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ON G-VECTOR BUNDLES AND INVARIANT VECTOR FIELDS

Hiromichi MATSUNAGA

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Introduction

The main purpose of this note is to exhibit an isomorphism of semi-groups between the equivalence classes of G-vector bundles over a G-manifold with one orbit type and some classes of vector bundles over the orbit space. The article is a continuation of the author's preceding paper [4]. In which the author has proposed a too restrictive condition, i. e. the normalizer of the isotropy subgroup is the direct product, (C_2) in § 2 of [4]. For example, in § 4 of Chapter 4, [2], $SO(n)$, $SU(n)$-actions have been investigated. In these cases, the normalizers are semi-direct products, which are shown in § 1 of this note. In this note we attain to some kind of vector bundles over orbit spaces, called *local H-vector bundles*, which behave in a rather different manner than the usual H-vector bundles. We treat in this note only G-manifolds with one orbit type for a simplicity. We could reformulate the theorem 2 in [4] in a semi-direct product case, but the verification is too long, and so we will omit it. Thus this note is a theory concerning fiber bundles with Lie group actions of one orbit type.

In § 2, we reconstruct the characterization of G-vector bundles along the line of Part 1, [6]. A pair of transition functions is obtained.

§ 3 contains a proof of the continuity of them, and the main theorem is given.

In § 4, we calculate Grothendieck group of local H-vector bundles over spheres.

As in [4], the invariant fields problem is treated in § 5. Tangent bundles over G-manifolds are typical examples of G-vector bundles. The structure of them as coordinate bundles is analized, and applied to the investigation of invariant fields. The Stiefel manifold is a suitable example for a concrete calculation. In this section we discuss about the total space of a Stiefel manifold bundle over a Stiefel manifold.

1. Semi-direct products

Consider the standard imbedding $SO(n-k)=I_k\times SO(n-k)\subset SO(n)$, where I_k is the k-th identity matrix. Let $N(SO(n-k))$ be the normalizer of $SO(n-k)$ in $SO(n)$ and $\Gamma(SO(n-k))$ be the quotient group $SO(n-k)\backslash N(SO(n-k))$. We have the extention $SO(n-k)\longrightarrow N(SO(n-k))\longrightarrow\Gamma(SO(n-k))$. For a given $\begin{pmatrix}A & C\\ B & D\end{pmatrix}\in N(SO(n-k))$, where the types of A, B, C, D are $k\times k$, $(n-k)\times k$, $k\times(n-k)$, $(n-k)\times(n-k)$, respectively and for each $\begin{pmatrix}I_k & 0\\ 0 & S\end{pmatrix}\in SO(n-k)$, there exists $\begin{pmatrix}I_k & 0\\ 0 & S'\end{pmatrix}\in SO(n-k)$ such that $\begin{pmatrix}I_k & 0\\ 0 & S\end{pmatrix}\begin{pmatrix}A & C\\ B & D\end{pmatrix}=\begin{pmatrix}A & C\\ B & D\end{pmatrix}\begin{pmatrix}I_k & 0\\ 0 & S'\end{pmatrix}$. Then for each $S\in SO(n-k)$, $SB=B$, hence, $B=0$. Similarly $C=0$. By the relation $\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1} & 0\\ 0 & D^{-1}\end{pmatrix}=\begin{pmatrix}{}^tA & 0\\ 0 & {}^tD\end{pmatrix}$, we have $A\in O(k)$, $D\in O(n-k)$, then $N(SO(n-k))\subset O(k)\times O(n-k)$. Thus

$$N(SO(n-k))=(SO(k)\times I_{n-k})\cdot(I_k\times SO(n-k))\cup(SO(k)\times\begin{bmatrix}-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix})\cdot(\begin{bmatrix}-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\times SO(n-k)).$$

Hence the projection $\pi:\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\longrightarrow A$ induces an isomorphism $\Gamma(SO(n-k))=I_k\times SO(n-k)\backslash N(SO(n-k))\longrightarrow O(k)$. Define a homomorphism $s: O(k)\longrightarrow N(SO(n-k))$ by $s(A)=\begin{bmatrix}A & & & & \\ & \det A & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix}$, then we have $\pi\circ s=$ the identity map of $O(k)$. Thus we have

Proposition 1. *The normalizer $N(SO(n-k))$ is isomorphic to the semi-direct product $(I_k\times SO(n-k))\cdot(O(k)\times I_{n-k})$.*

Next, let $N(SU(n-k))$ be the normalizer of $SU(n-k)$ in $SU(n)$ and $\Gamma(SU(n-k))$ be the quotient group. Consider the extension $SU(n-k)\longrightarrow N(SU(n-k))\longrightarrow\Gamma(SU(n-k))$. As in SO case

$$N(SU(n-k))=\left\{\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\in U(k)\times U(n-k)\,;\,(\det A)\cdot(\det D)=1\right\}$$
$$=\left\{A\times\begin{bmatrix}(\det A)^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix};\,A\in U(k)\right\}\cdot(I_k\times SU(n-k)).$$

The map $\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\longrightarrow A$ gives an isomorphism $\pi: I_k\times SU(n-k)\backslash N(SU(n-k))$

$\longrightarrow U(k)$, and $A \longrightarrow \begin{bmatrix} A & & & \\ & (\det A)^{-1} & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}$ gives a section $s : U(k) \longrightarrow N(SU(n-k))$.

Thus we have

Proposition 2. *We have the isomorphism*

$N(SU(n-k)) \cong (I_k \times SU(n-k)) \cdot (U(k) \times I_{n-k})$, *a semi-direct product.*

Remark. The normalizer of $O(k)$ in $O(n)$ is the product $N(O(n-k)) \cong (I_k \times O(n-k)) \cdot (O(k) \times I_{n-k})$. [4]. Similarly we have $N(Sp(n-k)) \cong (I_k \times Sp(n-k)) \cdot (Sp(k) \times I_{n-k})$, direct product.

2. Transition functions

Suppose G to be a compact Lie group. Let M be a right G-manifold with one orbit type (H). Put $M_H = \{x \in M ; G_x = H\}$, then we have isomorphisms of G-manifolds $M \cong M_H \times_{\Gamma(H)} (H \backslash G) \cong M_H \times_{N(H)} G$, where $N(H)$ is the normalizer of H in G and $\Gamma(H) = H \backslash N(H)$. For each G-vector bundle $E \longrightarrow M$, the restriction $E \mid M_H \longrightarrow M_H$ is an $N(H)$-vector bundle, and we have an isomorphism of semi-groups

$\pi_*^{(1)} : Vect_G(M_H \times_{N(H)} G) \cong Vect_{N(H)}(M_H)$, §1 in [4].

Since $M_H/\Gamma(H)$ is a differentiable manifold, there exists an open covering $\bigcup_{i \in I} U_i = M_H/\Gamma(H)$ such that each U_i is contractible to a point x_i in U_i for each $i \in I$. Thus for the differentiable principal bundle

(1) ... $\Gamma(H) \longrightarrow M_H \longrightarrow M_H/\Gamma(H)$,

we have $\Gamma(H)$-equivalenes $\varphi_i : U_i \times \Gamma(H) \cong M_H \mid U_i$. For any $N(H)$-vector bundle $E \longrightarrow M_H$, we can choose $N(H)$-vector bundles $E_i \longrightarrow U_i \times \Gamma(H)$ with $E_i \cong \varphi_i^* \{E \mid \varphi_i(U_i \times \Gamma(H))\}$. Using the $N(H)$-equivariant contraction $U_i \times \Gamma(H) \longrightarrow (x_i) \times \Gamma(H)$, we have $E_i \cong U_i \times (E_i \mid \{x_i\} \times \Gamma(H))$ as $N(H)$-vector bundles. Let V_i be $E_i \mid \{x_i\} \times \{e\}$, e is the unit of $\Gamma(H)$, then by G. Segal $E_i \mid \{x_i\} \times \Gamma(H) \cong V_i(H) \times_H N$ as $N(H)$-vector bundles. Denote by q the projection $N(H) \longrightarrow \Gamma(H)$, then each projection $p_i : V_i \times_H N(H) \longrightarrow \Gamma(H)$ is given by $(v, n) \longrightarrow q(n)$, and we get an isomorphism of $N(H)$-vector bundles

$$\begin{array}{ccc} E \mid \varphi_i(U_i \times \Gamma(H)) & \xleftarrow{\psi_i} & U_i \times (V_i \times_H N(H)) \\ \downarrow & & \downarrow \\ \varphi_i(U_i \times \Gamma(H)) & \longleftarrow & U_i \times \Gamma(H). \end{array}$$

$\psi_j^{-1}\circ\psi_i : (U_i \cap U_j)\times V_i \times_H N(H) \longrightarrow (U_i \cap U_j)\times V_j \times_H N(H)$ is an isomorphism of $N(H)$-vector bundles. We put $(\psi_j^{-1}\circ\psi_i)(x, (v, n)) = (x, G_{ji}(x)(v, n))$, where $(v, n) = (vh, h^{-1}n)$ as an equivalence class for each $v \in V_i$, $h \in H$, $n \in N(H)$. Then $G_{ji} : U_i \cap U_j \longrightarrow \mathrm{Iso}_{N(H)}(V_i \times_H N(H), V_j \times_H N(H))$ is continuous map, where $Iso_{N(H)}(V_i \times_H N(H), V_j \times_H N(H))$ is equipped with the compact open topology. Now we obtain the following

Proposition 1. *For each $N(H)$-vector bundle $E \longrightarrow M_H$, we have an equivalence $E \cong \bigcup_{i\in I}(U_i \times V_i \times_H N(H))/(G_{ji})$, where we denote by $/(G_{ji})$ the pasting.*

From the commutativity of the middle square of the diagram

$$\begin{array}{ccccc}
(U_i \cap U_j)\times V_i \times_H N(H) & & \xrightarrow{\psi_j^{-1}\circ\psi_i = G_{ji}} & & (U_i \cap U_j)\times V_j \times_H N(H) \\
\downarrow \psi_i & \searrow p_i & & p_j \swarrow & \downarrow \psi_j \\
E\,|\,\varphi_i((U_i \cap U_j)\times \Gamma(H)) & & (U_i \cap U_j)\times \Gamma(H) \xrightarrow{\varphi_j^{-1}\circ\varphi_i} (U_i \cap U_j)\times \Gamma(H) & & E\,|\,\varphi_j((U_i \cap U_j)\times \Gamma(H)) \\
\downarrow p & \swarrow \varphi_i & & \varphi_j \searrow & \downarrow p \\
\varphi_i((U_i \cap U_j)\times \Gamma(H)) & & & & \varphi_j((U_i \cap U_j)\times \Gamma(H))
\end{array}$$

we have $p_j\circ(\psi_j^{-1}\circ\psi_i) = (\varphi_j^{-1}\circ\varphi_i)\circ p_i$, hence $p_j(G_{ji}(x)(v, n)) = \gamma_{ji}(x)q(n)$, where (γ_{ji}) is the set of transition functions of (1).

For another $N(H)$-vector bundle $E' \longrightarrow M_H$, let $\psi'_k : U'_k \times V_k \times_H N(H) \longrightarrow E'\,|\,\varphi'_k(U'_k \times \Gamma(H))$ be local trivialities. Consider an isomorphism of $N(H)$-vector bundles over M_H, $f : E \longrightarrow E'$. For isomorphisms $(U_i \cap U'_k)\times V_k \times_H N(H) \xrightarrow{\psi_i} E\,|\,\varphi_i((U_i \cap U'_k)\times \Gamma(H) \xrightarrow{f} E\,|\,\varphi'_k((U_i \cap U'_k)\times \Gamma(H) \xleftarrow{\psi'_k} (U_i \cap U'_k \times V'_k \times_H N(H)$, put $(\psi_k'^{-1}\circ f\circ\psi_i)(x, (v, n)) = (x, \bar{G}_{ki}(x)(v, n))$, then $\bar{G}_{ki} : U_i \cap U'_k \longrightarrow Iso_{N(H)}(V_i \times_H N(H), V'_k \times_H N(H))$ are continuous maps for each pair (i, k). By definition

$$(*)\begin{cases}\bar{G}_{kj}(x)G_{ji}(x) = \bar{G}_{ki}(x) \text{ on } U_i \cap U_j \cap U'_k, \\ G'_{lk}(x)\bar{G}_{kj}(x) = \bar{G}_{lj}(x) \text{ on } U_j \cap U'_k \cap U'_l.\end{cases}$$

Conversely, let G_{ji} and G'_{lk} be transition functions of $N(H)$-vector bundles $E \longrightarrow M_H$ and $E' \longrightarrow M_H$ respectively, and suppose that there are given $(\bar{G}_{kj})$ which fulfill the condition (*). Define

$h_{kj} : E \supset \psi_j((U_i \cap U'_k \times V_j \times_H N(H)) \longrightarrow \psi'_k((U_j \cap U'_k)\times V'_k \times_H N(H))$

by $h_{kj}(\psi_j(x, (v, n))) = \psi'_k(x, \bar{G}_{kj}(x)(v, n))$ on $U_j \cap U'_k$, then on $U_i \cap U_j \cap U'_k$,

$$\begin{aligned} h_{ki}(\psi_i(x, (v, n))) = \psi'_k(x, \bar{G}_{ki}(x)(v, n)) &= \psi'_k(x, \bar{G}_{kj}(x)G_{ji}(x)(v, n)) \\ &= h_{kj}(\psi_j(x, G_{ji}(x)(v, n))). \end{aligned}$$

By the definition, $\psi_j(x, G_{ji}(x)(v, n)) = \psi_i(x, (v, n))$, hence, $h_{ki} = h_{kj}$ on $U'_k \cap U_i \cap$

U_j. Further the second term of the above equalities is equal to

$$\psi'_l(x, \bar{G}_{li}(x)(v, n)) = h_{li}(\psi_i(x, (v, n))),$$

therefore, $h_{ki} = h_{lj}$ on $U'_k \cap U'_l \cap U_i \cap U_j$, and we obtain an isomorphism of $N(H)$-vector bundles $f : E \longrightarrow E'$, thus we have

Proposition 2. *Two $N(H)$-vector bundles (E, G_{ji}) and (E', G'_{lk}) are $N(H)$-equivalent if and only if there exist continuous functions $(\bar{G}_{ki})$ which fulfill the condition* (*).

To proceed more, we put the following hypothesis

(H) *there exist continuous maps $n_{ji} : U_i \cap U_j \longrightarrow N(H)$ with $q(n_{ji}(x)) = \gamma_{ji}(x)$ and $n_{kj}(x)n_{ji}(x) = n_{ki}(x)$, $n_{ii}(x) = e$, the unit.*

We call (n_{ji}) a lift of γ_{ji}). For each $(x, (v, n)) \in (U_i \cap U_j) \times V_i \times_H N(H)$, we have $G_{ji}(x)(v, n) = (v', n_{ji}(x)n)$ for some $v' \in V_j$. V_i is isomorphic to V_j as a vector space, so we can put $v' = g_{ji}(x)v$ and $g_{ji} : U_i \cap U_j \longrightarrow Iso\,(V_i, V_j)$. Since (G_{kj}) and (n_{kj}) fulfill the property of transition functions, then the functions (g_{ji}) do except for the continuity. We discuss the continuity in the next section.

Remark. Suppose $N(H)$ to be a semi-direct product $H\cdot\Gamma(H)$, then $\Gamma(H)$ is a subgroup of $N(H)$, $\Gamma(H) \approx (e)\cdot\Gamma(H) \subset N(H)$, then $n_{ji}(x) = (e)\cdot\gamma_{ji}(x)$ fulfill the hypothesis (H), where (γ_{ji}) is the set of transitions of (1).

3. The continuity of (g_{ji})

In this section, we consider the case, $N(H) = H\cdot\Gamma(H)$, a semi-direct product. Let $k_i : V_i \times \Gamma(H) \subset V_i \times H\cdot\Gamma(H) \longrightarrow V_i \times_H H\cdot\Gamma(H)$ be the composition of the inclusion and the projection, where V_i is a vector space for each i. The map k_i is a continuous injection and a fiberwise isomorphism of V_i-bundles over $\Gamma(H)$, then it is an isomorphism of bundles. We can define an $N(H)$-action on $V_i \times \Gamma(H)$ by $(v, \gamma)(h', \gamma') = (v\cdot I(\gamma)h', \gamma\gamma')$, where $(v, \gamma) \in V_i \times \Gamma(H)$, $(h', \gamma') \in H\cdot\Gamma(H)$ and $I(\gamma)h'$ denotes $\gamma h'\gamma^{-1}$. By the relation$(v, (I(\gamma)h'\cdot\gamma')) = (v\cdot I(\gamma)h', \gamma')$, k_i is $N(H)$-equivariant. Since $1\times g_{ji}\times\gamma_{ji} = (1\times k_j)^{-1}\circ(1\times G_{ji})\circ(1\times k_i)$, $g_{ji} : U_i \cap U_j \longrightarrow Iso\,(V_i, V_j)$ is continuous for each (i, j).

For $h \in H$, $(x, (v, h\gamma)) \in (U_i \cap U_j) \times V_i \times_H H\cdot\Gamma(H)$,

$$\begin{aligned}\psi_j^{-1}\circ\psi_i(x, (v, h\gamma)) &= (x, (g_{ji}(x)v, \gamma_{ji}(x)h\gamma)) \\ &= (x, (g_{ji}(x)v, I(\gamma_{ji}(x)(h))\cdot\gamma_{ji}(x)\gamma) \\ &= (x, (g_{ji}(x)v)\cdot I(\gamma_{ji}(x)(h)), \gamma_{ji}(x)\gamma),\end{aligned}$$

$$\psi_j^{-1}\circ\psi_i(x, (v, h\gamma)) = \psi_j^{-1}\circ\psi_i(x, (vh, \gamma)) = (x, g_{ji}(x)(vh), \gamma_{ji}(x)\gamma),$$

then we have

$$(2) \ldots g_{ji}(x)(vh) = \{g_{ji}(x)(v)\} I(\gamma_{ji}(x))(h).$$

Thus we get

Proposition 1.S. *Suppose $N(H)$ to be a semi-direct product $H \cdot \Gamma(H)$, then for any $N(H)$-vector bundle $E \longrightarrow M_H$, $E \cong [\bigcup_{i \in I} U_i \times V_i \times_H H \cdot \Gamma(H)]/(g_{ji}, \gamma_{ji})$.*

Let (γ'_{lk}) be another set of transition functions of the principal bundle $M_H \longrightarrow M_H/\Gamma(H)$, and $(\bar{\gamma}_{lk})$ be the equivalence between (γ_{ji}) and (γ'_{lk}). For another $N(H)$-vector bundle $E' \cong [\bigcup_{k \in K} U'_k \times V'_k \times_H H \cdot \Gamma(H)/(g'_{lk}, \gamma'_{lk})$, $\bar{G}_{li}$ in §2 can be represented as $(\psi'^{-1}_k \circ f \circ \psi_i)(x, (v, h\gamma)) = (x, \bar{g}_{ki}(x)(v), \bar{\gamma}_{ki}(x)h\gamma)$.

We can check that $\bar{g}_{ki} : U'_k \cap U_i \longrightarrow Iso\ (V_i, V'_k)$ is continuous for each k, i, and the relations

$$(*S) \begin{cases} \bar{g}_{ki}(x)(vh) = \{g_{ki}(x)(v)\} I(\bar{\gamma}_{ki}(x))(h), \\ \bar{g}_{kj}(x) g_{ji}(x) = \bar{g}_{ki}(x), \\ g'_{lk}(x) \bar{g}_{kj}(x) = \bar{g}_{lj}(x). \end{cases}$$

From the proposition 2, we have

Proposition 2.S. *Two $N(H)$-vector bundles (E, g_{ji}, γ_{ji}) and $(E', g'_{lk}, \gamma'_{lk})$ are equivalent if and only if there exist continuous functions $\bar{g}_{ki} : U'_k \cap U_i \longrightarrow Iso\ (V_i, V'_k)$ with the property $(*S)$.*

Let $\widehat{Vect}_{H^\Gamma}(M_H/\Gamma(H))$ be the family of vector bundles $\hat{E}$ with the property that for any contractible open covering $M_H/\Gamma(H) = \bigcup_{i \in I} U_i$ and transition functions $\gamma_{ji} : U_i \cap U_j \longrightarrow \Gamma(H)$ of (1), there are local trivialities $\hat{\psi}_i : \hat{E} \mid U_i \cong U_i \times V_i$ which fulfill the next conditions

(i) each V_i is an H-module,

(ii) define g_{ji} by $\hat{\psi}_j^{-1} \circ \psi_i(x, v) = (x, g_{ji}(x)(v))$, then $g_{ji} : U_i \cap U_j \longrightarrow Iso\ (V_i, V_j)$ is continuous and satisfies the relation $g_{ji}(x)(vh) = \{g_{ji}(x)(v)\} I(\gamma_{ji}(x))(h)$.

We call each element of $\widehat{Vect}_{H^\Gamma}(M_H/\Gamma(H))$ a *local H-vector bundle* and denote by (E, g_{ji}, γ_{ji}), or $\bigcup_i U_i \times V_i/(g_{ji}, \gamma_{ji})$.

Definition 1. Two local H-vector bundles (E, g_{ji}, γ_{ji}), $(E', g'_{lk}, \gamma'_{lk})$ are *related* if and only if there exist $(\bar{g}_{ki})$ with the property $(*S)$.

We can verify that the relation in the definition is an equivalence relation.

For each local H-vector bundle $\bigcup_i U_i \times V_i/(g_{ji}, \gamma_{ji})$, define

$$U_i \times V_i \times_H N(H) \ni (x, (v, n)) \equiv (x, (g_{ji}(x)(v)), \gamma_{ji}(x)n) \in U_j \times V_j \times_H N(H),$$

then

$$\begin{aligned}(x, vh, n) &\equiv (x, (g_{ji}(x)(vh), \gamma_{ji}(x)n)) = (x, g_{ji}(x)(v), I(\gamma_{ji}(x))(h) \cdot \gamma_{ji}(x)n) \\ &= (x, g_{ji}(x)(v), \gamma_{ji}(x)hn) \equiv (x, (hn)),\end{aligned}$$

and so the above $\equiv$ gives an equivalence relation in $\bigcup_i U_i \times V_i \times_H N(H)$. The quotient $\bigcup_i U_i \times V_i \times_H N(H)/(g_{ji}, \gamma_{ji})$ is an $N(H)$-vector bundle over M_H which we denote by $\pi^*_{(2)}(\hat{E})$, where $\hat{E} = \bigcup_i U_i \times V_i/(g_{ji}, \gamma_{ji})$. Conversely each $N(H)$-vector bundle $E = \bigcup_i U_i \times V_i \times_H N(H)/(g_{ji}, \gamma_{ji})$ gives a local H-vector bundle $\bigcup_i U_i \times V_i/(g_{ji}, \gamma_{ji})$, which we denote by $\pi^{(2)}_*(E)$. By Definition 1 and Proposition 2. S, if E is equivalent to E as $N(H)$-vector bundle, then $\pi^{(2)}_*(E)$ is related to $\pi^{(2)}_*(E')$, also if $\hat{E}$ is related to $\hat{E}'$, then $\pi^*_{(2)}(E)$ is equivalent to $\pi^*_{(2)}$ (E'). We denote by $Vect_{H^\Gamma}(M_H/\Gamma(H))$ the semi-group of equivalence classes of local H-vector bundles. The above consideration yields

Theorem. *If* $N(H) = H\cdot\Gamma(H)$, then

$$\pi^{(2)}_* : Vect_{N(H)}(M_H) \longrightarrow Vect_{H^\Gamma}(M_H/\Gamma(H))$$

is an isomorphism of semi-groups.

Denoting $\pi^{(1)}_*\cdot\pi^{(2)}_*$ by π_*, we have

Corollary. *If $N(H) = H\cdot\Gamma(H)$, then*

$$\pi_* : Vect_G(M) \cong Vect_{H^\Gamma}(M/G).$$

4. Local H-vector bundles over spheres

Let $\gamma_{ji} : S^n_i \cap S^n_j \longrightarrow \Gamma(H)$ be transition functions of a principal bundle $\Gamma(H) \longrightarrow \mathrm{P} \longrightarrow S^n$, where $i, j = 1, 2$ and S^n_1, S^n_2 is the upper, the lower hemisphere respestively. For any local H-vector bundle $\hat{E} \in Vect_{H^\Gamma}(S^n)$, we can choose H-modules V_1, V_2 and local trivialities $\hat{\phi}_i : S^n_i \times V_i \longrightarrow E \mid S^n_i$, $i = 1, 2$. The transition function $g_{12} : S^n_1 \cap S^n_2 = S^{n-1} \longrightarrow Iso\,(V_1, V_2)$ satisfies the relation $g_{12}(x)(vh) = \{g_{12}(x)(v)\}\, I(\gamma_{12}(x))(h)$ for $v \in V_1$, $h \in H$, see § 3. Let $\gamma_{12}(x_0) = \gamma_0$, $g_{12}(x_0) = g_0$ for a base point $x_0 \in S^{n-1}$. We can choose $\bar{\gamma}_{ji}$, $\bar{g}_{ji}$, g'_{ji}, especially $\gamma'_{12}(x) = \gamma_{12}(x)\,\gamma_0^{-1}$, to obtain

$$\begin{aligned} g'_{12}(x)(vh) &= g_{12}(x)\, g_0^{-1}(vh) = g_{12}(x)\{(g_{21}(x_0)(v))\, I(\gamma_{21}(x_0))\} \\ &= \{g_{12}(x)g_{21}(x_0)(v)\}\, I(\gamma_{12}(x)\,\gamma_{21}(x_0))(h) = \{g'_{12}(x)(v)\}\, I(\gamma'_{12}(x))(h) \\ g'_{21}(x)(vh) &= g_0 g_{21}(x)(vh) = g_0\{(g_{21}(x)(v))\, I(\gamma_{21}(x))(h)\} \\ &= \{g_0 g_{21}(x)(v)\}\, I(\gamma_0\,\gamma_{21}(x))(h) = (g'_{21}(x)(v))\, I(\gamma'_{21}(x))(h), \end{aligned}$$

hence $(S^n_i \times V_i,\ g'_{ji},\ \gamma'_{ji},\ i, j = 1, 2)$ is also a local H-vector bundle, which is related to the original one. Now $\gamma'_{12}(x_0) = e \in \Gamma(H)$, the unit, $g'_{12}(x_0) =$ the identity map of $V_1 = V_2$ as vector spaces. We denote by $h^{(i)} \in H$, H-actions on V_i, $i = 1, 2$, then

$v\cdot h^{(1)} = g'_{12}(x_0)(v\cdot h^{(1)}) = \{g'_{12}(x_0)(v)\}\, I(\gamma'_{12}(x_0))(h^{(2)}) = v\cdot h^{(2)}$, hence $V_1 = V_2$ as an H-module. There are two distinct cases.

Case $n \geqq 2$. $\gamma_{12}(S^{n-1}) \subset \Gamma_0$, the connected component of the unit e of $\Gamma(H)$. For example, let $G = SF(n)$, $H = SF(n-k)$, as in §1, where $F = O$ or U, then $\Gamma_0 = SF(k) \times I_{n-k} \subset \Gamma = F(k) \times I_{n-k}$, and the action of Γ_0 by the conjugation is trivial, hence $Vect_{H^\Gamma}(S^n) = Vect_H(S^n)$.

Case $n = 1$.

Lemma. *$Vect_{H^\Gamma}(S^1) = \hat{H}^\Gamma$, the semi-group of isomorphism classes of $\gamma_{12}(-1)$ invariant H-modules.*

Proof. For any local H-vector bundle $\hat{E} \in Vect_{H^\Gamma}(S^1)$, we can choose a normal form snch that $\gamma_{12} : S^0 \longrightarrow \Gamma(H)$ satisfies $\gamma_{12}(+1) = e$, $\gamma_{12}(-1) = \gamma_0 \in \Gamma(H)$, and $g_{12}(+1) =$ the identity map of V, $g_{12}(-1) = G \in Iso\,(V, V)$. Define $S : Vect_{H^\Gamma}(S^1) \longrightarrow \hat{H}^{\gamma_0}$ by $S(E) = S(S^1_1 \times V \cup S^1_2 \times V) = [V] \in H^{\gamma_0}$, the isomorphism class of V. For another choice of $g'_{12}(-1) = G'$, let

$\bar{g}_{11} = g_{11} = e$, $\bar{g}_{22} = G'^{-1}G$, $\bar{g}_{12} = g_{12}$, $\bar{g}_{21} = G'^{-1}Gg_{21} = G'^{-1}$, then (V, g', γ) is related to (V, g, γ).

Conversely, for any $[V] \in \hat{H}^{\gamma_0}$, we have an isomorphism of H-modules $G : V \longrightarrow V^{\gamma_0}$. Let $g_{12} = G$, then $(V, g_{12}, \gamma_{12}) \in Vect_{H^{\gamma_0}}(S^1)$. If $A : V \longrightarrow V'$ is an isomorphism of H-modules, then setting $g'_{12} = AGA^{-1}$, $(V', g'_{12}, \gamma_{12})$ is related to (V, g_{12}, γ_{12}). In fact it is enough to define

$$\bar{g}_{11}(x) = A,\ \bar{g}_{12}(x) = AG,\ \bar{g}_{21}(x) = A\cdot g_{21}(x),\ \bar{g}_{22}(x) = A.$$

Thus the inverse $T : \hat{H}^\gamma \longrightarrow Vect_{H^{\gamma_0}}(S^1)$ is defined and $S \circ T =$ the identity map of $\hat{H}^{\gamma_0}$, $T \circ S =$ the identity map of $Vect_{H^{\gamma_0}}(S^1)$. Hence we proved the lemma.

For example, suppose that $\gamma_{12} : S^0 \longrightarrow O(k)$ satisfies $\gamma_{12}(+1) = I_k$, the identity of $O(k)$, $\gamma_{12}(-1) = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$, then

$$\begin{bmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & a_{11} \cdots a_{1\,n-k} & \\ & & & a_{n-k\,1} \cdots a_{n-k\,n-k} & \end{bmatrix} \begin{bmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & a_{11} & -a_{12} \cdots & -a_{1\,n-k} \\ & & -a_{21} & a_{22} \cdots & a_{2\,n-k} \\ & & \vdots & \vdots & \vdots \\ & & -a_{n-k\,1} & a_{n-k\,2} & a_{n-k\,n-k} \end{bmatrix}.$$

Hence, by the action of $\gamma_{12}(-1)$, the standard maximal torus $\mathrm{Diag}\left\{\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}, \ldots, \begin{pmatrix}\cos\theta_l & -\sin\theta_l\\ \sin\theta_l & \cos\theta_l\end{pmatrix}\right\}$ is transformed into $\mathrm{Diag}\left\{\begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix}, \ldots, \begin{pmatrix}\cos\theta_l & -\sin\theta_l\\ \sin\theta_l & \cos\theta_l\end{pmatrix}\right\}$, where $n-k = 2l+1$, or $2l$. It is well known that

$R(T) = Z[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_l, \alpha_l^{-1}]$, $R(SO(2l+1) = Z[\lambda^1, \ldots, \lambda^{-1}]$,

$R(SO(2l)) = Z[\lambda^1, \ldots, \lambda^{l-1}, \lambda^l_+, \lambda^l_-]/(\sim)$, where $\alpha_j : \begin{pmatrix}\cos\theta_j & -\sin\theta_j\\ \sin\theta_j & \cos\theta_j\end{pmatrix} \longrightarrow \exp 2\pi i\,\theta_j$, $\lambda^k = \sigma^k[\alpha_1, \alpha_1^{-1}, \ldots, \alpha_l, \alpha_l^{-1}]$ is the k-th elementary symmetric function, and $\lambda_\pm = {}_{i(1)\leqq\cdots\leqq i(l)},\ \Sigma_{\varepsilon(1)\cdots\varepsilon(l)=\pm 1}\, \alpha_{i(1)}^{\varepsilon(1)} \cdots \alpha_{i(l)}^{\varepsilon(l)}$, 13, [3]. Since$(\alpha_1)^{\gamma_0} = \alpha_1^{-1}$, $(\alpha_k)^{\gamma_0} = \alpha_k$ for $k = 2, \ldots, l$, then $(\lambda^k)^{\gamma_0} = \lambda_k$ and $(\lambda^l_\pm)^{\gamma_0} \neq \lambda^l_\pm$, thus for complex vector bundles the lemma yields

Proposition 3. $K(Vect^C_{SO(2l)\gamma_0}S^1) = Z[\lambda^1, \ldots, \lambda^l] \subsetneqq R(SO(2l))$.

5. Tangent bundles and invariant vector fields.

At first we give a formula about the tangent bundle of a G-manifold with one orbit type (H), and some propositions. We apply them to investigate the existence of nowhere vanishing invariant vector fields on these manifolds.

Let M be a right G-manifold with one orbit type (H). Then the principal bundle of the fiber bundle $\mathfrak{F} : H\backslash G \longrightarrow M \longrightarrow M/G$ is $\Gamma(H) \longrightarrow M_H = \{x \in M,\ G_x = H\} \longrightarrow M/G$, and we have the isomorphism $M \cong M_H \times_{\Gamma(H)} (H\backslash G)$. Choosing a G-invariant Riemannian metric, as a G-vector bundle; $T(M) \cong \pi^*(T(M)) \oplus M_H \times_{\Gamma(H)} T(H\backslash G)$, where π^* denotes the induced bundle and $T(N)$ does the tangent bundle of N. The second term of the above right hand side is the fiber bundle along the fibers

$\hat{\mathfrak{F}} : R^n \longrightarrow M_H \times_{\Gamma(H)} T(H\backslash G) \longrightarrow M_H \times_{\Gamma(H)} (H\backslash G)$.

Since $M_H = M_H \times_{\Gamma(H)} (H\backslash N(H)) \subset M_H \times_{\Gamma(H)} (H\backslash G)$, $\{M_H \times_{\Gamma(H)} T(H\backslash G)\} \mid M_H \times_{\Gamma(H)} \Gamma(H) = M_H \times_{\Gamma(H)} \{T(H\backslash G) \mid \Gamma(H)\}$, where $\mid$ denotes the restriction. From the semi-direct product assumption, $\Gamma(H)$ is a subgroup of $N(H)$. $H\backslash G$ is a right G-manifold and so a right $\Gamma(H)$-manifold. For the right $N(H)$-vector bundle $T(H\backslash G) \mid \Gamma(H) \longrightarrow \Gamma(H)$, we obtain the equalities

$$F: T(H\backslash G) \mid \Gamma(H) \cong T_{\{H\}}(H\backslash G) \times {}_H N(H), \text{ [5]},$$
$$= T_{\{H\}}(H\backslash G) \times {}_H H\cdot\Gamma(H),$$
$$= T_{\{H\}}(H\backslash G) \times \Gamma(H), \text{ see } k_i \text{ in § 3.}$$

On the other hand, $H\backslash G$ is a left $\Gamma(H)$-manifold. Consider the composition of the left $\Gamma(H)$-action and the isomorphism F above,

$$\Gamma(H) \times \{T_{\{H\}}(H\backslash G)\} \longrightarrow T(H\backslash G) \mid \Gamma(H) \longrightarrow T_{\{H\}}(H\backslash G) \times \Gamma(H),$$

then each $(\gamma, v) \in \Gamma(H) \times \{T_{\{H\}}(H\backslash G)\}$ yields the unique $w \in T_{\{H\}}(H\backslash G)$ such that $F(\gamma\cdot v) = (w, \gamma)$. Define $f: \Gamma(H) \longrightarrow \text{Aut}\ \{T_{\{H\}}(H\backslash G)\}$ by $w = f(\gamma)\cdot v$, then f is a representation. In fact

(1) $\gamma v = \{f(\gamma)v\}\gamma$

(2) $(\gamma'\gamma)v = \{f(\gamma'\gamma)v\}\ (\gamma'\gamma)$

(3) $\gamma'\{f(\gamma)v\} = \{f(\gamma')[f(\gamma)v]\}\gamma'$

(4) $\{\gamma'[f(\gamma)v]\}\gamma = \{f(\gamma')[f(\gamma)v]\}\gamma'\gamma$, by (3),

then from (1),

$$\gamma'\gamma v = \gamma'\ \{[f(\gamma)v]\cdot\gamma\} = \{\gamma'[f(\gamma)v]\}\cdot\gamma = \{f(\gamma')f(\gamma)v\}\gamma'\gamma, \text{ by (4)}$$
$$= \{f(\gamma'\gamma)v\}\ (\gamma'\gamma) \text{ by (1),}$$

hence $f(\gamma'\gamma) = f(\gamma')f(\gamma)$. Evidently $f(e) =$ the identity map, thus f is a representation.

Now we attend to the local H-vector bundle $\pi_*(\hat{\mathfrak{F}} \mid M_H)$. Let (γ_{ji}) be transition functions of the principal bundle $\Gamma(H) \longrightarrow M_H \longrightarrow M/G$. We want to determine transition functions (g_{ji}) of $M_H \times_{\Gamma(H)} \{T(H\backslash G) \mid \Gamma(H)\} \longrightarrow M_H$, see §§ 2 and 3. As a right $N(H)$-vector bundle over $U_i \times \Gamma(H)$, we have isomorphisms

$$g^{(i)}: U_i \times T_{\{H\}}(H\backslash G) \times \Gamma(H) \longrightarrow \{U_i \times \Gamma(H)\} \times_{\Gamma(H)} \{T(H\backslash G) \mid \Gamma(H)\},$$

which is obtained from F^{-1} above, explicitly, for the composition

$$(\hat{\mathfrak{F}} \mid M_H) \mid \varphi_i(U_i \cap U_j \times \Gamma(H)) \xleftarrow{\varphi_i} \{U_i \cap U_j \times \Gamma(H)\} \times_{\Gamma(H)}$$
$$\{T(H\backslash G) \mid \Gamma(H)\} \overset{g^{(i)}}{\longleftarrow} U_i \cap U_j \times T_{\{H\}}(H\backslash G) \times \Gamma(H)$$
$$= (\hat{\mathfrak{F}} \mid M_H) \mid \varphi_j(U_i \cap U_j \times \Gamma(H)) \xleftarrow{\varphi_j} \{U_i \cap U_j \times \Gamma(H)\} \times_{\Gamma(H)}$$
$$\{T(H\backslash G) \mid \Gamma(H)\} \xleftarrow{g^{(j)}} U_i \cap U_j \times T_{\{H\}}(H\backslash G) \times \Gamma(H),$$
$$\{\varphi_j \circ g^{(j)}\}^{-1}\{\varphi_i\, g^{(i)}\}(x, v, \gamma) = (g^{(j)})^{-1}(\varphi_j^{-1}\circ\varphi_i)(x, \gamma, f(\gamma^{-1})v)$$
$$= (g^{(j)})^{-1}(x, \gamma_{ji}(x)\gamma, f(\gamma^{-1})v)$$
$$= (x, f(\gamma_{ji}))f(\gamma^{-1})v, \gamma_{ji}(x)\gamma)$$
$$= (x, f(\gamma_{ji}(x))v, \gamma_{ji}(x)\gamma),$$

hence $g_{ji}(x) = f(\gamma_{ji}(x))$, thus we obtain

Proposition 4. *The π_*-image of the tangent bundle along the fibers is given by* $\bigcup_i U_i \times T_{\{H\}}(H\backslash G)/(f(\gamma_{ji}), \gamma_{ji})$.

Definition 2, [4]. Let G be a compact Lie group. A vector field X on a G-manifold M is called *G-invariant* if it satisfies the equality

$(dg)_x X_x = X_{xg}$ for all $x \in M$ and $g \in G$.

We have already prepared the next propositions in [4].

Proposition 5. *A G-manifold M admits a G-invariant vector field without singularities if and only if the tangent bundle $T(M)$ of M has a G-vector bundle decomposition $T(M) = E \oplus \theta^1$, where E is a G-vector bundle and θ^1 is the product G-line bundle over M.*

Proposition 6. *Let X be a vector field on a G-manifold M and $\{\varphi_t\}$ be the one parameter group of transformations generated by X. Then X is a G-invariant if and only if $g \cdot \varphi_t = \varphi_t \cdot g$ for each $t \in R$ and $g \in G$.*

Consider the standard imbedding $U(N) \subset SO(2N)$, which is given by $A + Bi \longrightarrow \begin{pmatrix} A & -B \\ A & B \end{pmatrix}$. The center of $U(N)$ is Diag (exp $2\pi it$), which induces a nowhere vanishing vector field on the sphere $S^{2N-1} \subset C^N$, the complex N-space. We call this field the canonical field.

Corollary of Proposition 6. *The canonical vector field on the sphere S^{2N-1} is invariant under an orthogonal action of a compact connectd Lie group if and only if the action is a complex unitary action.*

Proof. Suppose $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(2N)$ to commute with each element of the center $\mathfrak{C}$ of $U(N)$. Let $\exp 2\pi it = a + bi$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \text{Diag } a & \text{Diag } -b \\ \text{Diag } b & \text{Diag } \quad a \end{pmatrix} = \begin{pmatrix} aA+bB & -bA+aB \\ aC+bD & -bC+aD \end{pmatrix},$$

$$\begin{pmatrix} \text{Diag } a & \text{Diag } -b \\ \text{Diag } b & \text{Diag } \quad a \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA-bC & aB-bD \\ bA+aC & bB+aD \end{pmatrix},$$

hence $bB = -bC$, $-bA = -bD$, thus $C = -B$, $D = A$, and so $G \in U(N) \subset SO(2N)$ and the corollary is proved.

Next as an example we choose the real Stiefel manifold $SO(n-k) \backslash SO(n) = V_{n,k}$. Denote by V^{n-k} the standard real representation space of $SO(n-k)$, then we have

Proposition 7.

$$T(SO(n-k) \backslash SO(n)) = \bigoplus_k V^{n-k} \times_{SO(n-k)} SO(n) \oplus R^{k(k-1)} \times (SO(n-k) \backslash SO(n)).$$

Corollary of Propositions 5 and 7. *On the $SO(n)$-manifold $SO(n-k)\backslash SO(n)$, there exist just $k(k-1)/2$-liniearly independent invariant fields.*

Proof of Proposition 7.

Since $T(SO(n\text{-}k)\backslash SO(n)) \cong T_{\{SO(n-k)\}}(SO(n-k)\backslash SO(n)) \times_{SO(n-k)} SO(n)$, it is sufficient to determine the isotropy representation $SO(n-k) \longrightarrow \text{Aut}\,\{T_{\{SO(n-k)\}}(SO(n-k)\backslash SO(n))\}$. It is known that

$$T_{\{e\}}(SO(n)) \cong \mathfrak{M}^s(R, n), \text{ skew symmetric } n \times n\text{-matrices},$$
$$\cong R^{n-1} \oplus R^{n-2} \oplus \ldots \oplus R^1 \cong R^{n(n-1)/2} \text{ as a vector space},$$
$$T_{\{e\}}(SO(n)) \cong T_{\{e\}}(SO(n-k)) \oplus T_{\{SO(n-k)\}}(SO(n-k)\backslash SO(n)).$$

Let x_{ij} be a coordinate system of $n \times n$-matrices $\mathfrak{M}_n(R)$ and define $T_{\{e\}}(SO(n)) \longrightarrow \mathfrak{M}_n(R)$ by $X_e \longrightarrow (X_e(x_{ij}))$. Let g_1 be a variable in a neighborhood of the unit in $SO(n)$. Since we have concerned with the right action,

$$(dR_g X_e)(x_{ij}) = X_e(x_{ij} R_g) = X_e(x_{ij}(g_1 g)) = X_e(\Sigma_k x_{ik}(g_1) x_{ki}(g))$$
$$= \Sigma_k X_e(x_{ik}) x_{kj}(g),$$

and so for each $g = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & g_{k+11} \cdots g_{k+1n} \\ & & & \vdots \\ & & & g_{n1} \cdots g_{nn} \end{bmatrix} \in SO(n-k) \subset SO(n)$, the isotropy representation is given by

$$\begin{bmatrix} 0 & y_{12} & y_{13} & \cdots & y_{1k} & y_{1k+1} & \cdots & y_{1n} \\ & 0 & y_{23} & \cdots & & & & \\ & & & 0 & y_{k-1k} & y_{k-1k+1} & \cdots & y_{k-1n} \\ & & & & 0 & y_{kk+1} & \cdots & y_{kn} \\ & & & & & & 0 & y_{n-1n} \\ & & & & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & g_{k+11} \cdots g_{k+1n} \\ & & & \\ & & & g_{n1} \cdots g_{nn} \end{bmatrix}$$

which is equal to $\{(k-1)\theta \oplus \rho_{n-k}\} \oplus \{(k-2)\theta \oplus \rho_{n-k}\} \oplus \cdots \oplus \rho_{n-k} = \{k(k-1)/2\}\theta \oplus k\rho_{n-k}$, where θ is the one dimensional trivial representation and ρ_{n-k} is the standard one. Thus we proved the proposition.

Now since $\Gamma(SO(n-k)) = O(k) \subset N(SO(n-k)) \subset SO(n)$, from the principal bundle $O(k) \longrightarrow O(k+r) \longrightarrow O(k+r)/O(k) = V_{k+r,r}$,

we obtain an associated bundle

$$SO(n-k)\backslash SO(n) \longrightarrow O(k+r) \times_{O(k)} [SO(n-k)\backslash SO(n)] \xrightarrow{p} V_{k+r,r}.$$

The total space is an $SO(n)$-manifold with one orbit type $(SO(n-k))$, (Proposition 4. 1, [2]). We denote by $V_{n,k,r}$ the total space. By Proposition 7, we have

$$T(V_{n,k,r}) = O(k+r) \times_{O(k)} \{R^{k(k-1)/2} \times (SO(n-k)\backslash SO(n)\} \oplus$$
$$O(k+r) \times_{O(k)} \{\oplus_k V^{n-k} \times_{SO(n-k)} SO(n)\} \oplus p^*T(V_{k+r,r}).$$

By W. A. Sutherland, [7], $V_{k+r,r}$ is parallelizable whenever r is greater than 1. Thus ${}_{n,k,r}$ admits at least dim $V_{k+r,r}=r(r+2k-1)/2$-linearly independent invariant fields. Now we recall the representation f in Proposition 4. Since $\gamma v=\{f(\gamma)v\}\gamma$, then $f(\gamma)v=\gamma v\gamma^{-1}$. In the case of our example, for $A\in O(k)$,

$$\begin{pmatrix}\alpha & \beta\\ -\beta & \delta\end{pmatrix}\in\mathfrak{M}^s(R,n),\ \begin{pmatrix}A & 0\\ 0 & I_{n-k}\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\beta & \delta\end{pmatrix}\begin{pmatrix}{}^tA & 0\\ 0 & I_{n-k}\end{pmatrix}=\begin{pmatrix}A\alpha{}^tA & A\beta\\ -\beta{}^tA & \delta.\end{pmatrix}$$

Thus $V_{n,k,r}$ admits at least $r(r+2k-1)/2$-linearly independent invariant fields.

Remark. The homomorphism $T_{\{e\}}(SO(n))\longrightarrow\mathfrak{M}_n(R)$ defined above is the restriction of the isomorphism in § 3 of Chapter 4, [1], which is an isomorphism between the Lie algebra of $GL(n, C)$ and $\mathfrak{M}_n(C)$, the Lie algebra of all $n\times n$-matrices of complex entries.

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