Mem. Fac. Lit. & Sci., Shimane Univ.,
Nat. Sci., 10, pp. 19–25 Dec. 20, 1976

On the Left Translations of Homogeneous Loops

Michihiko KIKKAWA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 6, 1976)

The notion of homogeneous systems introduced in §1 of this paper is an abstraction of homogeneity of homogeneous loops and groups. In consideration of the geometric approach tried in [1], the left translations of homogeneous Lie loops lead us to the notion of (parallel) displacements of homogeneous systems. This paper is aimed at the axiomatic construction of homogeneous systems which will be useful to the study of a certain class of locally reductive spaces including homogeneous Lie loops and Lie groups. In §2 the homogeneous systems of homogeneous loops are found. In §3 it is shown that any automorphism of a homogeneous system is an automorphism fixing the origin followed by a displacement from the origin. As the isotropy subgroup of the group of displacements the notion of holonomy groups is obtained which is closely related with (non-) associativity of the binary system induced from the homogeneous system. Symmetric homogneous systems are introduced in §4 which should be combined with symmetric homogeneous spaces.

§1. Homogeneous Systems

Let G be a non-empty set. A ternary system $\eta: G\times G\times G\to G$ on G will be called a *homogeneous system* if it satisfies the following conditions;

(1.1) $$\eta(x, x, y)=\eta(x, y, x)=y$$

(1.2) $$\eta(x, y, \eta(y, x, z))=z \quad \text{and}$$

(1.3) $$\eta(x, y, \eta(u, v, w))=\eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$$

for $x, y, z, u, v, w\in G$.

For a homogeneous system η on G we consider the maps $\eta(x, y): G\to G$ by setting $\eta(x, y)z: =\eta(x, y, z)$ for $x, y, z\in G$. Then from (1.1) and (1.2) it follows that the map $\eta(x, y)$ is a permutation of G for any pair $x, y\in G$, which will be called the *displacement* of η from x to y. Indeed the second equality of (1.1) shows that $\eta(x, y)$ brings x to y. The remaining conditions for the homogeneous system are rewritten as

(1.1′) $$\eta(x, x)=1_G \text{ (the identity map of } G)$$

(1.2′) $$\eta(x, y)^{-1}=\eta(y, x)$$

(1.3′) $$\eta(x, y) \text{ is an automorphism of } \eta.$$

For a fixed $a\in G$ we define a binary operation $\mu^{(a)}$ with respect to the homogeneous system η on G as follows:

$$\mu^{(a)}(x, y)=\eta(a, x, y), \quad x, y\in G. \tag{1.4}$$

Then (1.3′) shows

PROPOSITION 1. *Each displacement $\eta(x, y)$ of a homogeneous system induces isomorphisms of binary systems defined above, that is, $\eta(x, y)$: $(G, \mu^{(a)})\to(G, \mu^{(a')})$, $a'=\eta(x, y)a$, is an isomorphism.*

REMARK 1. If a homogeneous system on a topological space G is continuous, then the multiplication $xy=\mu^{(a)}(x, y)$ for a fixed $a\in G$ defines an H-space on G with the unit a.

EXAMPLE (Homogeneous system of a group). Let G be a group. Then the ternary system η defined by

$$\eta(x, y, z)=yx^{-1}z \quad \text{for} \quad x, y, z\in G, \tag{1.5}$$

is a homogeneous system on G which will be called the *homogeneous system of the group* G. The binary operation $\mu^{(e)}$ defined by (1.4) is just equal to the group operation of G, where e is the identity. In this case, η satisfies the transition relation

$$\eta(y, z)\eta(x, y)=\eta(x, z) \quad \text{for} \quad x, y, z\in G. \tag{1.6}$$

It is shown that the homogeneous system of a group is characterized by this condition, that is;

PROPOSITION 2. *A homogeneous system η on a set G is that of a group if and only if η satisfies* (1.6).

PROOF. It is sufficient to show that if the homogeneous system η satisfies (1.6) then $(G, \mu^{(e)})$ is a group for any fixed $e\in G$. For the multiplication $xy:=\mu^{(e)}(x, y)$, (1.1) implies $ex=xe=x$ for any $x\in G$. By (1.3) and (1.6) we have $x(yz)=\eta(e, x, \eta(e, y, z))=\eta(\eta(e, x)e, \eta(e, x)y, \eta(e, x)z)=\eta(x, xy)\eta(e, x)z=\eta(e, xy)z=(xy)z$. The inverse x^{-1} of x is given by $x^{-1}=\eta(x, e, e)$. Hence $(G, \mu^{(e)})$ is a group. In this case, the homogeneous system of the group is equal to η. *q.e.d.*

REMARK 2. Since (1.1) and (1.6) implies (1.2), it can be said that to give a group operation on a set G is equal to give a ternary operation η on G satisfying (1.1), (1.3) and (1.6) and a fixed element e as the identity.

§2. Homogeneous Loops

In the preceding paper [1] we have introduced the notion of homogeneous loops.

A *homogeneous loop* (G, μ) is a non-empty set G with a multiplication $\mu(x, y)=xy$ satisfying the following conditions (2.1), (2.2) and (2.3):

(2.1) (G, μ) is a loop, i.e., it has the two-sided identity e and, for each $x \in G$, the left translation L_x: $y \mapsto xy$ and the right translation R_x: $y \mapsto yx$ are permutations of G.

(2.2) Each $x \in G$ has its inverse x^{-1} such that $L_{x^{-1}}=(L_x)^{-1}$.

(2.3) For each pair $x, y \in G$ the *left inner mapping* $L_{x,y}=L_{xy}^{-1}L_xL_y$ is an automorphism of (G, μ).

Now we consider the homogeneous system of a homogeneous loop. Since any group is a homogeneous loop whose left inner mappings are equal to the identity map, the discussion on groups in the last section is a special one of the following:

THEOREM 1. *On a homogeneous loop (G, μ) there is a unique homogeneous system η: $G\times G\times G\to G$ such that*

$$\mu(x, y)=\eta(e, x, y), \tag{2.4}$$

where e is the identity of μ. η is given by

$$\eta(x, y, z)=x((x^{-1}y)(x^{-1}z)) \quad \text{for} \quad x, y, z \in G. \tag{2.5}$$

This homogeneous system will be called the *homogeneous system of the loop* (G, μ).

PROOF. Let η be a homogeneous system on G satisfying (2.4). Then we have $L_x=\eta(e, x)$ and $\eta(e, x^{-1})=\eta(x, e)$. By using (1.2′) and (1.3) we get (2.5) as $x((x^{-1}y)(x^{-1}z))=\eta(e, x)\eta(e, x^{-1}y, x^{-1}z)=\eta(x, y, z)$ for $x, y, z \in G$. Next we show that η defined by (2.5) is actually a homogeneous system on G. (1.1) is an immediate consequence of (2.5). From the formulae for the left inner mappings of homogeneous loops;

$$L_{x,x^{-1}y}=L_{y^{-1},x}=(L_{x^{-1},y})^{-1} \qquad \text{(cf. Lemma 1.8 of [1])} \tag{2.6}$$

we obtain (1.2′), that is, $\eta(x, y)\eta(y, x)=L_yL_{x,x^{-1}y}L_{y,y^{-1}x}L_y^{-1}=1_G$ for $x, y \in G$. From (2.6) we get also

$$\eta(x, y)=L_y L_{y^{-1},x} L_x^{-1}.$$

Then by using (2.3) we obtain

$$\eta(x, y)\eta(x, z)w=\eta(y, \eta(x, y)z, \eta(x, y)w) \tag{2.7}$$

which is one of the special cases of (1.3). Since $\eta(x, y)\eta(u, v, w)=\eta(x, y)\eta(x, u)\eta(u,$

$x)\eta(u, v)w$, (1.3) can be shown by using (2.7) repeatedly. *q.e.d.*

REMARK 3. If η is the homogeneous system of a homogeneous loop (G, μ), then (2.5) shows that the binary system $(G, \mu^{(a)})$ defined by (1.4) is the transposed loop of (G, μ) centered at $a \in G$ (cf. § 1 of [1]). It is used in the definition of the canonical connection of homogeneous Lie loops (cf. also § 3 of [1]). From the above theorem it follows that the homogeneous system of a homogeneous loop (G, μ) can be regarded as the apparatus which assigns to each $a \in G$ the transposed loop $\mu^{(a)}$ at a. In this case each displacement $\eta(x, y)$ induces isomorphisms of transposed loops as shown in Proposition 1.

REMARK 4. Homogeneous systems are not always those of homogeneous loops. For instance, define a ternary operation η on $\mathbf{R}^n$ by

$$\eta(x, y, z) = \begin{cases} x+y-z & \text{if } x \neq y \\ z & \text{if } x=y, \end{cases}$$

for $x, y, z \in \mathbf{R}^n$. Then η is a homogeneous system on $\mathbf{R}^n$. However, $\mu(x, y)=\eta(0, x, y)$ does not form a loop since $\mu(0, x)=\mu(2x, x)$ for $x \neq 0$, that is, the right translation R_x $(x \neq 0)$ is not a permutation of $\mathbf{R}^n$.

THEOREM 2. *Let (G, η) be a homogeneous system. It is the homogeneous system of a homogeneous loop if and only if there exists an element $e \in G$ such that the map $\rho_x^{(e)}$ defined by $\rho_x^{(e)}y: =\eta(e, y, x)$ is a permutation of the set G.*

PROOF. Let (G, μ) be a homogeneous loop. Then its homogeneous system is defined by (2.5) so the map $\rho_x^{(e)}$ for the identity e is the right translation R_x of (G, μ), which is a permutation of G. Conversely, suppose that a homogeneous system (G, η) has an element $e \in G$ such that $\rho_x^{(e)}$ is a permutation of G. We show that the binary system $\mu=\mu^{(e)}$ defined by (1.4) for e is a homogeneous loop with the identity e. Then Theorem 1 implies that η is the homogeneous system of (G, μ). From (1.1) it is clear that e is the two-sided identity of μ. The left translation L_x of μ is the displacement $\eta(e, x)$ from e to x which is a permutation of G and the right translation $R_x=\rho_x^{(e)}$ is assumed to be a permutation of G so (2.1) is satisfied by μ. Set $x^{-1}: =\eta(x, e, e)$ for each $x \in G$. Then we have $L_{x^{-1}}=(L_x)^{-1}$. For any pair $x, y \in G$, the left inner mapping $L_{x,y}$ is expressed by displacements of η as

$$L_{x,y}=\eta(xy, e)\eta(x, xy)\eta(e, x)=\eta(x^{-1}, e)\eta(y, x^{-1})\eta(e, y), \tag{2.8}$$

where $xy=\mu(x, y)$. By (1.3′) each displacement of η is an automorphism of η, so the equality $L_{x,y}\eta(e, u, v)=\eta(L_{x,y}e, L_{x,y}u, L_{x,y}v)$ holds which shows that the left inner mapping $L_{x,y}$ is an automorphism of μ. Thus (G, μ) is shown to be a homogeneous loop. *q.e.d.*

§3. Automorphisms of Homogeneous Systems

An *automorphism* σ of a homogeneous system (G, η) is a permutation of G such that $\sigma\eta(x, y, z)=\eta(\sigma x, \sigma y, \sigma z)$ for all $x, y, z \in G$. The automorphism group of η will be denoted by $A(\eta)$ and its isotropy subgroup at $e \in G$ by $A_e(\eta)$. Since the results about semi-direct products of homogeneous loops given in [1] are founded mostly on their left translations, they lead us to the corresponding results of automorphisms of homogeneous systems. For instance;

THEOREM 3. *Let η be a homogeneous system on a set G. For a fixed element $e \in G$ denote by $\bar{G}_e$ the subset of $A(\eta)$ consisting of all displacements $\eta(e, x)$, $x \in G$, from e. Then $A(\eta)$ is uniquely factorized as*

$$A(\eta)=\bar{G}_e A_e(\eta) \tag{3.1}$$

that is, each automorphism $\sigma \in A(\eta)$ of η is factored in a unique way as $\sigma=\eta(e, x)\alpha$, $\alpha \in A_e(\eta)$.

PROOF. For any $\sigma \in A(\eta)$ set $x=\sigma(e)$. Then $\alpha=\eta(x, e)\sigma$ is an element of $A_e(\eta)$ and $\sigma=\eta(e, x)\alpha$. On the other hand, if $\eta(e, x)\alpha=\eta(e, y)\beta$ for $\alpha, \beta \in A_e(\eta)$, then operating it on e we get $x=y$ and so $\alpha=\beta$. *q.e.d.*

As a matter of fact the composition of two automorphisms of η is related with the binary system $\mu^{(e)}$ at e as follows; if $\sigma=\eta(e, x)\alpha$ and $\tau=\eta(e, y)\beta$ are elements of $A(\eta)$ then

$$\sigma\tau=\eta(e, x\cdot\alpha y)\lambda_{x,\alpha y}\alpha\beta, \tag{3.2}$$

where $x\cdot\alpha y=\mu^{(e)}(x, \alpha y)$ and $\lambda_{x,y}$ denotes the element of $A_e(\eta)$ given by

$$\lambda_{x,y}=\eta(x\cdot y, e)\eta(e, x)\eta(e, y) \tag{3.3}$$

which is also equal to each of the last two terms of (2.8).

Let $D(\eta)$ be the group of displacements of η, i.e., the subgroup of $A(\eta)$ generated by all displacements of η and set $\Lambda_e(\eta):=A_e(\eta)\cap D(\eta)$ which will be called the *holonomy group* of η at $e \in G$.

The followings are easily proved:

PROPOSITION 3. *The holonomy groups of a homogeneous system η are in the same conjugacy class in $A(\eta)$, that is, $\Lambda_y(\eta)=\eta(x, y)\Lambda_x(\eta)\eta(y, x)$ for any $x, y \in G$.*

PROPOSITION 4. *A homogeneous system η is that of a group if and only if the holonomy group of η is trivial.*

From Theorem 3 we have the following

COROLLARY 1. $D(\eta)=\bar{G}_e A_e(\eta)$ *(uniquely factored).*

COROLLARY 2. *If η is the homogeneous system of a homogeneous loop (G, μ), then the group $A_e(\eta)$, $D(\eta)$ and $A_e(\eta)$ (e being the identity of μ) are the automorphism group, the left translation group and the left inner mapping group of (G, μ), respectively.*

(2.8), (3.2) and (3.3) also imply

COROLLARY 3. *Under the same assumption as in* Corollary 2, *$A(\eta)$ (resp. $D(\eta)$) is isomorphic to the semi-direct product $G \times A_e(\eta)$ (resp. $G \times A_e(\eta)$) of the homogeneous loop G by the group $A_e(\eta)$ (resp. $A_e(\eta)$).* Cf. Corollary 2.2 of [1].

REMARK 5. From Theorem 3 and Corollary 1, it follows that the set G with a homogeneous system η is identified with the quotient space $A(\eta)/A_e(\eta)$ (resp. $D(\eta)/A_e(\eta)$) for any fixed e. If G is a group and η its homogeneous system, then $A_e(\eta)$ is the automorphism group of G which is normal in $A(\eta)$.

EXAMPLE. Let η be the homogeneous system of the additive group of $\mathbf{R}^n$. Then $w=\eta(x, y, z)$ is the fourth vertex of the parallelogram $zxyw$ with the parallel edges $xz /\!/ yw$ (which may be degenerate) and the displacement $\eta(x, y)$ is the parallel displacement of $\mathbf{R}^n$ from x to y. The automorphism group $A(\eta)$ is the affine transformation group of $\mathbf{R}^n$.

§4. Symmetric Homogeneous Systems

A homogeneous system η on G will be called *symmetric* if, for each $x \in G$, the map S_x: $G \to G$ defined by $S_x y: =\eta(y, x, x)$, is an automorphism of η. By definition of S_x the equalities $S_x x = x$ and $S_x S_x = 1_G$ are clear, and so each S_x is a permutation of G. If η is symmetric the equality $S_x S_y z = \eta(S_x z, S_x y, S_x y)$ is also valid. From these equalities we have

PROPOSITION 5. *Let (G, η) be a symmetric homogeneous system. Under the multiplication defined by $x*y: =S_x y$, G is a reflection space of* O. Loos [2], *i.e., it satisfies $x*x=x$, $x*(x*y)=y$ and $x*(y*z)=(x*y)*(x*z)$.*

For a fixed $e \in G$, the left translation L_x with respect to the multiplication $\mu^{(e)}$ satisfies $(L_x)^{-1}=L_{x^{-1}}$ for $x^{-1}=\eta(x, e, e)$. Hence we get $\eta(e, x)^{-1}=\eta(e, S_e x)$ from which we can prove the following theorem, similarly as in Theorem 6.1 of [1];

THEOREM 4. *Let $A(\eta)=\bar{G}_e A_e(\eta)$ be the decomposition of the automorphism group of a homogeneous system (G, η) for a fixed $e \in G$* (cf. Theorem 3). *Then the homogeneous system η is symmetric if and only if the mapping of $A(\eta)$ onto itself sending $\eta(e, x)\alpha$ to $\eta(e, x)^{-1}\alpha$ is an automorphism of the group $A(\eta)$, where $\alpha \in A_e(\eta)$.*

COROLLARY (cf. Theorem 6.1 of [1]). *Suppose that* (G, η) *is the homogeneous system of a homogeneous loop* (G, μ). *Then* η *is symmetric if and only if* (G, μ) *is a symmetric homogeneous loop, that is by definition,* $(xy)^{-1}=x^{-1}y^{-1}$ *holds in* (G, μ) *for* $x, y \in G$.

References

[1] M. KIKKAWA, Geometry of homogeneous Lie loops, Hiroshima Math. J. **5** (1975), 141–179.

[2] O. Loos, Spiegelungsräume und homogene symmetrische Räume, Math. Z. **99** (1967), 141–170.