Against the Tyranny of the Senses
―Banning the Visual Aids from Mathematical Argumentation―

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ABSTRACT

In this paper I highlight an essential distinction between the Euclidean deductive method and the axiomatic method and I present some implications for the pedagogy of proof in school geometry. In particular, I examine the role of visualization and visual representations in the Euclidean deductive method and the axiomatic method. Further, considering how proof is canonically taught in schools, I propose that visual representations are not merely pedagogical props. Instead, I argue that visuals are essential bricks that shape the proof process in geometry classrooms. Therefore, this paper cautions about projecting the ideal of rigorous proofs independent of visual representations to students, when the Euclidean deductive method is used for proving.

【Keyword：Mathematics Education, Pedagogy, Pedagogical Instrument】

【キーワード：数学教育 , 教材 , 教育工学】

In the last one hundred years or so of modern public schooling, mathematics has seen a very interesting map of tendencies, radicalized and often opposing positions over the nature of mathematical knowledge and, subsequently, over the pedagogy of school mathematics. Some milestones of this map are the development of the axiomatic method and high degree of abstract logic to frame deductive reasoning, the Bourbaki movement attempting to implement new standards of rigor and formalization in Western curriculum of the 60's, a set theoretical approach of school curriculum in the 70's and early 80's, and in the last two decades, a rise of constructivist positions on both the philosophy of mathematics and its pedagogy. The past decade saw a turmoil in Western schools of mathematics education (turmoil known in USA as the Math Wars), due to harsh criticisms that claim, amongst other things, that extreme constructivist approaches attempted to deny deductive reasoning its historical central role in mathematics curricula.[i] With respect to reasoning and validation of math knowledge, constructivist curricula were accused of advocating an unwarranted emphasis on visual thinking and intuition at the expense of mathematical rigor and a loss in knowledge.[ii] With respect to the use of visual representations in school mathematics, at the peak of Math Wars in the USA, the most radicalized positions on the development of mathematics education inadvertently promoted a false dichotomy in curricular choice: "constructivist reasoning" versus "deductive/ formal reasoning." Thus, at that time, in educational parlay of schoolteachers and administrators, mathematics curricula were being identified as "constructivist," or "traditional" . The "constructivist" curricula were assimilated to "a visual thinking approach curriculum," often associated with an emphasis on mathematical modeling and a heavy use of computer technology in math classrooms. At the other end of the spectrum, a "traditional approach" was associated with an emphasis on writing "two-column proofs".[iii] In this context, visualization in mathematics is regarded by the "constructivists" as an essential tool for learning mathematics by exploring, conjecturing, modeling and problem solving. Those who oppose the technology-minded radical constructivists, subscribed to relegating visualization to a desirable, even necessary, yet lower, intuitive stage in the process of reasoning and proof. At superior stages of learning they place deductive proof framed by adequate levels of formalism, and possibly the rigor of the axiomatic method, at least in spirit, if not in language. It is well known that the fundamental difference between these two positions is epistemological.[iv] The two pedagogies were based on different perspectives on the nature of mathematical knowledge and on the nature of learning, hence their differences in the nature of validation, the role of validation in generating new knowledge and on what mathematical skills and abilities should be emphasized in school.

More severely, at the peak of Math Wars, this perceived dichotomy tended to polarize classroom practices of teaching mathematics in USA, whether the curricula actually intended such radicalization or

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not. Thus, what actually happened in many classrooms was an "either/or" choice between a "constructivist approach" that overemphasized the reliance on the visual evidence and deemphasized the symbolic and "rigorous" mathematical language, or the opposite choice of focusing on traditional math skills supported by some appropriate degree of mathematical formalism and rigor, that limit (but not eliminate) the role of visual representations in proving. Traditional curricula have been criticized for (over)emphasize the two column proof, whilst their some of their most radical opponents attempted to replace the "two-column proof" writing with visual reasoning and even to rename the process with the weaker alternative of "reasoning and demonstration".[v]

Nonetheless, regardless of epistemological positions of the curricula, classrooms tend to use heavily visualization and visual reasoning. How these visual aids are used will make a great difference in classroom routines of proof and reasoning. Most geometry classrooms teach deductive reasoning and proof.[vi] In these lessons there must be always a message (explicit or implicit) about what counts as evidence (including visual representations and whether or not we can rely on them for proving). The teachings about what is a correct proof inevitably hint at the axiomatic construction. Research on classroom discourse found that geometry teachers may promote in some elementary form the underlining principles of the axiomatic method as ideals of rigor in proof. Yet, in geometry classrooms, it can be very challenging to "talk the talk" of axiomatic method ideals for a correct proof, when the process of reasoning itself relies so heavily on visuals.[vii] But can one correct this pitfall by teaching students to follow their deductive reasoning with writing a proof that becomes independent of the visuals they used?

The purpose of this paper is threefold: First, I invite in particular teachers of mathematics to reflect upon some epistemological distinctions between the axiomatic method and the deductive method of reasoning, and how these two are reflected in school approaches to proof and reasoning. Both these methods entail an axiomatic system frame, and both entail deductive processes of reasoning. Research indicates that these apparent communalities between the two methods may cause teachers and students to collapse the two methods and thus, to carry into the classroom practices some principles of rigor from the axiomatic method.[viii] Since the axiomatic method in fact changes fundamentally the rules of the game with respect to what constitutes evidence in a proof, classroom discourse would see tensions between the proof practices (which even in the two-column proof curricula rely on visual evidence) and the principles by which discourse about mathematics is organized in classroom.

Second, it is of interest for future teachers in particular, to understand the place that visual representations play in the classical deductive method canonically used in classrooms. This paper argues that visual representations play a much more fundamental role than just helping students understand the content easier.

Third, it is important to reflect on some important principles of the axiomatic method itself. Although the axiomatic method is not directly applied in school curricula, however, some principles are reflected in the ways we promote in classrooms an idea of rigor in mathematics. Research suggests that classroom routines of reasoning and proof are influenced to teachers' beliefs about mathematical reasoning and mathematical rigor.[ix]

This paper is organized as follows: First it offers a brief overview of the development of the axiomatic method and its epistemological underpinnings regarding visual representations. Second, it presents the role that visualization and visual reasoning played historically in deductive reasoning. Finally, it presents some implications of the above for the process of argumentation in school geometry.

Reason without Image: The Quiet Colors of the Axiomatic Method

Logical reasoning, often wrongly equated to deductive reasoning, may well develop without undergoing the experience of being schooled, of organized education. On the other hand, school in some form is needed to introduce us to *deduction* as a process of reasoning. Deduction is a disciplined process that supposes a more educated specificity in its' practices than the mere association of propositions by some accepted rules of inference. The latter practices at large constitute "logical reasoning." Deduction is indeed a process of the Logic type in the sense that it include associating propositional knowledge by rules of inference in its' practice except that deduction also presumes two important structural elements that do not necessarily bear with priority in the general "logical reasoning" processes:

One is that the deductive reasoning harbors strictly defined vertical *hierarchies*—parents-children type of relationships that develop from a well defined starting point—of texts associated by established conventions. In these hierarchies, we start the process from a base level consisting of primary texts called axioms and build up levels by derivative associations so that knowledge is organized on levels of inference counted as a second, third, …nth …degree of inferences down from the axioms. Secondary texts are the ones that derive directly from axioms, tertiary texts from secondary texts or from a combination between secondary texts and axioms, etc.

The second important presumption of a deductive system is that there exists a "starting level" in the reasoning system-that is, we admit and define a family of axioms from which all results within a system may be derived by inference. Both the vertical/hierarchical nature of the deductive structure and the necessity of identifying the family of axioms that makes the foundations of all deductive results suggest that deduction requires knowing some articulations conventionally signified (like the choice of *specific* axioms that frame a system, or the specificity of the rules of inference).

As such, deduction depends upon specific cultural artifacts (like formal language) that require a form of explicit education to make it accessible. Moreover and perhaps most importantly, in order for deduction to instrument the thinking process, a particular environment is required that withstands the specificity of structure and of language.

Deduction developed as a particular practice in argumentation. Western cultures place the origins of deductive reasoning in ancient Greece, as a process that started by coalescing even more ancient logical inferences and physical representations of space and then organized these in a system of relationships for the first time articulated as the *method of geometry* by Euclid and his disciples. It was not until the 19th century that the deductive system underwent any major development. At the beginning of the 20th century the deductive system was formalized under the school of Hilbert: all the geometrical texts of Euclid were reformulated on the basis of rigorously defined systems of axioms that made explicit all the assumptions left unspecified by Euclid.[x]

The project of Hilbert's axiomatic mathematics was to infer any result by formal logical associations from axioms, with a particular care to obtain consistent geometries (that is, all important results were obtained exclusively on the basis of elements of the system within which they are formulated). We say that geometric axiomatic systems of geometries are closed, that is, any result may be obtained within them based solely on inference relationships amongst texts. A complete axiomatic system stays by itself, and may be taught in the regime of " tabula rasa" that is, one need not know any prior mathematics or any mathematics outside of these axiomatic systems in order to learn them.[xi] "Closed systems" means they are sufficient in themselves, their elements are self-explanatory, and therefore the formal axiomatic process of deduction requires no visual representations, in the sense that comprehension and reasoning do not depend on visual (diagrammatic) representations of the objects on which we work.

Canonical discourses in mathematics consider Hilbert's formal axiomatic systems to be the culmination of the deductive process and the foundation of rigorous reasoning:[xii] The fact that visual and intuitive elements were not determinant factors in the process of validation represented for many mathematicians the very substance of mathematical argument.[xiii] In fact, historically, axiomatic systems were not a product of mathematical practices—for mathematicians' everyday work was not going by the axiomatic method. Instead, the axiomatic method was a meta-mathematics discourse, a "modus ponens."[xiv] Axiomatic discourse came as a response to the quest for a method to access objective truths of the World and avoid relying on one's senses for reaching "the truth." The old historical struggle of Rationality and Enlightenment to make "the truth" objective, finally received a solid theoretical foundation when Axiomatic eliminated from argumentation any legitimacy of visual evidence and eliminated, with formal language, the "natural intuition."[xv] Establishing a system free of these two "shortcomings" intended to shift the basis of reasoning from the " tyranny of senses to the safe reliability on the rational".[xvi]

Let us note here that an important difference between the Greek "deduction" and Hilbert's "axiomatic method" is the relevance and the place of the visual representations in the deductive argument. The school established by Euclid in geometry is widely considered the parent of the deductive method. However, unlike the axiomatic method, the Greeks placed a major emphasis on the visual representation of the mathematical discussions (particularly by diagrams and 3-D constructions). The *dia'grammon*— "the method using lines"—was the predominant way of approaching problems in Greek mathematics, even arithmetical problems.[xvii]

The "method using lines" consisted in developing logical inferences on the basis of a graphical representation obtained by combining three basic elements of space: the point, the line and the circle. Both the power and the limitations of diagrams came from the fact that they were exclusively the product of construction by the straight-edge and the compass, which meant they imbedded in their identity and structure the geometrical properties coming from lines and circles. These properties of lines and circles had the same influence in shaping the dia'grammon that genetic markers have in shaping the bodies that carry them.

Reasoning by dia'grammon, framed by the rules that govern straight lines and circles, enabled some powerful inferences and mathematical constructs, yet limited the possible conjecturing and conceiving of space and time to the universe of straight line and circle. Objects were being conceived of in relation to constructions with rule and compass. The basic geometrical features and properties of lines and circles, *accepted by visual evidence*, were constituted in basic axioms on which the deductive reasoning was then built.

It is important to mention that the formal axiomatics of Hilbert turns away from the Greek tradition at the point where axioms were accepted by visual evidence. Axiomatic system comes to eliminate the visual identity of axioms by spelling out *conceptually* the basic features of point, line, circle and space. Some of these basic features were compressed in the diagram and taken for granted at an intuitive level by the ancient Greeks. These hidden assumptions evaded articulation in the times of dia'grammon.

Consider this diagram (Diagram 1) representing the following problem formulated by the Greek mathematician Menelaus:

A'B .B'C C'A = 1
A'C B'A C'B

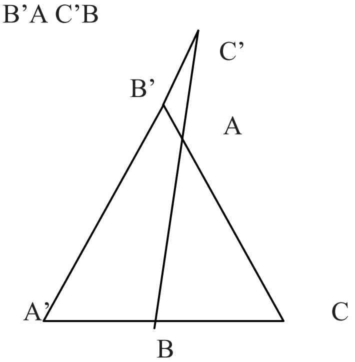

Diagram 1

Note that the problem has a subdued assumption that line {BAC'} in fact cuts the triangle A' B' C in two points and furthermore, it must cut the third side (A'B') in a point as well, and that point (C') is outside of the triangle A' B' C. In fact, this composite assumption, not spelled out by Greeks and taken as diagrammatically obvious, is decomposed and proven later on in the axiomatic construction of geometry.

In fact, the basic assumption on which the problem above relies is what is today known as the "Axiom of Pash" : *A line touching a triangle and passing inside it, touches that triangle in two points* (Diagram 2). Considered by Greeks diagrammatically obvious, this axiom and all her derivatives never were spelled out by Greeks.[xviii]

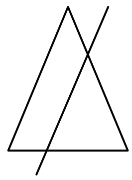
Diagram 2

Axiomatics comes to challenge the "taken for granted" status of these implicit assumptions imbedded in the visual diagrams. In this respect, Hilbert used formal language to spell out all those assumptions hidden in the diagrams in order to replace the "natural (sense)" with the "(product of) conceptual articulations." But how is this "conceptual articulation" working differently than the diagram?

Let us take a look at how the Pash axioms, not articulated at all by Greeks as diagrammatically obvious, but well articulated in the axiomatic system of Hilbert:

Let A,B and C be three points non co-linear. We call a triangle the set

{A,B,C}U{BC}U{CA}U{AC} where BC,CA and AC are the subsets of all points co-linear situated between A,B, and C.

Let ABC be a triangle set and let d be a line that does not go through any vertex A,B or C. If the line d has a common point with one side of the triangles, then it will have one common point with at least another side. That is:

$d \cap /AB/ \neq \emptyset \Rightarrow (d \cap /AC/ = \emptyset) \vee (d \cap /BC/ \neq \emptyset)$

Note that in the axiomatic formulation above in fact *there is no need for any visual representation*. The definitions and theorem are based on sets of points related to each other by logical operators (in the case above $\cap$ represents intersection, and $\vee$ represents the logical operator "AND") which are formally defined. We do know what choices we make when we operate on sets because the operation itself is formally defined: we know for example that the intersection of two sets will be either a set of points common to both initial sets, or the Empty set; but this definition of intersection formalized as an operation (just like Sum) in symbolic language circumvents the need for any visual representation in order to resolve any problem that requires intersection of sets because we can apply the formal operation and generate the result without making a diagram.

However, I suggest that operating on formalized texts as the above is much more difficult and less natural, since it requires particular cognitive abilities together with knowledge of concepts and procedures. Thus this kind of text may not be readily available to everyone as it takes us out of regular language and "natural" intuitions.

Why, then, would anybody endeavor to replace the Greek dia'grammon, apparently accessible to everyone through naked eye, with formalization so complex that it requires complex cognitive abilities to access it? The answer to this is important for understanding the role of the visual in mathematical argumentation: The dia'grammons were limiting the objects of geometry to the particulars of the Euclidian space that fell short of being a closed system (postulate 5 was not proven and in fact proved to be non-universal). Moreover, the Euclidian space was found to be only *one possibility of interpreting space in general* (and non-Euclidian geometries started from here). On the other hand, the axiomatic method of Hilbert served as an instrument to expand exploration of space in a disciplined manner-note the emphasis on *disciplined*, which is meant to maintain the frames of reference within the domain of mathematics. Whilst the axiomatic method stressed the specificity of the results and their dependence on the choice of foundational axioms, it also liberated concepts from the constraints of uncritically accepted visual representations. Diagrams themselves were readjusted to represent objects of geometry in other than Euclidian spaces:

For example, to Greeks a line was an object with length but no width—embodied by a diagram that obviated any need to change its shape (Diagram 3):

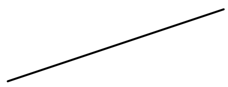

Diagram 3

…whilst for Hilbert's axiomatics, a line is a 1-dimensional infinite subset of a defined space with defined parameters and properties. Hilbert's approach keeps the concept of "line" as an object determined by 2 points in any space, but eliminates the straightness imbedded in the Greek line. The axiomatic formulation of the line concept allows a circle drawn on the surface of a sphere and going through the poles *to satisfy the definition of a line*. So we may have circles as *lines* (Diagram 4).

The triangle that the Greek Menelaus talks about in the theorem presented above—that to him was accessible in the Euclidian space as shown in the Diagram 1—the Pash axiom mentioned above, might now both be applied onto the surface of the sphere for example, using axiomatic formulations. Lines are now the big circles of the sphere. Can you stretch your imagination to see how the Menelaus theorem may look like on the sphere and beyond, in spaces that may not even have a graphical representation accessible to us?

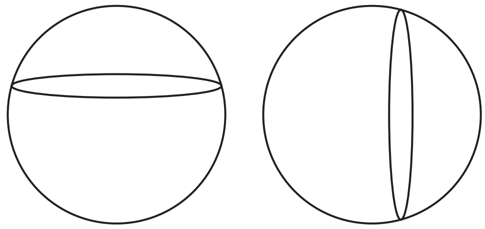

Diagram 4

Putting the Visual Back into the Argument

In the section above I pointed out an important distinction between Greek dia'grammon and the axiomatic method with respect to the use of visual representations: The Greek visual representations of constructible objects were legitimate objects on which one reasoned, whilst for Hilbert's system the objects on which one reasoned shifted to eliminate visual representations and replace them with propositional knowledge, where the objects were formalized by symbols and connecting operators.

Let us now examine how diagrams constituted for Greeks an essential tool in reasoning:

First of all, the elements of the diagrams are geometrical constructions that have been obtained with basic tools of drawing like the straightedge and the compass (for example, points appear as intersections of lines or arcs). As such, *a diagram is a finite and discrete representation*—and hence mentally graspable— of a structure meant to represent geometric objects that in fact are continuous in nature and have an infinite number of points which makes them not graspable or manipulable in their entire essence by our minds. The functional role of the diagram in geometry is not that of a picture of the geometrical object on which a problem is formulated, but instead the diagram is a picture of the problem itself. That is, the elements emphasized to the eye and mind in the diagram are physical representations of the givens in the hypothesis of a problem: a diagram "at work" appeals to deductive reasoning through the special points defined as intersection of particular lines. Measures of angles and properties of congruence, parallelism and perpendicularity reveal particular properties of shapes in one eye-grasp. The number of givens in the hypothesis being finite, so is the diagram. Thus the diagram in fact is a text illustrating a finite number of givens and their spatial relationships. The diagram is relied upon as a finite and manageable system of relations because it emphasizes a finite number of intersection points and segments to which a problem refers, which makes it a lucrative instrument for exploring and organizing structures and models in an infinite universe.[xix]

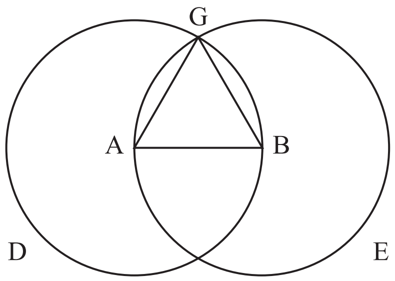

Diagram 5

Second, *the diagram encodes the text in a certain manner*.[xx] For example, since assumptions are encoded implicitly in the diagram, they will constitute the background for reasoning without having too many explicit elements in our way. The advantage to encoding these assumptions implicitly in the schema that a diagram represents is that all these assumptions, all these characteristics may be grasped *together* by our minds *as one system of issues* that *do not need specification of identity, shape and role, done for each of them one by one*. Consider for example the very first proposition of Euclid's *Elements*. This proposition, based on the diagram that represents it in the Elements, contains the assumption that the circles intersect (Diagram 5).

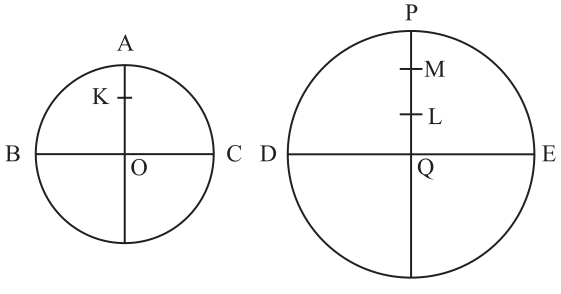

Diagram 6

This diagram contains a series of assumptions on the basis of Pash axioms: for example that circles intersect in two points to begin with. Pash axioms allows us to establish that in an intersection of certain figures we do have *a determined* number of points, which helps reasoning because we know we deal only with these many points and we can grasp them in sight all at the same time.

The diagram organized the text: One important feature of the diagram is that the cases for discussion are set by the choice of references fixed in the diagram.

Let us consider for example this problem of Archimedes. Let (BAC) and (DPE) be two similar sections (Diagram 6). K and M are their gravity centers.

Prove that A K: KO:PM:MQ

The proof is by *reductio ad absurdum,* assuming there exists a point L so that PL:LQ:: AK:KO

The problem presents two cases for discussion, depending on where you place point L—above or below M. The cases are asymmetrical therefore they are distinct.

A diagram was a metonym of a proposition, the core of the proposition itself. In Greek geometry the diagrams were intrinsic part of the problem text: they actually constituted the language by which the problem and the solution were formulated and communicated. Moreover, the text and the diagram are interdependent: assertions derive from a combination of text and diagram.[xxi]

Argumentation in School Geometry

Today the discourses about mathematics like to present a dichotomy between rigorous and intuitive mathematics, where visualization and visual reasoning is associated with intuitive phases of thinking and with imagination, with the exploring and conjecturing phase, but also with imprecision.[xxii] Whilst the deductive and axiomatic methods are associated with (more) rigorous mathematics, namely with the rigor of proof, it is important to note that, although the axiomatic method endeavors to circumvent the subjectivity of visual representations, it does not replace the visual representations at the core of mathematical thinking.[xxiii]

On the basis of this dichotomy, critics of traditional curricula, hold visualization and visual reasoning as a flagship of constructivist pedagogy. In their critique of traditional pedagogy they emphasize the pursuit two-column proof as the opposite element to progressive constructivism.[xxiv] As Schoenfeld points out, school geometry more often than not looses its appealing to students because lessons are organized as processes of establishing two-column proofs for problems, that is, putting problems in a formalized form with emphasis on inferences and connections of the text.[xxv]

However, in most modern schools in this world, the canonical approach to proof is a modernized version of the classic Greek dia'grammon.[xxvi] Moreover, school mathematics has a tendency to represent the diagrammatic method of deductive reasoning as *the sole embodiment of logical reasoning*.[xxvii]

Deductive arguments are special types of arguments, because they are framed in well-defined axiomatic systems (even if the frame behind the rules is not made explicit in school mathematics for example). When a family of axioms is assumed at the basis of a mathematical discussion, the mathematical knowledge involved in discussion carries a certain specificity defined and bounded by those axioms. Thus, in everyday speech, it is easy to make confusion between deductive method and axiomatic method.

As seen in the previous section, visual reasoning is organically imbedded in this dia'grammon method. Thus, the canonical approaches to proof in school should not be regarded as opposite to visual reasoning. They are in fact fundamentally relying on visual reasoning in the same subtle way that the dia'grammon does. On the other hand, the axiomatic method has a complex role in construction of knowledge. In this respect, whilst never directly incorporated in school curricula, the axiomatic method has an important role in curricular thinking because it does inform what are the standards of rigor in validation of knowledge, it supports and frames abstractization, formalization and generalization of mathematical knowledge.[xxviii] The latter three are fundamental processes of knowledge construction in mathematics and as such are reflected, albeit to a primitive degree, in school mathematics as well. To be precise, we do not apply the axiomatic method per se, and tedious elements of the axiomatic method are not incorporated in school curricula. However, we go by some of the fundamental tenets in the spirit of the axiomatic method when we extend a reasoning from concrete to abstract, when we do work with some formalized language and rigorous formal definitions and when we go through the process of generalizing a result. One important example, in this respect, is the fact that whilst we avoid the process of axiomatic reasoning and its formalized language in schools, however, we do use formal and abstract definitions for mathematical objects, definitions that tend to settle the object outside its visual representation. Moreover, whilst in a Geometry class the dia'grammon method is used for the actual process of reasoning and explicating one's reasoning, however, we use formal language to write the reasoning and the result. As we do so, we undergo a process of abstractization and/or generalizing the result to the extend possible at that level of knowledge. For generalizing and abstractization, the procedures and the language are those of the Axiomatic method. Writing mathematics, after doing the reasoning, is the final phase of the process of solving a problem. It is in this phase that the spirit of the Axiomatic method is visible and the influence of it may grow. Thus, the canonical process of learning in school usually undergoes two phases: a phase of reasoning and explaining, in which there has to be a fundamental reliance on visual representations (cf. the dia'grammon used in school), and a phase of writing the solution/proof. The latter is framed by principles of the axiomatic method (stated in schools in a more or less elementary form). These principles support the use of formal language. The use of formal language supports the struggle to eliminate the dependence on visual representations in the process of proof writing and to attain the highest degree of rigor possible. School mathematics incorporates both these processes in proof construction: one process relies on visual representations to explore, scaffold reasoning and explain, the other strives to not rely on visuals for conceptualizing and presenting a final form of knowledge. Incorporating both these processes, school mathematics reflects closely the process of knowledge production established today in the discipline.

Given the degree of abstraction to which axiomatics is being formulated, in school curricula the axiomatic constructions are rarely encouraged. Only rudiments of the axiomatic process may be present in the form of noting the axioms at the origins of a particular result. However, this is not true of the spirit of axiomatic deduction and the prevalence of deductive argumentation viewed as the hinge of geometry. Thus it is important for teachers to understand what kind of reasoning process is that which they engage students in, and the differences between the stages of this process with respect to visual representations.

The process of mathematisation is an organic syncretism of visual representations and logical articulations in which a dividing line between the former and the latter, practicing mathematicians find much more difficult to draw nowadays in the post-Hilbert era.[xxix] Although the uses of dia'grammon change depending on the method engaged, however, the *usability of visuals* bridge between different approaches to geometry, thus making geometry, as Hilbert himself admits, distinctive as a domain of mathematics.

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論文要旨

本論は、ユークリッドの演繹法と公理的方法との根本的な違いを指摘し、幾何学において証明を用いた数学教育の指導法にその違いが与えた影響について論じる。特に、ユークリッドの演繹法と公理的方法との間で図式化や描画モデルを用いた方法が担う役割について考察した。また、学校教育に於いて数学的証明がどのように指導されているのかを検討した結果、数学教育に於いて用いられる描画モデルが単なるモデル以上の意味を持つことを論じる。本論は、数学教育に可視化したイメージを用いることが、特に幾何学分野の指導において欠かすことのできない一段階であることを示し、ユークリッドの演繹法を用いた指導で可視化したイメージを用いることに否定的な数学教育の指導法を批判的に検討する。