Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
32 (1999), pp. 57–68

GEOMETRY OF HOMOGENEOUS LEFT LIE LOOPS AND TANGENT LIE TRIPLE ALGEBRAS

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(Received: December 25, 1998)

Abstract. On occasion of retirement from Department of Mathematics, Shimane University, in March 1999, looking back upon the researching life over thirty years, which has been devoted to establishing an extensive theory of non-associative generalization of the theory of Lie groups, the author would like to present here a summary of his scientific works.

1. Geodesic Homogeneous Left Lie Loops

1.1. **Geodesic local loops.** (Akivis [1], Kikkawa [10], [11], [16], Sabinin [50])
The concept of geodesic local loops has been introduced by Kikkawa [10] in 1964, which is a kind of local loops defined on any manifold with a linear connection. The multiplication of the local loop is given by parallel displacements of geodesic curves along geodesic curves passing through some fixed point which plays a role of unit element of the local loop.

Exactly, for any linearly connected manifold with any fixed point e, the local binary operation is introduced by

$$\mu(x, y) = \mathrm{Exp}_x \circ \tau_{e,x} \circ \mathrm{Exp}_e^{-1}(y)$$

in some normal neighbourhood of e . Here, $\tau_{e,x}$ denotes the parallel displacement of tangent vectors along the geodesic arc joining e to x. It has been proved that this local multiplication forms a local loop, called *geodesic local loop* at e ([10]). Denote by L_x the left translation by x. Non-associativity of μ is indicated by left inner maps $L_{x,y} := L^{-1}_{\mu(x,y)} \circ L_x \circ L_y$ which is not always equal to the identity map unless μ is associative.

In [10], it has been shown that the curvature tensor vanishes at the unit e if the left inner maps satisfy some relations in the case of linear connections without torsion. This fact suggests us that the left inner maps of geodesic local loops are in deep conection with the curvature tensor.

1991 *Mathematics Subject Classification.* 15A69, 17A40, 20N05, 53C05, 53C30.
Key words and phrases. Loop, Reductive homogeneous space, Connection, Lie algebra, Lie triple system, Ternary compositions, Lie triple algebra.

1.2. **Homogeneous left Lie loops.** (HOFMANN-STRAMBACH [8], KIKKAWA [16], [17], [19], [34], [36], [40])

Motivated by the concept of geodesic local loops on affinely connected manifolds, the algebraic concept of homogeneous loops and the concept of homogeneous Lie loops on manifolds have been introduced by the author in 1975 [16]. Since then, he has intended and established an extensive theory of non-associative generalization of the well-known theory of Lie groups and Lie algebras .

A *loop* (G, μ) is an algebraic binary operation on a set G with the unit e such that all left translations $L_x : G \to G; L_x y := \mu(x, y)$ and right translations $R_x : G \to G; R_x y := \mu(y, x)$, for any $x \in G$, are bijections of G. If the left translation L_x satisfies $(L_x)^{-1} = L_{x^{-1}}$ for $x^{-1} = L_x^{-1} e$, $x \in G$, the loop is said to have the *left inverse property* . The loop μ with the left inverse property is said to be *homogeneous* if all left inner maps $L_{x,y}$ are automorphisms of μ. The subgroup of the automorphism group $Aut(\mu)$ generated by all left inner maps is called the *left inner mapping group* of (G, μ). *Homogeneous Lie loop* is a homogeneous loop defined on a differentiable manifold whose multiplication μ is differentiable.

Later, in 1988 [36], the author introduced the concept of homogeneous *left* loops, binary systems for which right translations are not required to be bijective but required all the other properties for homognenous loops. So, homogeneous left loops are not always loops. Nevertheless, it is easy to check that almost all of the results on homogeneous Lie loops (e.g. in [16] above) are valid for left ones since they are concerned only with left translations. Since 1988, the author has treated homogeneous *left* loops instead of homogeneous loops. Of course, it need not to distinguish them when the local Lie loops are discussed, since any homogeneous left Lie loop is a homogeneous local Lie loop, that is, the right translations of homogeneous left Lie loops around the unit e are local diffeomorphism in some neighbourhood of e.

1.3. **Homogeneous left Lie loops as reductive homogeneous spaces.** (KIKKAWA [16], NOMIZU [49])

For any homogeneous left loop (G, μ), the concept of semi-direct product $A = G \times K$ of G by a group K is introduced, where K is a subgroup of $Aut(G, \mu)$ containing the left inner mapping group of (G, μ). That is, for any (x, α) and (y, β) in A, define their product by;

$$(x, \alpha)(y, \beta) := (\mu(x, \alpha y), L_{x,\alpha y} \circ \alpha \circ \beta).$$

Then, A forms a group which is called the *semi-direct product* of G and K.

Let (G, μ) be a homogeneous left Lie loop , K_e the closure of the left inner mapping group in the (differentiable) automorphism group of (G, μ). Then, the Lie group A of semi-direct product of G and K_e is called the *enveloping Lie group* of (G, μ).

By using this, it is shown that any homogeneous left Lie loop is regarded as a reductive homogeneous space $G = A/K_e$.

Settle on A/K_e the canonical connection of the reductive homogeneous space of NOMIZU [49]. Then the homogeneous left Lie loop G is said to be *geodesic* if the multiplication μ is coincident with the geodesic local loop of the canonical connection, in some neighbourhood of the unit e.

Later, in Section 1.6, the canonical connection of homogeneous left Lie loops will be defined explicitly.

Especially, any Lie group (G, μ) is a geodesic homogeneous left Lie loop whose left inner maps are equal to the identity map. In this case, the canonical connection is reduced to the (-)-connection of E. Cartan. Therfore, we can say: *The theory of geodesic homogeneous left Lie loops is an exact generalization of the theory of Lie groups.*

On the other hand, assume that a homogeneous left Lie loop (G, μ) satisfies the relation;

$$\mu(x, y)^{-1} = \mu(x^{-1}, y^{-1}) \quad \text{for any } x, y \in G.$$

This relation means that the *inversion* $J : G \to G; J(x) := x^{-1}$, is an automorphism of (G, μ). Then, it is shown that the reductive homogeneous space $G = A/K_e$ is reduced to an affine symmetric space. So, it is called a *symmetric loop* (cf. [16]). It has been shown that any symmetric loop is geodesic ([16]).

As a matter of facts, the idea of *homogeneous* loop was born and grew up from some particular study of the properties of geodesic local loops of affine symmetric spaces, in a series of articles in 1973–1975 [14], [15], [18]. It has been shown that, by the transvections of affine symmetric space G obtained from composition of reflections (cf. LOOS [48]), a local multiplication xy on some neighborhood of any point e of G can be defined, in which the relations

$$(xy)^{-1} = x^{-1}y^{-1} \quad \text{and} \quad L_{x,y}(uv) = L_{x,y}u \, L_{x,y}v$$

are characteristic, that is, it forms a symmetric local loop.

1.4. **Homogeneous systems associated with homogeneous left loops.** (KIKKAWA [19], [20], [22], [24], [27], [31], [32])

Let (G, μ) be a homogeneous left loop with the unit e. Set a ternary operation $\eta : G \times G \times G \to G$ on G by

$$\eta(x, y, z) := L_x \mu(L_x^{-1} y, L_x^{-1} z).$$

Then (G, η) satisfies the following characteristic properties:

$$\begin{aligned} &(\text{ i}) \quad \eta(x, x, y) = \eta(x, y, x) = y \\ &(\text{ii}) \quad \eta(x, y, \eta(y, x, z)) = z \\ &(\text{iii}) \quad \eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)) \end{aligned}$$

for any $x, y, z, u, v, w \in G$. This ternary system (G, η) is called the *homogeneous system* associated with (G, μ) ([19]). It determines the multiplication μ of the left loop by;

$$\mu(x, y) = \eta(e, x, y).$$

Since 1976, the theory of homogeneous loops has been converted and developed into the theory of homogeneous systems ([19] – [43]) by the author. In 1993

([42]), it was shown that to give a homogeneous left loop (G, μ) on a set G is equivalent to give a homogeneous system (G, η) with a fixed element e on G. Extensive theory of homogeneous left Lie loops are based on this fact([35], [36], [37], [38], [42], [44]).

The homogeneous system (G, η) of a homogeneous left Lie loop (G, μ) is said to be *geodesic* if (G, μ) is geodesic. In [31], it has been shown that any geodesic homogeneous system can be regarded as a totally geodesic submanifold of the enveloping Lie group $A = G \times K_e$ with the $(-)$-connection of E. Cartan, and characterization of geodesic homogeneous systems as totally geodesic submanifolds of some Lie groups has been clarified.

1.5. **Normal left subloops of homogeneous left loops.** (Kikkawa [17], [24])

By converting algebraic treatment of left loops into homogeneous systems, a clear manner to introduce the concept of invariant left subloops and normal left subloops of any homogeneous left loops has been found by the author ([24], [27], [42], [43], [44]):

Let (G, μ) be a homogeneous left loop and (H, μ_H) a homogeneous left subloop. Let (G, η) and (H, η_H) the homogeneous systems associated with μ and μ_H, respectively. Then, (H, μ_H) is an *invariant* left subloop of (G, μ) if the following relation holds for any $x, y \in G$:

$$\eta(x, y, xH) = yH,$$

where

$$xH := \eta(H, x, H) \quad \text{for} \quad x \in G.$$

An invariant (left) subloop H is *normal* if

$$\eta(x, y, z)H = \eta(xH, yH, zH)$$

hold for $x, y, z \in G$,

It has been shown ([24]) that a left subloop H of a homogeneous left loop is normal if and only if it is the kernel of a homomorphism from (G, μ) into some homogeneous left loop.

1.6. **Canonical connection of Lie loops.** (Kikkawa [16], [33], [34], [35])

Let G be a manifold of dimension n. For any left Lie loop (G, μ), it can be defined a ternary system $\eta : G \times G \times G \to G$ by the same way as the case of homogeneous left loops([35]), i.e. $\eta(x, y, z) := x((L_x^{-1}y)(L_x^{-1}z))$ by denoting $xy = \mu(x, y)$. Various tangential foumulas are presented in [34] in terms of certain new notation of differentiating operations for vector fields which is valid for any local coordinate system on G: For instance, assume that two vector fields X and Y are expressed in some local coordinates $(x^1, \cdots, x^n)$ as

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}.$$

Here, we abbreviate the symbol of summing up $\sum_{i=1}^{n}$. Then, we denote

$$X_x Y := X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

and, if u, v, w are considered as independent coordinate variables in the $3n$-variable functions $\eta(u,v,w)^k,\ k = 1, \cdots, n$, we denote

$$\eta(x, X_x, Y_x) := X^i Y^j \frac{\partial^2 \eta^k}{\partial v^i \partial w^j}_{|u=v=w=x} \frac{\partial}{\partial x^k}_{|x}.$$

Then, it has been shown that a linear connection ∇ is given explicitly by the formula below, which is called the *canonical connection* of the left Lie loop (G, μ) :

$$(\nabla_X Y)_x = X_x Y - \eta(x, X_x, Y_x) \quad \text{at} \quad x \in M,$$

for any vector fields X, Y on G. Especially, this is valid for homogeneous left Lie loops.

Let (G, μ) be a homogeneous left Lie loop with the canonical connection ∇ . Then, by using Lemma in [34], it is shown that ∇ is locally reductive, that is, the equations

$$\nabla S = 0 \quad \text{and} \quad \nabla R = 0.$$

hold for the torsion S and the curvature R .

2. Tangent Lie triple algebras

2.1. **Bianchi's identities and Ricci's identities for torsion and curvature.**

Here we recall some identities for linear connections which are related to the tangent algebras of homogeneous left Lie loops.

Let (M, ∇) be a differentiable manifold M with a linear connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M); (X, Y) \mapsto \nabla_X Y$ for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the module of all differentiable vector fields on M over the ring $\mathfrak{F}(M)$ of real-valued differentiable functions on M. Then, the torsion tensor field S and the curvature tensor field R of ∇ are given by the following equations:

$$S(X, Y) := [X, Y] - \nabla_X Y + \nabla_Y X$$
$$R(X, Y)Z := \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z,$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

The following identities are well-known in Differential Geometry:

$$\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z + S(S(X,Y),Z) - (\nabla_X S)(Y,Z)\} = 0$$

(Bianchi's 1st identity)

$$\mathfrak{S}_{X,Y,Z}\{R(S(X,Y),Z)W - (\nabla_X R)(Y,Z)W\} = 0$$

(Bianchi's 2nd identity)

$$(\nabla^2 S)(;X;Y) - (\nabla^2 S)(;Y;X) = R(X,Y)S - \nabla_{S(X,Y)} S$$

(Ricci's identity for$S$)

$$(\nabla^2 R)(;X;Y) - (\nabla^2 R)(;Y;X) = R(X,Y)R - \nabla_{S(X,Y)} R$$

(Ricci's identity for$R$).

Here, we denote by $\mathfrak{S}_{X,Y,Z}$ the cyclic sum with respect to X, Y, Z.

2.2. **Tangent Lie triple algebras of homogeneous left Lie loops.** (Akivis [2], Akivis-Shelekhov [4], Hofmann-Strambach [8], Kikkawa [16], [17], [25], [26], [41])

In [35], it has been shown that the tangent Akivis algebra (cf. Akivis [2], Hofmann-Strambach [8]) $(\mathfrak{g}; [X, Y], \langle X, Y, Z\rangle)$ of any geodesic left Lie loop (G, μ) is related with the canonical connection ∇ by the following eqations:

$$S_e(X, Y) = [X, Y], \quad R_e(X, Y)Z = \langle X, Y, Z\rangle - \langle Y, X, Z\rangle,$$

where $\mathfrak{g}$ is identified with the tangent space of G at the unit e, and S , R are the torsion and the curvature tensors of the canonical connection ∇, respectively.

Assume that (G, μ) is a homogeneous left Lie loop. Then it is shown that the tangential algebra $(\mathfrak{g}; [X, Y], [X, Y, Z])$ given by

$$[X, Y] = S_e(X, Y), \qquad [X, Y, Z] = R_e(X, Y)Z$$

forms a *Lie triple algebra* (*general Lie triple system* of Yamaguti [55]) which is called the *tangent Lie triple algebra* of (G, μ) ([16]). In fact, since the equations $\nabla S = 0$ and $\nabla R = 0$ hold for the torsion S and curvature R of the canonical connection, the identities of Bianchi and Ricci above evaluated at the point e imply the following relations;

$$\begin{aligned}
&[X, Y] = -[Y, X]\\
&[X, Y, Z] = -[Y, X, Z]\\
&\mathfrak{S}_{X,Y,Z}\{[X, Y, Z] + [[X, Y], Z]\} = 0\\
&\mathfrak{S}_{X,Y,Z}\{[[X, Y], Z, W]\} = 0\\
&[U, V, [X, Y]] = [[U, V, X], Y] + [X, [U, V, Y]]\\
&[U, V, [X, Y, Z]] = [[U, V, X], Y, Z]\\
&\qquad\qquad + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]],
\end{aligned}$$

those which are exactly the axiom of Lie triple algebra.

It should be noted that any Lie triple algebra is reduced to a Lie algebra if the triple product $[X, Y, Z]$ vanishes identically, and is reduced to a Lie triple system of E. Cartan if the bracket product $[X, Y]$ vanishes identically.

General theory of abstract Lie triple algebras on vector spaces over fields of characteristic zero has been treated in [23], [25], [28], [29]. Especially, decomposition problem of finite dimensional real Lie triple algebras into their ideals has been treated in [25] by introducing a new concept of *Killing Ricci forms* which is a generalization of both of Killing forms for Lie algebras and Ricci tensors for symmetric spaces. The results have been applied to the decomposition theory of homogeneous left Lie loops ([26]), and of homogeneous systems with naturally reductive metrics ([32]). They are based on the early results [12] on integrability of distributions given by decomposition of torsion tensor and curvature tensor on affinely connected manifolds.

Recently, the author generalized the concept of Lie triple algebra to an algebraic system which has more triple product ([45]).

2.3. **Lie groups as associative homogeneous left Lie loops and symmetric homogeneous left Lie loops.** (Kikkawa [14], [15], [16], [18], [22], [40], [41])

Let (G, μ) be a Lie group. Then it forms an associative homogeneous left Lie loops whose canonical connection is reduced to the $(-)$-connection of E. Cartan, i.e., it satisfies $\nabla S = 0$ and $R = 0$. So the tangent Lie triple algebra is reduced to the Lie algebra of (G, μ). In fact, the Lie bracket is given exactly by $[X, Y] = S_e(X, Y)$ and $[X, Y, Z] = 0$.

On the other hand, assume that a homogeneous left Lie loop (G, μ) is symmetric, that is, it satisfies the following relations:

$$(\mu(x, y))^{-1} = \mu(x^{-1}, y^{-1}) \quad \text{for} \quad x, y \in G.$$

Then, the canonical connection of (G, μ) satisfies $S = 0$ and $\nabla R = 0$, so that the tangent Lie triple algebra is reduced to a *Lie triple system* with the ternary product $[X, Y, Z] = R_e(X, Y)Z$.

The problem of imbedding of symmetric homogeneous systems into their enveloping Lie groups has been dicussed in [31].

2.4. **Akivis left Lie loops in Lie groups.** (Akivis [2], Akivis [3], Kikkawa [37], [38], [54], [40], [44])

Let (G, μ), $\mu(x, y) = xy$, be a Lie group with the Lie algebra $(\mathfrak{g}, [X, Y])$. For any integer p, set a new multiplication

$$\mu_p(x, y) := x^{p+1} y\, x^{-p}.$$

Then, (G, μ_p) forms a homogeneous left Lie loop in G, which is called an *Akivis left loop*. Especially, if $p = 0$ we get $\mu_0(x, y) = \mu(x, y)$. The tangent Lie triple algebra $(\mathfrak{g}; [X, Y]_p, [X, Y, Z]_p)$ of (G, μ_p) is given by

$$[X, Y]_p = (1 + 2p)[X, Y], \qquad [X, Y, Z]_p = -p(1 + p)[[X, Y], Z],$$

where $[X, Y] = [X, Y]_0$ is the Lie bracket in the Lie algebra of (G, μ). More generally, homogeneous local Lie loops μ_p , called *Akivis local loops* ([54]), can be defined in some neighbourhood of the unit e of the Lie group, by setting $\mu_p(x, y) = x^{p+1} y x^{-p}$ for any *real number* p. This local loop was found by Akivis [3] in 1978. If $p = -\frac{1}{2}$, we get a symmetric homogeneous local Lie loop $\mu_{-\frac{1}{2}}$ whose tangent Lie triple algebra is reduced to a Lie triple system ([39]).

2.5. **Generalized theory of the theory of Lie groups and Lie algebras.** (Kikkawa [16]–[42], Sagle-Schumi [53], [53])]

Non-associative generalization of the well-known theory of Lie groups and Lie algebras has been established consistently by the author. By means of the concept of homogeneous left Lie loops, the theory has been developed including the theory of Lie subloops and subalgebras of the tangent algebras, as the *theory of geodesic homogeneous left Lie loops*([16], [17], [42]).

Let (G, μ) be a homogeneous left Lie loops which is assumed to be geodesic, that is, the multiplication μ coinsides with the geodesic local loop of the canonical connection, in some neighbourhood of the unit e. The following results have been shown by the author ([16], [17], [20], [22], [24], [26], [30], [40]) :

Any homomorphism of homogeneous left loops induces a homomorphism of their tangent Lie triple algebras. Two geodesic homogeneous left Lie loops are locally isomorphic if and only if their tangent Lie triple algebras are isomorphic. Moreover, if the homogeneous left Lie loops are analytic and the underlying manifolds are connected and simply connected, then they are isomorphic if and only if their tangent Lie triple algebras are isomorphic.

Let H be an invariant left Lie subloop of G. Then, its tangent Lie triple algebra $\mathfrak{h}$ is an invariant Lie triple subsystem of $\mathfrak{g}$. Conversely, any invariant subsystem $\mathfrak{h}$ of $\mathfrak{g}$ is the tangent Lie triple algebra of an invariant left Lie subloop H of G. An invariant and closed left Lie subloop H of G is normal if and only if its tangent Lie triple algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. The proof of this fact depends on the early results in [13] .

A condition for existence of a homogeneous left Lie loop whose tangent Lie triple algebra be isomorphic to any given finite dimensional real Lie triple algebra has been investigated in [30] .

The decomposition problem of homogeneous systems which is equivalent to the same problem of homogeneous left Lie loops has been treated by the author in [26], [32], which is based on the results in [12] and [25].

2.6. **3-webs and local Lie loops.** (Akivis [1], [2], Akivis-Shelekhov [4], Bol [5], Chern [7], Kikkawa [33], [34])

By transferring the theory of differentiable local loops to the theory of differentiable 3-webs, the author has introduced the concept of *Chern connection* of any 3-web in [33] explicitly, as a linear connection on the product manifold on which the 3-web is settled, and the interrelations between the torsion and curvature tensors of 3-webs and those of Chern connection has been clarified ([33], [34]).

By using this linear connection, the concept of canonical connections of differentiable local loops is introduced, and some explicit formulars of the torsion and the curvature are given for left I.P. loops and homogeneous loops, by means of the loop multiplication and its left inner maps. These formulas clarified that the torsion tensor presents the differential measure of non-commutativity of the multiplication, while the curvature tensor presents that of non-associativity, as be suggested in the result of the first work ([10]) of the author.

3. Projectivity of homogeneous (left) Lie loops

3.1. **Projective relation of geodesic homogeneous left Lie loops.** (Kikkawa [35], [36]–[46])

Let (G, μ) and $(G, \tilde{\mu})$ be two homogeneous left Lie loops on the same underlying manifold G, ∇ and $\tilde{\nabla}$ be the canonical connections, η and $\tilde{\eta}$ the associated homogeneous systems of μ and $\tilde{\mu}$, respectively . For convenience, two left loops are assumed to have the same unit element e. Then, they are said to be *in projective relation* if the following two conditions are satisfied [36], [38] :

(1) Any geodesic curve of ∇ is a geodesic curve of $\tilde{\nabla}$, and vice versa.

(2) The following mutual relations hold:

$$\tilde{\eta}(x, y, \eta(u, v, w)) = \eta(\tilde{\eta}(x, y, u), \tilde{\eta}(x, y, v), \tilde{\eta}(x, y, w))$$
$$\eta(x, y, \tilde{\eta}(u, v, w)) = \tilde{\eta}(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$$

By introducing the differential geometric concept of *affine homogeneous structure* on linearly connected manifolds, the author investigated in [35] various relations of torsion and curvature for affine homogeneous structures, which imply some characteristic relations for two geodesic homogeneous left Lie loops to be in projective relation ([35]). The projectivity relation can be considered for geodesic homogeneous local Lie loops, that is, two geodesic homogeneous local Lie loops with the same unit e are in projective relation if (1) and (2) above are valid in some neighborhood of e.

Projectivity problem of subsystems of homogeneous systems is treated in [43].

3.2. **Projectivity of Lie groups and Akivis homogeneous left loops.** (KIKKAWA [36], [37], [39], [40], [54], [41], [43], [44])

The problem of finding geodesic homogeneous local Lie loops which are in projective relation with any given Lie group G has been investigated by the author ([36], [37], [38], [40], [54]) . It has been shown that any Akivis local loop in G is in projective relation with the Lie group G and that these Akivis local loops are in projective relation with each other. Moreover, it has been shown that, if the Lie group G is simple and of odd dimension, just only these Akivis local loops are the geodesic homogeneous local Lie loops in projective relation with G ([54]).

3.3. **Symmetrizability of geodesic homogeneous left Lie loops.** (KIKKAWA [39], [41])

A geodesic homogeneous left Lie loop (resp. local Lie loop) is said to be *symmetrizable* if it is in projective relation with some symmetric homogeneous left(resp. local) Lie loop. Any Akivis local loop in any Lie group is symmetrizable. The author investigated the condition for geodesic homogeneous left(resp. local) Lie loops to be symmetrizable:

A geodesic homogeneous left Lie loop (G, μ) is locally symmetrizable if and only if its tangent Lie triple algebra $\{\mathfrak{g}; [X, Y], [X, Y, Z]\}$ satisfies the following conditions;

(1) $(\mathfrak{g}, [X, Y])$ forms a Lie algebra.

(2) $(\mathfrak{g}, [X, Y, Z])$ forms a Lie triple system.

(3) $[X, [U, V, W]] = [[X, U], V, W] + [U, [X, V], W] + [U, V, [X, W]]$ holds for any $X, U, V, W \in \mathfrak{g}$.

3.4. **Projectivity of Lie triple algebras.** (KIKKAWA [46])

In [46], the author introduced the concept of projectivity of Lie triple algebras.

Let $\mathfrak{g} = (\mathbf{V}; [X, Y], [X, Y, Z])$ be a Lie triple algebra with the underlying vector space $\mathbf{V}$. A Lie algebra $\mathfrak{l} = (\mathbf{V}; L(X, Y))$ will be called a *Lie algebra of projectivity of a Lie triple algebra* $\mathfrak{g}$ if it satisfies the following relations:

$$\begin{aligned} L(X, [Y, Z]) &= [L(X, Y), Z] + [Y, L(X, Z)] \\ L(X, [Y, Z, W]) &= [L(X, Y), Z, W] \\ &\quad + [Y, L(X, Z), W] + [Y, Z, L(X, W)] \\ [U, V, L(X, Y)] &= L([U, V, X], Y) + L(X, [U, V, Y]) \end{aligned}$$

Two Lie triple algebras

$$\mathfrak{g} = (\mathbf{V}; [X, Y], [X, Y, Z]) \quad \text{and} \quad \tilde{\mathfrak{g}} = (\mathbf{V}; [X, Y]^{\sim}, [X, Y, Z]^{\sim})$$

with the same underlying vector space $\mathbf{V}$ are *in projective relation* if there exists a Lie algebra $\mathfrak{l} = (\mathbf{V}; L(X,Y))$ of projectivity of $\mathfrak{g}$ such that $\tilde{\mathfrak{g}}$ is related with $\mathfrak{g}$ by;

$$[X,Y]^{\sim} := [X,Y] + 2L(X,Y)$$
$$[X,Y,Z]^{\sim} := [X,Y,Z] - L([X,Y],Z) - L(L(X,Y),Z),$$

for $X, Y, Z \in \mathbf{V}$.

In [47], it has been proved that two geodesic homogeneous left Lie loops (G,μ) and $(G,\tilde{\mu})$ on the same connected analytic manifold G are in projective relation if and only if their tangent Lie triple algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are in projective relation.

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