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On the Centroid Method in Factor Analysis (II)

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In the previous paper [1], we have mentioned the problem of convergence to the unique structure for the iterative centroid method and to examine this problem we have proposed the non-adjusted complete centroid method. We have also shown that in the case of known true communalities we could get the true structure by this method without iterations. In the present paper, we describe some numerical results in the case of unknown true communalities. It is very complicated problem to examine the problem of unique convergence irrespective of initial estimates of communalities numerically. In the present paper, we consider only three cases of initial estimates, that is, (1) unit diagonal elements (UDE), (2) maximum absolute row elements (MRE), (3) squared multiple correlation coefficient of a variable with remaining others (SMC). We cut off our iterations with following criterions : (1) the mean residual sum of squares of off-diagonal parts and as the critical value of which we have taken the value 0.00004 (2) communalities which have 4 reliable numbers with respect to decimal fraction part, (3) given maximal numbers of iterations, and (4) maximum absolute difference of off-diagonal elements between given correlation matrix and the reproduced correlation matrix using estimates of factor loadings, and for which we have 4 reliable numbers with respect to decimal fraction part. In the case of given factor numbers, we use criterions (2), (3), (4), while in the case of unspecified factor numbers, we use these four criterions and in addition to these, (5) factor number to guarantee one degree of freedom for the significance test of fit. We also describe some results of sampling experiments and some of them are compared with corresponding results in the maximum likelihood estimates, and some results with respect to the factorial invariance will be described. These latter two problems will be further studied and described in the next paper. In the present calculations, some of results do not satisfy the above mentioned convergence criterions exactly, but they may be sufficient results approximately. These calculations are made with HITAC5020 at Computing Center of Tokyo University and with RIMS-C-1 at Research Institute For Mathematical Sciences in Kyoto University. Some of these results have gotten during our activities as research fellows at the above mentioned institute in Kyoto University. Now, we describe some results and for details we refer to attached tables.

(A) At present, we have examined 28 cases with given true factor numbers and among these we have gotten unique convergence for 11 cases. For other 17 cases, we have gotten varying results with initial estimates of communalities. Among 11 cases above mentioned, we have gotten convergences to given true structures for 8 cases. These 28 cases are unique structures with given factor numbers.

(B) With respect to sampling experiments, we have the following results. With respect to the generating procedure of samples, we refer to Browne [2].

(1) *Series 14*

For these cases, the population correlation matrix is the Pop. 07 (see attached table). For Pop. 07, we have gotten the unique convergence to the true structure irrespective of initial estimates of communalities.

(a) *14S02*

For this case we have the sample size 100 and we have gotten unique convergence with mean residual sum of squares 0.000646. The number of factors is specified being two.

(b) *14S03*

For this case we have also the sample size 100 and we may find variations with respect to variables 1 and 2. The number of factors is also specifyed being two. Mean residual sum of squares are (1) 0.000713 for SMC, (2) 0.000731 for MRE, and (3) 0.000713 for UDE.

(2) *Series 12*

For these cases, the population correlation matrix is the matrix 12T, and we could not get the unique convergence.

(a) *12S02 with sample size 50*

We have specified the number of factos being two, and have gotten the mean residual sum of squares (1) 0.001590 for SMC, (2) 0.001773 for MRE, and (3) 0.001778 for UDE. In this case, we have also the following comparison with the maximum likelihood estimates (we describe here only communalities).

initial	estimate	MLE (5+4)	Centroid (39 cycles)
1	0.828	0.9495	0.9689
2	0.535	0.4355	0.3176
3	0.591	0.5033	0.6980
4	0.635	0.5804	0.5823
5	0.551	1.0000	0.6355

The significance of numbers in bracket is the following:

for MLE figures before + sign tell us the numbers of iteration cycles needed at which we got an improper solution and total cycles in the above is 9 cycles. The mean residual sum of squares for the centroid method is 0.001783.

initial	estimate	MLE (1+5)	Centroid (39 cycles)
1	0.4	1.0000	1.0400
2	0.5	0.3880	0.3177
3	0.8	0.7177	0.7311
4	0.8	0.5853	0.5823
5	0.4	0.4172	0.4275

The residual sum of squares for the centroid method is 0.001077.

(b) *12S05 with sample size 100*

We have specified the number of factors being two, and gotten the mean residual sum of squares (1) 0.000755 for MRE, (2) 0.000550 for SMC, and (3) 0.000259 for UDE. In this case, we have the following comparison with the maximum likelihood estimates.

	initial	MLE (13)	Centroid (15)
1	0.829	0.9383	0.9436
2	0.513	0.9877	0.5363
3	0.525	0.4228	0.5279
4	0.702	0.6553	0.6786
5	0.625	0.5644	0.4836

The mean residual sum of squares for the centroid is 0.000731.

	initial	MLE (1+2)	Centroid (15)
1	0.2	1.0000	0.9336
2	0.8	0.9950	0.7453
3	0.8	0.3759	0.4489
4	0.5	0.5938	0.6717
5	0.5	0.4789	0.5383

The mean residual sum of squares for the centroid method is 0.000235.

	initial	MLE (1+8)	Centroid (15)
1	0.4	1.0000	1.0315
2	0.5	0.3744	0.3547
3	0.8	0.9734	0.8000
4	0.8	0.6268	0.6674
5	0.4	0.5221	0.4756

The mean residual sum of squares for the centroid is 0.000449.

(C) With respect to factorial invariance, we have the following results.

(1) *The case of Pop. 05*

In this case, for the case of Pop. 05 we have the unique convergence to the true structure.

(a) Deleting variable with the highest communality (=0.61), and extracting two factors, we could not get the unique convergence and also the factorial invariance (cf. Pop. 05-2).

(b) Deleting variable with the lowest communality (=0.17), we could not get the unique convergence and also the factorial invariance.

(2) *The case of Pop. 03*

For the Pop. 03, we have gotten the unique convergence to a structure different from the true.

(a) Adding one variable to the original, we have gotten the unique convergence to the nearly true, and we may find the factorial invariance with respect to variables 2 to 8. (cf. Pop. 03-1)

(b) Deleting variable 1, we have gotten the factorial invariance and unique convergence for the case of SMC and MRE initials. (cf. Pop. 03-2)

(3) *The case of Pop. 04 and Pop. 04-1.*

In these cases, we have gotten the (almost) unique convergence with respect to original matrices, that is, for Pop. 05, Pop. 03, and Pop. 04. In the following cases, we could not get the unique convergence with respect to the original matrices.

(1) *The case of Pop. 01, Pop. 01-2 and Pop. 01-3.*

(2) *The case of Pop. 02, Pop. 02-1* and *Pop. 02-2.*

The followings are tables of our numerical results.

(1) *Pop. 01* ; $p=5$, $k=2$

(a) True structure

	F1	F2	Comm.	Cor. Matrix			
1	0.02	0.9	0.8104				
2	0.1	−0.01	0.0101	−0.007			
3	0.3	0.6	0.4500	0.546	0.024		
4	0.8	−0.2	0.6800	−0.164	0.082	0.12	
5	0.4	0.5	0.41	0.458	0.035	0.42	0.22

(b) Squared Multiple Correlation

1	2	3	4	5
0.4421	0.0071	0.3595	0.1731	0.3166

(c) Final Results

(1) SMC initial (43 cycles)

	Comm.	F1	F2
1	0.8578	0.7058	0.5997
2	0.0126	0.0612	−0.0942
3	0.4414	0.6476	0.1483
4	0.5387	0.3326	−0.6543
5	0.4205	0.6485	0.0005

(2) MRE initial (43 cycles)

Comm.	F2	F1
0.8551	0.5991	0.7044
0.0125	−0.0934	0.0611
0.4418	0.15	0.6475
0.5459	−0.6583	0.3355
0.4199	0.0026	0.648

(3) UDE initial (43 cycles)

1	0.8571	0.7055	0.5995
2	0.0122	0.0607	−0.0925
3	0.4413	0.6473	0.1494
4	0.546	0.3351	−0.6586
5	0.4202	0.6482	0.0022

(2) *Pop. 01-2, Deleted variable 1*

(a) Squared Multiple Correlation

2	3	4	5
0.0071	0.1773	0.0548	0.2057

(b) Final Results

(1) SMC initial (13 cycles)

	Comm.	F1	F2
1	0.0605	0.1186	0.2155
2	0.3861	0.5591	−0.2711
3	0.1519	0.3378	0.1946
4	0.4868	0.6837	−0.139

(2) MRE initial (11 cycles)

Comm.	F1	F2
0.0448	0.1089	0.1815
0.4406	−0.5888	−0.3064
0.1828	0.3545	0.2390
0.4406	0.6539	−0.1142

(3) UDE initial (13 cycles)

1	0.1122	0.1485	0.3002
2	0.4408	0.5892	−0.306
3	0.1122	0.3133	0.1184
4	0.4408	0.6543	−0.1126

(3) *Pop. 01-3, Deleted variable 2*

(a) Squared Multiple Correlation

1	3	4	5
0.4421	0.3594	0.1692	0.3165

(b) Final Results

(1) SMC initial (13 cycles)

	Comm.	F1	F2
1	0.6196	0.6391	0.4595
2	0.5499	0.7163	0.192
3	0.3399	0.2259	−0.5374
4	0.5076	0.7026	−0.1141

(2) MRE initial (11 cycles)

Comm.	F1	F2
0.5775	0.6225	0.4358
0.5775	0.7306	0.2092
0.2956	0.2072	−0.5029
0.5338	0.7166	−0.1422

(3) UDE initial (13 cycles)

	Comm.	F1	F2
1	0.5746	0.6157	0.4422
2	0.5746	0.7227	0.2286
3	0.4651	0.279	−0.6223
4	0.4651	0.6803	−0.0484

(4) *Pop. 02*; $p=6$, $k=2$

(a) True structure

	F1	F2	Comm.	Cor. Matrix				
1	0.1	−0.3	0.1					
2	0.2	0.1	0.05	−0.01				
3	0.4	0.5	0.41	−0.11	0.13			
4	0.2	0.3	0.13	−0.07	0.07	0.23		
5	0.5	−0.8	0.89	0.29	0.02	−0.2	−0.14	
6	0.4	0.2	0.2	−0.02	0.1	0.26	0.14	0.04

(b) Squared Multiple Correlation

1	2	3	4	5	6
0.0876	0.0249	0.1508	0.0725	0.1317	0.0885

(c) Final Results

(1) SMC initial (19 cycles)

	Comm.	F1	F2
1	0.2693	0.2069	−0.476
2	0.0515	0.214	0.0751
3	0.4141	0.4288	0.4799
4	0.1334	0.2152	0.2951
5	0.3134	0.1915	−0.5261
6	0.2099	0.4322	0.1519

(2) MRE initial (19 cycles)

Comm.	F1	F2
0.291	0.2197	−0.4927
0.0515	0.2141	0.0751
0.4141	0.4288	0.4798
0.1334	0.2152	0.2951
0.291	0.1783	−0.5091
0.2099	0.4323	0.1518

(3) UDE initial (19 cycles)

	Comm.	F1	F2
1	0.2761	0.2109	−0.4813
2	0.0515	0.2141	0.0751
3	0.4141	0.4288	0.4798
4	0.1334	0.2152	0.2951
5	0.306	0.1871	−0.5205
6	0.2099	0.4323	0.1518

(5) *Pop. 02-1, adding variable 7 to Pop. 02*

(a) Squared Multiple Correlation

1	2	3	4	5	6	7
0.0902	0.0424	0.3542	0.1143	0.1617	0.1686	0.4695

(b) Variable 7 has the communality 0.85 and loadings are 0.6 and 0.7 for F1 and F2, respectively.

(c) Final Results

(1) SMC initial (18 cycles)

	Comm.	F1	F2
1	0.2578	−0.2951	0.4132
2	0.0522	0.173	0.1492
3	0.4118	0.628	0.1323
4	0.1322	0.3615	0.0389
5	0.3292	−0.3768	0.4327
6	0.2114	0.3483	0.3003
7	0.8494	0.8937	0.2252

(2) MRE initial (18 cycles)

Comm.	F1	F2
0.2701	−0.2991	0.4251
0.0522	0.173	0.1492
0.4119	0.628	0.1321
0.1322	0.3616	0.0388
0.3152	−0.3723	0.4203
0.2115	0.3843	0.3003
0.8494	0.8938	0.2249

(3) UDE initial (18 cycles)

	Comm.	F1	F2
1	0.2703	−0.2992	0.4252
2	0.0522	0.173	0.1492
3	0.4119	0.628	0.1321
4	0.1322	0.3616	0.0388
5	0.315	−0.3722	0.4201
6	0.2115	0.3483	0.3003
7	0.8494	0.8938	0.2249

(6) *Deleted variable 2 in Pop. 02*

(a) Squared Multiple Correlation

1	2	3	4	5
0.0875	0.1413	0.071	0.1299	0.0852

(b) Final Results

(1) SMC initial (27 cycles)

	Comm.	F1	F2
1	0.2715	−0.3724	0.3644
2	0.4041	0.5889	0.2394
3	0.1321	0.3483	0.104
4	0.3138	−0.442	0.3441
5	0.2192	0.2931	0.3651

(2) MRE initial (27 cycles)

Comm.	F1	F2
0.2928	−0.3828	0.3824
0.4041	0.5888	0.2394
0.1321	0.3483	0.104
0.2928	−0.4317	0.3262
0.2192	0.2931	0.3651

(3) UDE initial (26 cycles)

	COMM.	F1	F2
1	0.2928	−0.3828	0.3824
2	0.4041	0.5888	0.2394
3	0.1321	0.3483	0.104
4	0.2928	−0.4317	0.3262
5	0.2192	0.2931	0.3651

(7) *Pop. 03*, $p=9$, $k=3$

(a) True Structure

	Comm.	F1	F2	F3
1	0.7	0.5	0.6	0.3
2	0.61	−0.6	0.4	0.3
3	0.69	0.2	−0.4	0.7
4	0.54	−0.6	0.3	0.3
5	0.81	0.7	−0.4	−0.4
6	0.45	0.5	−0.4	−0.2
7	0.9	−0.5	0.7	−0.4
8	0.33	−0.2	−0.2	0.5
9	0.75	0.5	−0.5	0.5

(b) Squared Multiple Correlation

1	2	3	4	5	6	7	8	9
0.0823	0.5423	0.5996	0.4772	0.6626	0.4088	0.7774	0.2772	0.6841

(c) Final Results

(1) SMC initial (34 cycles)

	Comm.	F1	F2	F3
1	0.4734	−0.0991	0.1507	0.664
2	0.6107	−0.64	0.4234	−0.1475
3	0.6907	0.5252	0.6431	0.0358
4	0.5399	−0.5599	0.4212	−0.2214
5	0.8092	0.6869	−0.5415	0.21

(2) MRE initial (34 cycles)

Comm.	F1	F2	F3
0.4731	−0.0991	0.1507	0.6638
0.6107	−0.6401	0.4234	−0.1475
0.6907	0.5252	0.6431	0.0358
0.5399	−0.5599	0.4212	−0.2213
0.8092	0.6869	−.5415	0.2099

6	0.4523	0.5932	−0.3053	0.0851	0.4523	0.5932	−.3053	0.085
7	0.902	−0.907	−0.2819	−0.0017	0.9021	−0.907	−.2819	−0.0016
8	0.3371	0.104	0.5311	−0.2104	0.3371	0.104	0.5311	−0.2104
9	0.7569	0.7603	0.386	0.173	0.7569	0.7603	0.3859	0.173

(3) UDE initial (35 cycles)

	Comm.	F1	F2	F3
1	0.4722	−0.0989	0.1504	0.6632
2	0.6107	−0.6401	0.4234	−0.1473
3	0.6907	0.5252	0.6431	0.0359
4	0.5399	−0.5599	0.4212	−0.2212
5	0.8092	0.6869	−0.5416	0.2097
6	0.4523	0.5932	−0.3053	0.0849
7	0.9021	−0.907	−0.2818	−0.0015
8	0.3372	0.104	0.5311	−0.2105
9	0.757	0.7603	0.3859	0.1731

(8) *Pop. 03-1, adding one variable to Pop. 03*

(a) Added variable has the communality 0.89 and loadings are 0.6, 0.2 and 0.7 for F1, F2 and F3, respectively.

(b) Squared Multiple Correlations

1	2	3	4	5	6	7	8	9	10
0.5638	0.5432	0.6188	0.4772	0.6646	0.4096	0.7845	0.2776	0.7021	0.6998

(c) Final Results

(1) SMC initial (50 cycles)

	Comm.	F1	F2	F3
1	0.7058	0.2417	0.2846	0.7526
2	0.61	−0.5704	0.5331	0.0215
3	0.69	0.5886	0.5102	−.2885
4	0.54	−.5248	0.5103	−.0645
5	0.81	0.6178	−.6535	0.0355
6	0.45	0.523	−.4127	−.0785
7	0.9	−.8674	−.0804	0.3756
8	0.33	0.1002	0.476	−.3056
9	0.75	0.811	0.2196	−.21
10	0.8854	0.6424	0.5387	0.4271

(2) MRE initial (50 cycles)

	Comm.	F1	F2	F3
1	0.7133	0.2431	0.2861	0.7566
2	0.61	−.5704	0.5331	0.0215
3	0.69	0.5886	0.5101	−.2887
4	0.54	−.5248	0.5103	−.0645
5	0.81	0.6178	−.6535	0.0355
6	0.45	0.523	−.4127	−.0785
7	0.9	−.8674	−.0804	0.3757
8	0.33	0.1002	0.476	−.3056
9	0.75	0.8109	0.2196	−.2101
10	0.8796	0.6413	0.5375	0.4235

(c) UDE initial (50 cycles)

	Comm.	F1	F2	F3
1	0.7056	0.2416	0.2846	0.7525
2	0.61	−.5704	0.5331	0.0215
3	0.69	0.5886	0.5102	−.2885
4	0.54	−.5248	0.5103	−.0645
5	0.81	0.6178	−.6535	0.0355
6	0.45	0.523	−.4127	−.0785

7	0.9	−.8674	−.0804	0.3756
8	0.33	0.1002	0.476	−.3056
9	0.75	0.811	0.2196	−.21
10	0.8855	0.6424	0.5388	0.4272

(9) *Pop. 03-2, deleted variable 1 in Pop. 03*

(a) Squared Multiple Correlations

1	2	3	4	5	6	7	8
0.5420	0.5943	0.475	0.6624	0.4074	0.7671	0.2705	0.6702

(b) Final Results

(1) SMC initial (25 cycles)

	Comm.	F1	F2	F3
1	0.61	−.6407	0.4463	0.018
2	0.6903	0.5454	0.6236	0.063
3	0.54	−.5723	0.4578	−.054
4	0.81	0.6925	−.5747	0.0107
5	0.45	0.5889	−.3179	−.0467
6	0.8999	−.9061	−.2728	0.0673
7	0.3301	0.0892	0.5579	−.104
8	0.7496	0.7878	0.3424	0.1083

(2) MRE initial (25 cycles)

	Comm.	F1	F2	F3
1	0.61	−.6407	0.4463	0.018
2	0.6903	0.5454	0.6236	0.0631
3	0.54	−.5723	0.4578	−.0541
4	0.81	0.6925	−.5747	0.0107
5	0.45	0.5889	−.3179	−.0468
6	0.8997	−.906	−.2727	0.0668
7	0.3302	0.0892	0.558	−.1045
8	0.7496	0.7878	0.3424	0.1082

(3) UDE initial (25 cycles)

	Comm.	F1	F2	F3
1	0.6102	−.6371	0.4521	−.0000
2	0.6959	0.5434	0.6094	0.1709
3	0.5351	−.568	0.4609	−.0034
4	0.8112	0.6888	−.5802	−.0102
5	0.4464	0.5848	−.3232	0.0067
6	0.8822	−.8972	−.2498	−.1218
7	0.628	0.1501	0.6342	−.4508
8	0.7398	0.7814	0.323	0.1582

(10) *Pop. C4* ; $p=4$, $k=1$

(a) True structure

	Comm.	F1	Cor. matrix		
1	0.81	0.9			
2	0.0001	−.01	−.009		
3	0.000009	0.003	0.0027	−.00003	
4	0.000001	−.001	−.0009	0.00001	−.000003

(b) Squared Multiple Correlations

1	2	3	4
0.00009	0.00008	0.00001	0.00000

(c) Final Results

(1) SMC initial (50 cycles)

	Comm.	F1
1	0.067402	0.2596
2	0.001108	−.0333
3	0.000085	0.0092
4	0.000009	−.003

(2) MRE initial (50 cycles)

Comm.	F1
0.06746	0.2597
0.001108	−.0333
0.000085	0.0092
0.000009	−.003

(3) UDE initial (50 cycles)

	Comm.	F1
1	0.065372	0.2557
2	0.00114	−.0338
3	0.000087	0.0093
4	0.000009	−.0031

(11) *Pop. 04-1, adding one variable to Pop. 04*

(a) The added variable has the communality 0.36 and the loading 0.6.

(b) Squared Multiple Correlations

1	2	3	4	5
0.291636	0.000083	0.000008	0.000001	0.291601

(c) Final Results

(1) SMC initial (50 cycles)

	Comm.	F1
1	0.629042	0.793122
2	0.000104	−.010177
3	0.000009	0.003053
4	0.000001	−.001018
5	0.463231	0.680611

(2) MRE initial 50 cycles)

Comm.	F1
0.629035	0.793118
0.000104	−.010177
0.000009	0.003053
0.000001	−.001018
0.463236	0.680614

(3) UDE initial (50 cycles)

	Comm.	F1
1	0.62776	0.792313
2	0.000104	−.010178
3	0.000009	0.003053
4	0.000001	−.001018
5	0.464179	0.681307

(12) *Pop. 05* ; $p=6$, $k=3$

(a) True structure

	Comm.	F1	F2
1	0.37	0.1	−.6
2	0.20	0.2	0.4
3	0.18	0.3	0.3
4	0.17	0.4	−.1
5	0.29	0.5	0.2
6	0.61	0.6	0.5

(b) Squared Multiple Correlations

1	2	3	4	5	6
0.1067	0.1318	0.1266	0.0715	0.1808	0.3057

(c) Final Results

(1) Squared MC initial (50 cycles)

	Comm.	F1	F2
1	0.3697	−.3534	0.4947
2	0.2001	0.4243	−.1415
3	0.18	0.4243	−.0001
4	0.17	0.2122	0.3536
5	0.29	0.495	0.2121
6	0.61	0.7778	0.0707

(2) MRE initial (50 cycles)

	Comm.	F1	F2
1	0.3697	−.3534	0.4947
2	0.2001	0.4243	−.1415
3	0.18	0.4243	−.0001
4	0.17	0.2122	0.3536
5	0.29	0.495	0.2121
6	0.61	0.7778	0.0707

(3) UDE initial (50 cycles)

	Comm.	F1	F2
1	0.3697	−.3534	0.4948
2	0.2001	0.4243	−.1415
3	0.18	0.4243	−.0001
4	0.17	0.2121	0.3536
5	0.29	0.495	0.2121
6	0.61	0.7778	0.0707

(13) *Pop. 05-2 ; Deleted variable 6 in Pop. 05*

(a) Squared Multiple Correlations

1	2	3	4	5
0.0760	0.09	0.0808	0.0506	0.091

(b) Final Results

(1) SMC initial (25 cycles)

	Comm.	F1	F2
1	0.2565	−.3148	0.3968
2	0.2455	0.4567	−.1921
3	0.185	0.4301	0.0031
4	0.1816	0.2067	0.3726
5	0.2823	0.4867	0.2131

(2) MRE initial (23 cycles)

	Comm.	F1	F2
1	0.2497	−.3113	0.3909
2	0.2497	0.4591	−.1973
3	0.1852	0.4303	0.0026
4	0.1821	0.207	0.3732
5	0.2819	0.4867	0.2124

(3) UDE initial (25 cycles)

	Comm.	F1	F2
1	0.309	−.3414	0.4387
2	0.2202	0.4419	−.1576
3	0.1829	0.4276	0.0083
4	0.177	0.2036	0.3681
5	0.2849	0.4865	0.2198

(14) *Pop. 05-3 ; Deleted variable 4 in Pop. 05*

(a) Squared Multiple Correlation

1	2	3	4	5
0.0862	0.1318	0.1257	0.1708	0.29

(b) Final Results

(1) SMC initial (29 cycles)

	Comm.	F1	F2
1	0.1982	−.3542	0.2696
2	0.2593	0.4676	−.2016
3	0.1814	0.4241	0.0393
4	0.3019	0.4687	0.2868
5	0.6057	0.7646	0.1451

(2) MRE initial (29 cycles)

Comm.	F1	F2
0.1882	−.3502	0.256
0.2682	0.4712	−.2148
0.1814	0.4241	0.0391
0.3022	0.4688	0.287
0.6056	0.7646	0.1447

(3) UDE initial (29 cycles)

	Comm.	F1	F2
1	0.2357	−.369	0.3155
2	0.2357	0.4576	−.1621
3	0.1812	0.4236	0.0421
4	0.3001	0.4675	0.2856
5	0.6063	0.7641	0.1499

(15) *Pop. 17* ; $p=5$, $k=2$

(a) True Structure

	Comm.	F1	F2
1	0.6388	0.68	0.42
2	0.657	0.81	0.03
3	0.4001	0.49	0.4
4	0.578	0.62	0.44
5	0.8245	0.82	0.39

(b) Squared Multiple Correlations

1	2	3	4	5
0.5613	0.4718	0.3432	0.5016	0.6939

(c) FINAL RESULTS

(1) SMC initial (12 cycles)

	Comm.	F1	F2
1	0.64	0.7979	0.058
2	0.5546	0.7152	−.2076
3	0.4013	0.6181	0.1389
4	0.5773	0.7522	0.1068
5	0.8503	0.9171	−.0962

(2) MRE initial (17 cycles)

Comm.	F1	F2
0.6386	0.795	0.0812
0.6772	0.7451	−.3494
0.4	0.6158	0.1442
0.5781	0.75	0.1246
0.8223	0.9068	−.0006

(3) UDE initial (12 cycles)

	Comm.	F1	F2
1	0.6332	0.7932	0.0638
2	0.5163	0.7026	0.1505
3	0.5772	0.6619	−.373
4	0.5534	0.7432	0.0325
5	0.8509	0.9138	0.1261

(16) *Pop. 12* ; $p=9$, $k=3$

(a) True Structure

	Comm.	F1	F2	F3
1	0.7325	0.1		0.85
2	0.29	0.2	0.5	
3	0.5525	0.3	0.4	0.55
4	0.4525	0.4	0.3	0.45
5	0.4125	0.5	0.2	0.35
6	0.37	0.6	0.1	
7	0.5525	0.7		0.25
8	0.89	0.8	0.5	
9	0.8325	0.6		0.15

(b) Squared Multiple Correlations

1	2	3	4	5	6	7	8	9
0.4108	0.2106	0.4402	0.3957	0.3783	0.3383	0.4894	0.6904	0.6582

(c) Final Results

(1) SMC initial (29 cycles)

	Comm.	F1	F2	F3
1	0.7324	0.4777	−.6797	0.2058
2	0.2901	0.3412	0.087	−.4075
3	0.5525	0.6429	−.3318	−.1706
4	0.4525	0.6411	−.1918	−.0686
5	0.4125	0.6393	−.0519	0.0334
6	0.37	0.5208	0.303	0.0835
7	0.5525	0.6824	0.1421	0.2582
8	0.8898	0.826	0.3929	−.2305
9	0.8325	0.7973	0.3301	0.2963

(2) MRE initial (29 cycles)

Comm.	F1	F2	F3
0.7325	0.4777	−.6797	0.2058
0.2902	0.3412	0.0871	−.4076
0.5525	0.6429	−.3318	−.1706
0.4525	0.6411	−.1918	−.0686
0.4125	0.6393	−.0519	0.0334
0.37	0.5208	0.303	0.0835
0.5525	0.6824	0.1421	0.2582
0.8897	0.826	0.3929	−.2304
0.8325	0.7973	0.3301	0.2963

(3) UDE initial (29 cycles)

	Comm.	F1	F2	F3
1	0.7325	0.4777	−.6797	0.2058
2	0.2902	0.3412	0.0871	−.4077
3	0.5525	0.6429	−.3318	−.1706
4	0.4525	0.6411	−.1918	−.0686
5	0.4125	0.6393	−.0519	0.0334
6	0.37	0.5208	0.303	0.0835
7	0.5525	0.6824	0.1421	0.2582
8	0.8896	0.826	0.3929	−.2304
9	0.8325	0.7973	0.3301	0.2963

(17) *Pop. 18* ; $p=13$, $k=4$

(a) True Structure

	Comm.	F1	F2	F3	F4
1	0.9057708422	0.94136	− 0.08466	0.05721	0.09577
2	0.4427843445	0.18625	0.56248	0.0537	0.29804
3	0.9774263227	0.68772	0.44547	0.54903	−.06775
4	0.9233264625	0.90948	0.14378	0.24731	0.11974
5	0.621797387	0.31553	−.35165	−.09594	0.624
6	0.9092294403	0.81817	0.09031	0.40152	0.26543
7	0.7334244561	0.056	0.07856	0.84937	−.05184
8	0.9590822577	0.07302	0.3613	0.51107	0.74968
9	0.8664687786	0.06097	0.88721	0.14294	0.2349
10	0.657674987	−.63017	−.06738	0.49821	0.08836
11	0.9319979123	0.23439	0.68631	0.07004	0.63335
12	0.1907959301	0.04625	0.09716	0.01164	0.42318
13	0.4729353171	0.52781	0.37303	0.2014	0.12099

(b) Final Results

(1) Given initials (44 cycles)

	Given initial	F1	F2	F3	F4	Est. comm.
1	0.9096302	0.69654	−.64642	0.04317	−.02956	0.9057711822
2	0.7671258	0.52704	0.31049	−.23834	−.10864	0.4427846198
3	0.9145793	0.83581	−.1138	−.15228	0.49265	0.9774239775
4	0.135835	0.8461	−.4397	−.01121	0.11827	0.9233271374
5	0.8327057	0.31348	−.19641	0.4883	−.49651	0.6218118589
6	0.6019432	0.87141	−.30972	0.18403	0.1417	0.9092260142
7	0.8164857	0.32852	0.20509	0.32226	0.69254	0.733455059
8	0.7532471	0.7047	0.57096	0.36518	−.05572	0.9590591458
9	0.8052419	0.58096	0.57517	−.44446	0.02419	0.8664646723
10	0.5697763	−.279	0.56242	0.39501	0.3278	0.6576557755
11	0.9010009	0.76762	0.47921	−.15221	−.29993	0.9320058749
12	0.6326829	0.26447	0.18802	0.14757	−.25244	0.1907972298
13	0.0413178	0.75907	−.06047	−.157	0.10127	0.4729357654

(2) MRE initial (41 cycles)

	Comm.	F1	F2	F3	F4
1	0.905770471	0.69654	−.64642	0.04318	−.02956
2	0.4427840669	0.52704	0.31049	−.23834	−.10864
3	0.9774286979	0.83581	−.1138	−.15228	0.49266
4	0.9233257749	0.8461	−.43969	−.0112	0.11827
5	0.6217830651	0.31348	−.1964	0.48829	−.4965
6	0.9092328477	0.87141	−.30972	0.18404	0.1417
7	0.7333937962	0.32851	0.20507	0.32223	0.69252
8	0.9591053839	0.70471	0.57097	0.3652	−.05572
9	0.8664728374	0.58096	0.57518	−.44446	0.02419

10	0.6576942487	−.27901	0.56244	0.39502	0.32783
11	0.9319899691	0.76762	0.4792	−.1522	−.29992
12	0.1907946222	0.26446	0.18802	0.14757	−.25244
13	0.472934869	0.65907	−.06047	−.157	0.10127

(3) UDE initial (40 cycles)

	Comm.	F1	F2	F3	F4
1	0.9057704001	0.69654	−.64642	0.04317	−.02956
2	0.4427840632	0.52704	0.31049	−.23834	−.10864
3	0.9774287499	0.83581	−.1138	−.15228	0.49266
4	0.9233257509	0.8461	−.43969	−.0112	0.11827
5	0.6217835544	0.31348	−.1964	0.48829	−.4965
6	0.9092327737	0.87141	−.30972	0.18404	0.1417
7	0.7333939223	0.32851	0.20507	0.32223	0.69252
8	0.9591051984	0.70471	0.57097	0.3652	−.05572
9	0.8664726845	0.58096	0.57518	−.44446	0.02419
10	0.6576941969	−.27901	0.56244	0.39502	0.32783
11	0.9319900944	0.76762	0.47921	−.1522	−.29992
12	0.1907946159	0.26446	0.18802	0.14657	−.25244
13	0.4729348737	0.65907	−.06047	−.157	0.10127

(18) *Pop. 12T and Series 12*

(A) **Pop. 12T**

(a) True Structure

	Comm.	F1	F2
1	0.9941	0.71	0.7
2	0.45	0.6	0.3
3	0.41	0.5	0.4
4	0.52	0.4	0.6
5	0.58	0.3	0.7

(b) Final Results

(1) SMC initial (19 cycles)

	Comm.	F1	F2
1	0.9953	0.997	−.0362
2	0.4525	0.6293	−.2377
3	0.4095	0.6336	−.0896
4	0.5442	0.7188	0.166
5	0.537	0.7057	0.1975

(2) MRE initial (23 cycles)

Comm.	F1	F2
0.9952	0.9969	−.0368
0.4523	0.6292	−.2374
0.4095	0.6336	−.0899
0.5393	0.7174	0.1569
0.5434	0.7074	0.2073

(3) UDE initial (19 cycles)

	Comm.	F1	F2
1	0.995	0.9968	−.0377
2	0.4521	0.6291	−.2373
3	0.4096	0.6336	−.0905

4	0.5339	0.7159	0.1464
5	0.5514	0.7095	0.2191

(c) Squared Multiple Correlations

1	2	3	4	5
0.775	0.4123	0.4042	0.5005	0.5036

(B) **12S02 ; sample size = 50**

(a) Correlation matrix

1				
2	0.620679			
3	0.679312	0.445883		
4	0.739434	0.430012	0.604028	
5	0.637414	0.262143	0.384982	0.455227

(b) Final Results

(1) SMC initial (42 cycles)

	Comm.	F1	F2
1	0.9828	0.9899	−.0544
2	0.3176	0.5616	0.047
3	0.7108	0.7641	0.3563
4	0.5823	0.7603	0.0647
5	0.5569	0.6212	−.4135

(2) MRE initial (42 cycles)

Comm.	F1	F2
0.9696	0.9846	−.0156
0.3176	0.5606	0.0575
0.6988	0.7595	0.3492
0.5823	0.759	0.0789
0.6305	0.64	−.47

(3) UDE initial (39 cycles)

	Comm.	F1	F2
1	0.9691	0.9843	−.0137
2	0.3176	0.5606	0.0581
3	0.6978	0.7591	0.3485
4	0.5824	0.7589	0.0799
5	0.6344	0.641	−.4728

(4) Jöreskog's initial (39 cycles)

Init. comm.	Est. comm.	F1	F2
0.828	0.9689	0.9842	−.0132
0.535	0.3179	0.5605	.0582
0.591	0.698	0.7592	0.3487
0.635	0.5823	0.7589	0.0799
0.551	0.6355	0.6413	−.4736

(5) Given initial (39 cycles)

	Init. comm.	Est. comm.	F1	F2
1	0.4	1.04	1.0073	0.1595
2	0.5	0.3177	0.5627	−.0323
3	0.8	0.7311	0.771	−.3697
4	0.8	0.5823	0.7618	−.0449
5	0.4	0.4275	0.5873	0.2873

(C) **12S05 ; sample size = 100**

(a) Correlation matrix

1				
2	0.613669			
3	0.619881	0.35837		
4	0.783872	0.508601	0.571326	
5	0.721049	0.368286	0.483223	0.571997

(b) Final results

(1) SMC initial (34 cycles)

	Comm.	F1	F2
1	0.9703	0.9802	0.0973
2	0.425	0.601	0.2526
3	0.5595	0.6851	−.3002
4	0.6778	0.8229	−.0249
5	0.4828	0.6944	−.0247

(2) MRE initial (34 cycles)

Comm.	F1	F2
0.9429	0.971	−.0069
0.5371	0.6293	−.3755
0.5333	0.6768	0.2742
0.6775	0.8212	0.0566
0.483	0.693	0.0516

(3) UDE initial (14 cycles)

	Comm.	F1	F2
1	0.9409	0.9698	0.0189
2	0.5823	0.6408	−.4143
3	0.4499	0.6544	0.1472
4	0.6731	0.8195	0.0399
5	0.5465	0.7093	0.2083

(4) Given initial (15 cycles)

Given comm.	Est. comm.	F1	F2
0.2	0.9336	0.963	0.0782
0.8	0.7453	0.6804	−.5314
0.8	0.4489	0.6509	0.159
0.5	0.6717	0.815	0.0864
0.5	0.5383	0.7036	0.2078

(5) Given initial (15 cycles)

	Init. comm.	Est. comm,	F1	F2
1	0.4	1.0315	0.989	0.3313
2	0.5	0.3547	0.5781	0.1432
3	0.8	0.8	0.7432	−.4977
4	0.8	0.6674	0.8141	0.0677
5	0.4	0.4756	0.6874	0.0555

(6) Joreskog's initial (15 cycles)

Init. comm.	Est. comm.	F1	F2
0.829	0.9436	0.9713	−.0068
0.513	0.5364	0.6293	−.3748
0.525	0.5279	0.6752	0.2685
0.702	0.6786	0.8217	0.0595
0.625	0.4836	0.6933	0.0536

(19) *Pop. 07 and Series 14*

(A) Pop. 07 ; p=7, k=2

(a) True strucrure

	Comm.	F1	F2
1	0.64	0.8	
2	0.58	0.7	0.3
3	0.61	0.6	0.5
4	0.5	0.5	0.5
5	0.52	0.4	0.6
6	0.45	0.3	0.6
7	0.34	0.3	0.5

(b) Squared Multiple Correlations

1	2	3	4	5	6	7
0.3613	0.4782	0.5143	0.4232	0.4137	0.3425	0.273

(c) Final Results

(1) SMC initial (40 cycles)

	Comm.	F1	F2
1	0.6398	0.6145	−.512

(2) MRE initial (35 cycles)

Comm.	F1	F2
0.6398	0.6145	−.512

2	0.5801	0.7298	−.2178	0.5801	0.7298	−.2177
3	0.61	0.781	−.0000	0.61	0.781	−.0000
4	0.5	0.7042	0.064	0.5	0.7042	0.064
5	0.52	0.6914	0.2048	0.52	0.6914	0.2048
6	0.45	0.6146	0.2689	0.45	0.6146	0.2689
7	0.34	0.5506	0.192	0.34	0.5506	0.192

(3) UDE initial (38 cycles)

	Comm.	F1	F2
1	0.6398	0.6145	−.512
2	0.5801	0.7298	−.2178
3	0.61	0.781	−.0000
4	0.5	0.7042	0.064
5	0.52	0.6914	0.2048
6	0.45	0.6146	0.2689
7	0.34	0.5506	0.192

(B) 14S02 ; sample size = 100

(a) Correlation matrix

1						
2	0.705675					
3	0.574053	0.575998				
4	0.548642	0.653925	0.629275			
5	0.455391	0.537591	0.637338	0.606582		
6	0.315335	0.484872	0.556001	0.468090	0.479186	
7	0.357236	0.388025	0.536364	0.463707	0.504734	0.462690

(b) Final Results

	(1) SMC initial (36 cycles)			(2) MRE initial (36 cycles)		
	Comm.	F1	F2	Comm.	F1	F2
1	0.5806	0.6923	−.3184	0.5806	0.6923	−.3184
2	0.8596	0.8231	−.4267	0.8596	0.8231	−.4267
3	0.6978	0.8234	0.1409	0.6978	0.8234	0.1409
4	0.6069	0.7784	−.0308	0.6069	0.7784	−.0308
5	0.5662	0.7412	0.1297	0.5662	0.7412	0.1297
6	0.4452	0.6286	0.2237	0.4452	0.6286	0.2237

(3) UDE initial (36 cycles)

	Comm.	F1	F2
1	0.5806	0.6923	−.3184
2	0.8596	0.8231	−.4267
3	0.6978	0.8234	0.1409
4	0.6069	0.7784	−.0308
5	0.5662	0.7412	0.1297
6	0.4452	0.6286	0.2237
7	0.4666	0.6223	0.2817

(C) **14S03 ; sample size=100**

(a) Correlation matrix

1						
2	0.652762					
3	0.462435	0.5944				
4	0.463695	0.465033	0.55811			
5	0.357126	0.475477	0.563806	0.601052		
6	0.166958	0.188587	0.418228	0.338579	0.488395	
7	0.157438	0.305135	0.35847	0.328597	0.383889	0.278983

(b) Final Results

	(1) SMC initial (12 cycles)			(2) MRE initial (12 cycles)		
	Comm.	F1	F2	Comm.	F1	F2
1	0.5899	0.623	−.4491	0.6317	0.6321	−.4818
2	0.7317	0.746	−.4184	0.694	0.7377	−.3869
3	0.6062	0.7785	0.0104	0.6061	0.7784	0.011
4	0.5138	0.7145	0.0571	0.5138	0.7145	0.0576
5	0.6637	0.7724	0.2592	0.6638	0.7723	0.2594
6	0.3812	0.4942	0.3701	0.381	0.4941	0.3699
7	0.2281	0.4461	0.1707	0.2281	0.4460	0.1708

(3) UDE initial (11 cycles)

	Comm.	F1	F2
1	0.5918	0.6235	−.4507
2	0.7298	0.7456	−.4169
3	0.6062	0.7785	0.0104
4	0.5138	0.7145	0.0571
5	0.6637	0.7724	0.2592
6	0.3812	0.4942	0.3701
7	0.2281	0.4461	0.1707

(20) *Maxwell's Data* ; $p=10$, *sample size*=810

With respect to this data, we have applied our method with MRE initial, and have extracted 3, 4 and 5 factors.

(a) Correlation matrix

1									
2	0.345								
3	0.594	0.477							
4	0.404	0.338	0.498						
5	0.579	0.230	0.505	0.389					
6	−.280	−.159	−.251	−.168	−.151				
7	−.449	−.205	−.377	−.249	−.285	0.363			
8	−.188	−.120	−.186	−.173	−.129	0.359	0.448		
9	−.303	−.168	−.273	−.195	−.159	0.227	0.439	0.429	
10	−.200	−.145	−.154	−.055	−.079	0.260	0.511	0.316	0.301

(b) Final Results

(1) Three factors (24 cycles)

	Comm.	F1	F2	F3
1	0.6519	0.3570	0.6879	−.2264
2	0.2743	0.2684	0.4193	0.1629
3	0.6962	0.4732	0.6798	0.1011
4	0.3882	0.3642	0.4816	0.1536
5	0.4591	0.4205	0.4957	−.1912
6	0.2241	0.1312	−.4544	−.0201
7	0.7094	0.2801	−.7570	0.2405
8	0.5708	0.4105	−.5677	−.2829
9	0.3146	−.1895	−.5264	−.0401
10	0.3346	0.3371	−.4587	0.1027

Residual Mean S. S.=0.000873

(2) Four factors (50 cycles)

	Comm.	F1	F2	F3	F4
1	0.5487	0.307	0.6632	−.0957	0.0741
2	0.3242	0.2679	0.4222	0.1864	0.1984
3	0.7154	0.4522	0.6757	0.1506	0.1783
4	0.3697	0.3384	0.4742	0.1668	0.0498
5	1.1041	0.5831	0.589	−.4081	−.5007
6	0.2109	0.12	−.44	0.0165	−.0504
7	0.7221	0.2681	−.7391	0.2531	−.1997
8	1.2496	0.5835	−.6844	−.4525	0.4858
9	0.2905	0.1719	−.5074	−.0056	−.0592
10	0.3826	0.3322	−.4534	0.1885	−.1765

Residual Mean S. S.=0.000186

(3) Five factors (50 cycles)

	Comm.	F1	F2	F3	F4	F5
1	0.5831	0.3136	0.6728	−.0947	−.0477	−.1444
2	0.3744	0.2783	0.4342	0.1978	−.2112	0.1569
3	0.7004	0.4415	0.6806	0.1304	−.1588	0.0055
4	0.3616	0.3315	0.4788	0.1448	−.0387	−.0059
5	0.9707	0.534	0.5778	−.3784	0.4564	−.0122
6	0.2082	0.119	−.4342	−.0052	0.0446	0.0598
7	0.6751	0.2488	−.7163	0.1739	0.165	0.2065
8	1.3283	0.5955	−.6774	−.5074	−.5071	−.0122
9	0.2942	0.1708	−.5001	−.0126	0.0538	0.109
10	0.7748	0.44	−.5163	0.3513	0.2437	−.3631

Residual Mean S. S.=0.000058

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