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A Theorem to Construct 3-Way BIB Designs

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In this paper, I give an extension of a theorem due to A. Hedayat and D. Raghavarao to construct 3-way BIB designs. By making use of this extension, a 3-way BIB design associated with two inequivalent (31, 15, 7) cyclic difference sets, and also a 3-way BIB design associated with (16, 6, 2) abelian difference set are constructed.

THEOREM 1. *Let G be a group with the binary operation* $*$ *and let* v *be the order of G. Let* φ *be an automorphism of G. Suppose that* $\{d_i, i=1,\dots, k\}$ *is a* (v, k, λ) *difference set on G, and so is* $\{b_i, i=1,\dots, k\}$ *and* $\{\varphi(d_i^{-1})*b_i, i=1,\dots, k\}$. *Then, a 3-way BIB design can be constructed. And the* (g_j*d_i, g_j)*-component of this design is* $\varphi(g_j)*b_i$*, and other components are blank.*

PROOF. Let A be a matrix whose (g_j*d_i, g_j)-component is $\varphi(g_j)*b_i$, and other components are blank. The matrix derived from the matrix A, by replacing (g_j*d_i, g_j)-component by 1 and blank by 0, is clearly incidence matrix of a design derived from the (v, k, λ) difference set $\{d_i\}$. Since the g_j-column of A is consist of $\{\varphi(g_j)*b_i, i=1,\dots, k\}$, the condition about symbol-column is satisfied. The components, which are not blank, of the g_i-row of A are of the form (f_t*d_t, f_t). Consequently the $(g_i, g_i*d_t^{-1})$-component is not blank, and its value is $\varphi(g_i*d_t^{-1})*b_t$. Since $\varphi(g_i*d_t^{-1})*b_t = \varphi(g_i)*(\varphi(d_t^{-1})*b_t)$, and $\{\varphi(d_t^{-1})*b_t, t=1,\dots, k\}$ is a difference set, the condition about symbol-row is satisfied. Then the proof is completed.

When φ is simply the identity mapping of G in Theorem 1, we get

COROLLARY 1. *Let G be a group with binary operation* $*$ *and let* v *be the order of G. Suppose that* $\{d_i\}$ *is a* (v, k, λ) *difference set and so is* $\{b_i, i=1,\dots, k\}$ *and also* $\{d_i^{-1}*b_i, i=1,\dots, k\}$. *Then a 3-way BIB design can be constructed. And its* (g_j*d_i, g_j)*-component is* g_j*b_i.

EXAMPLE. Let D and B be two inequivalent (31, 15, 7) cyclic difference sets.

$$D=\{1, 2, 3, 4, 6, 8, 12, 15, 16, 17, 23, 24, 27, 29, 30\} \mod 31$$

and let d_i be defined by this order, i.e., $d_1=1$, $d_2=2,\dots, d_{15}=30$.

$$B=\{28, 1, 20, 2, 10, 16, 5, 7, 19, 18, 25, 9, 8, 4, 14\} \mod 31$$

and let b_i be defined by this order, i.e., $b_1=28$, $b_2=1,\dots$, $b_{15}=14$. Then $\{-d_i+b_i, i=1,\dots,15\}=\{27, 30, 17, 29, 4, 8, 24, 23, 3, 1, 2, 16, 12, 6, 15\}$ is the same cyclic difference set as D. Consequently, by Corollary 1, we can construct a 3-way BIB design associated with (31, 15, 7) difference sets.

If we consider only the $\{b_i, i=1,\dots,k\}$ of the form $\{\varphi(d_i)*d_i, i=1,\dots,k\}$ in Theorem 1, we obtain

COROLLARY 2. *Let G be a group with binary operation $*$ and let v be its order, and let φ be an automorphism of G. Suppose that $\{d_i, i=1,\dots,k\}$ is a difference set and also $\{\varphi(d_i)*d_i, i=1,\dots,k\}$. Then a 3-way BIB design can be constructed. And the (g_j*d_i, g_j)-component of this design is $\varphi(g_j)*\varphi(d_i)*d_i$, and other components are blank.*

REMARK. There is a simple method, almost same as the method of A. Hedayat and D. Raghavarao, which constructs the same 3-way BIB design. Superimposing the incidence matrix N of the design derived from (v, k, λ) difference set $\{d_i\}$ on the matrix M whose (g_i, g_j)-component is $\varphi(g_i)*g_j^{-1}*g_i$, and if $N_{ij}=1$, then replace 1 by M_{ij}, and if $N_{ij}=0$, then replace 0 by blank. The construction is completed.

EXAMPLE. Let us consider the (16, 6, 2) difference set $\{d_i, i=1,\dots,6\}=\{a, b, c, d, ab, cd\}$ on G, where G is an abelian group generated by a, b, c, d satisfying $a^2=b^2=c^2=d^2=1$. Consider an automorphism φ of G defined by $\varphi(a)=b$, $\varphi(b)=ab$, $\varphi(c)=cd$, $\varphi(d)=c$, then $\{\varphi(d_i)*d_i, i=1,\dots,6\}=\{ab, a, d, cd, b, c\}$ is a (16, 6, 2) difference set on G. Numbering $1, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd$, by the integers from 1 to 16, we obtain the next 3-way BIB design.

**	2	3	4	5	6	**	**	**	**	11	**	**	**	**	**
6	**	2	**	**	1	9	10	**	**	**	**	**	15	**	**
2	1	**	**	**	3	**	**	12	13	**	**	**	**	16	**
5	**	**	**	1	**	14	**	15	**	4	16	**	**	**	**
11	**	**	5	**	**	**	7	**	9	1	**	12	**	**	**
3	6	1	**	**	**	**	**	**	**	**	7	8	**	**	14
**	10	**	16	**	**	**	3	14	**	**	11	**	9	**	**
**	15	**	**	12	**	10	**	**	7	**	**	4	3	**	**
**	**	13	14	**	**	11	**	**	6	**	15	**	**	12	**
**	**	16	**	7	**	**	4	13	**	**	**	9	**	6	**
4	**	**	1	11	**	**	**	**	**	**	**	**	8	10	13
**	**	**	15	**	8	16	**	11	**	**	**	2	**	**	7
**	**	**	**	9	14	**	12	**	4	**	8	**	**	**	2
**	9	**	**	**	**	3	15	**	**	13	**	**	**	8	5
**	**	12	**	**	**	**	**	6	16	8	**	**	5	**	10
**	**	**	**	**	7	**	**	**	**	10	2	14	13	5	**

In Corollary 2, if we consider the automorphism φ of G, of the form $\varphi(g)=g^{b-1}$ for each integer b, we obtain Theorem 2.1 in the paper of A. Hedayat and D. Raghavarao [3].

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