty} \cdots \int_{-\nu_{ic}}^{\infty} n(\mathbf{0}, \boldsymbol{\Sigma}_i) \prod_{j \neq i} dx_j$$

where $\boldsymbol{\Sigma}_i$ is the submatrix of $c \times c$ by delating the i-th row and column of $\boldsymbol{\Sigma}' = [\sigma'_{jk}]_{j,\,k=0,\,1,\,\ldots,\,c}$

$$(12) \quad \sigma'_{jk} = \begin{cases} 1 & j=k \\ (n_j\, n_k/N_{ij}\ N_{ik})^{\frac{1}{2}} & j \neq k \end{cases}$$

and

$$(13) \quad \nu_{ik} = (n_i\, n_k/N_{ik}N)^{\frac{1}{2}}(\delta_i - \delta_k) \int_{-\infty}^{\infty} \frac{d}{dx} \Phi^{-1}(F(x))\, dF(x), \quad k \neq i.$$

Proof. (7) is evident from Lemma 3. For (8), we get

$$P_i = P[(n_0\, n_i/N_{i0})^{\frac{1}{2}}(T_{Ni} - T_{N0}) > z_\alpha,\ (n_0\, n_j/N_{j0})^{\frac{1}{2}}(T_{Nj} - T_{N0}) \leqq z_\alpha$$
$$\text{for all } j \neq i \mid \delta_i > 0,\ \delta_j \leqq 0 \text{ for all } j \neq i]$$
$$\sim \int_{z_\alpha - \mu_i}^{\infty} dx_i \int_{-\infty}^{z_\alpha - \mu_1} \cdots \int_{-\infty}^{z_\alpha - \mu_c} n(\mathbf{0}, \boldsymbol{\Sigma}) \prod_{j \neq i} dx_j$$

where the second equality is followed from Lemma 4.

The other relations (9) and (10) are proved similarly. For B, we first notice that $(n_i\, n_j/N_{ij})^{\frac{1}{2}}(T_{Ni} - T_{Nj})$ for $j \neq i$ are jointly normally distributed as $n(\boldsymbol{\nu}_i, \boldsymbol{\Sigma}_i$ (where $\boldsymbol{\nu}_i = (\nu_{i0}, \ldots, \nu_{ic})$ except ν_{ii}.

Then

$$P_i = P[(n_0\, n_i/N_{i0})^{\frac{1}{2}}(T_{Ni} - T_{N0}) > z_\alpha,\ T_{Ni} - T_{Nj} > 0 \text{ for all } j \neq i$$
$$\mid \delta_i > 0,\ \delta_j \leqq 0 \text{ for all } j \neq i]$$
$$\sim \int_{z_\alpha - \nu_{i_0}}^{\infty} dx_i \int_{-i_1}^{\infty} \cdots \int_{-\nu_{ic}}^{\infty} n(\mathbf{0}, \boldsymbol{\Sigma}_i) \prod_{j \neq i} dx_j\,.$$

Now assuming $n_i = n$, $i=1, \ldots, c$, we get $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 = [\sigma^0_{ii}]$, $\sigma^0_{ii} = \begin{cases} 1 & i=j \\ \frac{1}{2} & i \neq j \end{cases}$

and hence

$$(14) \quad 1-\alpha = \int_{-\infty}^{z_\alpha} \cdots \int_{-\infty}^{z_\alpha} n(\mathbf{0}, \boldsymbol{\Sigma}_0)\, dx_1 \ldots dx_c\,.$$

Then a constant z_α may be obtained from an existing table such as that of Dunnett, that is for $\alpha = 0.05$,

c	1	2	3	4	5	6
z_α	1.64	1.92	2.06	2.16	2.23	2.29

§ 4. **Asymptotic relative efficiency.** We first define the asymptotic relative efficiency of our procedure with respect to the normal procedure discussed by Dunnett or Paulson. As for the ARE e_{A,A_0} of A with respect to A_0, we assume that $m_i = \rho_i N$ and $n_0 / n_i \sim m_0 / m_i$ as $N \to \infty$ where m and n express the sample size for the normal and distribution-free procedures. Then e_{A,A_0} is defined by the common limit (if existing) of m_i / n_i as $N \to \infty$, satisfying the $2^c - 1$ equations $P_i = P_i^0$, $P_{ij} = P_{ij}^0$, ..., $P_{1\ldots c} = P_{1\ldots c}^0$ under $P_0 = P_0^0$ and $\Delta_i = \delta_i / \sqrt{N}$ where $P^{0\prime}s$ are the values in the procedure A_0 corresponding to $P's$ in the procedure A. The definition is quite analogious for e_{B,B_0}.

Theorem 2. The ARE e_{A,A_0} or e_{B,B_0} is given by the following,

$$(15) \quad e_{A,A_0} = e_{B,B_0} = \sigma^2 \left[\int_{-\infty}^{\infty} \frac{d}{dx} \Phi^{-1}(F(x))\, dF(x) \right]^2.$$

Proof. The probabilities $P^{0\prime}\,s$ in the procedure A_0 are easily computed by noticing that the correlation of $\overline{X}_i - \overline{X}_0$ and $\overline{X}_j - \overline{X}_0$ is $(m_i m_j / M_{i0} M_{j0})^{\frac{1}{2}}$ and $m_i / M_{i0} \sim n_i / N_{i0}$. First

$$(16) \quad P_0^0 = P\,[\overline{X}_i - \overline{X}_0 \leqq d\sigma\,(M_{i0}/m_i m_0)^{\frac{1}{2}} \text{ for all } i \mid \text{all } \Delta_i = 0]$$

$$\sim \int_{-\infty}^{d} \cdots \int_{-\infty}^{d} n\,(\mathbf{0}, \boldsymbol{\Sigma})\, dx_1 \ldots dx_c$$

Hence we get $P_0 = P_0^0$ from (7) and (16) by taking $d = z_\alpha$. Similarly we get under $\Delta_i = \delta_i / \sqrt{N}$

$$P_i^0 \sim \int_{d-si}^{\infty} dx_i \int_{-\infty}^{d-s1} \cdots \int_{-\infty}^{d-sc} n\,(\mathbf{0}, \boldsymbol{\Sigma}) \prod_{j \neq i} dx_j$$

$$(17) \quad P_{ij}^0 \sim \int_{d-si}^{\infty} \int_{d-sj}^{\infty} dx_i\, dx_j \int_{-\infty}^{d-s1} \cdots \int_{-\infty}^{d-sc} n\,(\mathbf{0}, \boldsymbol{\Sigma}) \prod_{k \neq i,j} dx_k$$

. . . .

$$P_{1\ldots c}^0 \sim \int_{d-s1}^{\infty} \cdots \int_{d-sc}^{\infty} n\,(\mathbf{0}, \boldsymbol{\Sigma})\, dx_1 \ldots dx_c$$

where

$$(18) \quad s_i = \sigma^{-1}\, \delta_i\, (m_0 m_i / M_{i0} N)^{\frac{1}{2}}.$$

From the relations (8), (9), (10) and (17), e_{A,A_0} is given by the limiting value of m_i / n_i satisfying $\mu_i = s_i$. Thus we get (15) by (6) and (18). The value of $e_{B,B0}$ is also obtained by noticing

$$(19) \quad P_i^0 \sim \int_{z\alpha - wi0}^{\infty} dx_i \int_{-wi1}^{\infty} \dots \int_{-wic}^{\infty} n\,(\mathbf{0},\ \boldsymbol{\Sigma}_i)\ \prod_{i \neq i} dx_j$$

where

$$(20) \quad w_{ik} = \sigma^{-1}\,(\delta_i - \delta_k)\ (m_i\, m_k / M_{ik} N)^{\frac{1}{2}}.$$

The expression (15) is known to be the ARE of the two-sample normal scores test respective to the Student t-test and is always $\geqq 1$ as shown by Chernoff-Savage [2]. Lastly we shall add that some multivariate extensions of these procedures are now preparing by the author along the line of Tamura [9].

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