Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
44 (2011), pp. 17–26

RULED LAGRANGIAN SUBMANIFOLDS IN COMPLEX EUCLIDEAN 3-SPACE

JONG TAEK CHO AND MAKOTO KIMURA

(Received: January 25, 2011)

Abstract. We show that if ruled Lagrangian submanifold M^3 in 3-dimensional complex Euclidean space is Einstein, then it is flat, provided that the map which gives direction of each ruling has constant rank. Also we give explicit construction of flat ruled Lagrangian submanifolds M^3 in $\mathbb{C}^3$, from some horizontal curves in S^5, such that M^3 is neither totally geodesic nor Riemannian product $\Sigma \times \mathbb{R}$.

1. Introduction

Lagrangian submanifolds plays important roles in differential geometry, symplectic geometry and mathematical physics. In particular, minimal Lagrangian submanifolds in Kähler manifolds and special Lagrangian submanifolds in Ricci-flat Kähler (i.e. Calabi-Yau) manifolds are distinguished important objects. In complex Euclidean spaces, minimal Lagrangian submanifolds and special Lagrangian submanifolds are coincide [5]. To construct *explicit* examples of special Lagrangian submanifolds in complex Euclidean 3-space, Joyce [6] investigated *ruled* Lagrangian submanifolds in $\mathbb{C}^3$. On the other hand, ruled submanifolds are studied extensively in not only differential geometry but also projective and algebraic geometry (cf. [9] and [10]). In this paper we study differential geometric properties of *ruled Lagrangian submanifolds in* $\mathbb{C}^3$.

Description of ruled 3-dimensional submanifolds in $\mathbb{C}^3$ are given by Joyce [6] as

$$M = \{r\phi(p) + \psi(p) : p \in \Sigma, \quad r \in \mathbb{R}\},$$

where Σ is a 2-dimensional surface, $\phi : \Sigma \to S^5$ and $\psi : \Sigma \to \mathbb{C}^3$ are smooth maps, and S^5 is the unit sphere in $\mathbb{C}^3$ (cf. §3). Then Joyce determined ruled 3-dimensional special Lagrangian submanifolds in $\mathbb{C}^3$ in the case that ϕ is an *immersion*. On the other hand, as a trivial case, when $\phi : \Sigma \to S^5$ is a *constant map*, the corresponding ruled submanifold is considered as a Riemannian product $\Sigma \times \mathbb{R}$, where Σ is a submanifold in $\mathbb{R}^5 = \{\phi(\Sigma)\}^\perp$ by the immersion $\psi : \Sigma \to \mathbb{R}^5 \subset \mathbb{C}^3$ cf. §4). Then the product submanifold $M^3 = \Sigma \times \mathbb{R}$ is Lagrangian in $\mathbb{C}^3$ if and only if

2000 *Mathematics Subject Classification.* 53C25, 53C40.
Key words and phrases. Lagrangian submanifolds, Ricci tensor, scalar curvature, complex Euclidean space.

$\psi : \Sigma \to \mathbb{C}^2 \subset \mathbb{C}^3$ is a Lagrangian immersion. Also the M^3 is special (i.e. minimal) Lagrangian in $\mathbb{C}^3$ if and only if Σ is minimal Lagrangian in $\mathbb{C}^2$.

Anyway ruled Lagrangian submanifold is interesting and important class among Lagrangian submanifolds in $\mathbb{C}^3$, and determination of *Einstein* (i.e., constant curvature) ruled Lagrangian 3-fold is a fundamental problem. When $\phi : \Sigma \to S^5$ is a constant map, if the ruled 3-fold $M^3 = \Sigma \times \mathbb{R}$ in $\mathbb{C}^3$ is Einstein, then M^3 is flat and Σ is a flat Lagrangian surface in $\mathbb{C}^2$ (cf. [1] and [3]). When ϕ is *an immersion*, the ruled 3-fold M^3 is Lagrangian provided that $\phi : \Sigma \to S^5$ is a *Legendre* immersion. Our first result is that if M^3 is Einstein then $\mathrm{Ric} = 0$ and the metric on Σ induced by ϕ has *constant Gaussian curvature* 1 (Proposition 2). Also if the scalar curvature R of M^3 is constant, then $R = 0$ (Proposition 3).

Next we discuss the case: $\operatorname{rank} d\phi = 1$. Then the ruled 3-fold M^3 is Lagrangian provided that the corresponding curve $\phi : I \to S^5$ is horizontal with respect to the Hopf fibration $S^5 \to \mathbb{CP}^2$. As a special case, we can construct flat ruled Lagrangian 3-fold M^3 explicitly, from a horizontal curve $\phi : I \to S^5$ satisfying $\|\phi'\| = 1$, $\langle \phi'', i\phi' \rangle = 0$, $\langle \phi''', i\phi'' \rangle \neq 0$ with non-vanishing geodesic curvature, and a function $\mu : J \times I \to \mathbb{R}$ satisfying $\mu \neq 0$ and $\mu_s \neq 0$ (I and J are intervals). Note that resulting Lagrangian submanifolds M^3 in $\mathbb{C}^3$ are neither totally geodesic, nor Riemannian product of a Lagrangian surface Σ^2 and a real line $\mathbb{R}$.

2. Lagrangian submanifolds in $\mathbb{C}^3$ and Legendrian submanifolds in S^5

Let $\mathbb{C}^3 = \{(z_1, z_2, z_3) | z_j \in \mathbb{C},\ j = 1, 2, 3\}$ be a 3-dimensional complex Euclidean space with standard real inner product $\langle (z_1, z_2, z_3), (w_1, w_2, w_3) \rangle =$ real part of $(z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3)$. Let $M = M^3$ be a Lagrangian submanifold in $\mathbb{C}^3$. Then Gauss equation is

$$\begin{aligned} g(R(X,Y)Z,W) &= c\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad + g(\sigma(Y,Z),\sigma(X,W)) - g(\sigma(X,Z),\sigma(Y,W)), \end{aligned}$$

where R and σ denote curvature tensor and second fundamental tensor of M for tangent vector field X, Y. Let T be a symmetric $(0,3)$-tensor field on M defined by $T(X,Y,Z) = \langle \sigma(X,Y), JZ \rangle$. Then the Ricci tensor of M is given by

$$\mathrm{Ric}(X,Y) = \sum_{i,j=1}^{3} \left(\langle T(X,Y,e_i), T(e_i,e_j,e_j) \rangle - \langle T(X,e_i,e_j), T(Y,e_i,e_j) \rangle \right) \tag{1}$$

where e_i $(i = 1, 2, 3)$ is an orthonormal basis of a tangent space of M. The scalar curvature ρ of M is

$$\rho = \|H\|^2 - \|\sigma\|^2. \tag{2}$$

Let S^5 be the unit sphere in $\mathbb{C}^3$ and let $f : M^2 \to S^5$ be an immersion. Then f is called a *Legendrian immersion* if for any $x \in M$, $\langle df(T_x(M)), if(x) \rangle = 0$, where

$i = \sqrt{-1}$. For a Legendrian immersion $f : M^2 \to S^5$, Gauss equation is given by

$$
\begin{aligned}
K &= 1 + \langle \sigma(e_1, e_1), \sigma(e_2, e_2)\rangle - \langle \sigma(e_1, e_2), \sigma(e_1, e_2)\rangle, \\
&= 1 + \langle \sigma(e_1, e_1), ie_1\rangle\langle \sigma(e_2, e_2), ie_1\rangle + \langle \sigma(e_1, e_1), ie_2\rangle\langle \sigma(e_2, e_2), ie_2\rangle \\
&\quad - \langle \sigma(e_1, e_2), ie_1\rangle\langle \sigma(e_1, e_2), ie_1\rangle - \langle \sigma(e_1, e_2), ie_2\rangle\langle \sigma(e_1, e_2), ie_2\rangle,
\end{aligned} \tag{3}
$$

where K denotes the Gaussian curvature of M^2 and e_1, e_2 is an orthonormal frame of a tangent space of M^2.

3. Ruled Lagrangian submanifolds of $\mathbb{C}^3$

According to §3 of [6] , we set up notation of ruled submanifolds in $\mathbb{C}^3$.

Definition 3.1. Let M be a real k-dimensional submanifold in $\mathbb{C}^3$. A *ruling* (Σ, π) of M is a $(k-1)$-dimensional manifold Σ and a smooth map $\pi : M \to \Sigma$, such that for each $p \in \Sigma$ the fibre $\pi^{-1}(p)$ is a real affine straight line in $\mathbb{C}^3$. A *ruled submanifold* is a triple (M, Σ, π), where M is a submanifold of $\mathbb{C}^3$ and (Σ, π) is a ruling of M.

Usually we will refer to the ruled submanifold as M, taking Σ, π to be given. As *r-orientation* for (Σ, π) is a choice of orientation for the real line $\pi^{-1}(p)$ for each $p \in \Sigma$, which varies continuously with p. A ruled submanifold (M, Σ, π) with an r-orientation is called *r-oriented ruled submanifold*.

Let (M, Σ, π) be an r-oriented ruled submanifold of N, and let S^5 be the unit sphere in $\mathbb{C}^3$. Define a map $\phi : \Sigma \to S^5$ such that $\phi(p)$ is the unique unit vector parallel to $\pi^{-1}(p)$ and in the positive direction with respect to the orientation on $\pi^{-1}(p)$, for each $p \in \Sigma$. Note that ϕ is a smooth map.

Define a map $\psi : \Sigma \to \mathbb{C}^3$ such that $\psi(p)$ is the unique vector in $\pi^{-1}(p)$ orthogonal to $\phi(p)$, for each $p \in \Sigma$. Then ψ is smooth and we have

$$M = \{r\phi(p) + \psi(p) : p \in \Sigma, \quad r \in \mathbb{R}\}. \tag{4}$$

Real 3-dimensional rules submanifolds M in $\mathbb{C}^3$ given by (4) are essentially included in one of the following classes: (i) rank of (differential of) $\phi : \Sigma \to S^5$ is equal to 0, i.e., ϕ is a constant map. (ii) rank of ϕ is equal to 1. (iii) rank of ϕ is equal to 2, i.e., ϕ is an immersion. In [6], only the case (iii) is discussed.

4. Riemannian product of Lagrangian surfaces in $\mathbb{C}^2$ and real lines

First we consider the case (i), i.e., $\phi : \Sigma \to S^5$ is a constant map. Then clearly the ruled submanifold M in $\mathbb{C}^3$ is given as the Riemannian product $\mathbb{R} \times \Sigma$, where Σ is considered as a surface in $\mathbb{R}^5 = \{\phi(\Sigma)\}^{\perp}$ by the immersion $\psi : \Sigma \to \mathbb{R}^5 \subset \mathbb{C}^3$. Moreover, in this case $M = \mathbb{R} \times \Sigma$ is a Lagrangian submanifold in $\mathbb{C}^3$ if and only if $\psi : \Sigma \to \mathbb{C}^2$ is a Lagrangian immersion, where $\mathbb{C}^2$ is the complex orthogonal complement of $\phi(\Sigma)$ in $\mathbb{C}^3$. Note that $M = \mathbb{R} \times \Sigma$ is a minimal Lagrangian submanifold (i.e. special Lagrangian submanifold) in $\mathbb{C}^3$ if and only if $\psi(\Sigma)$ is a minimal Lagrangian surface in $\mathbb{C}^2$. Also if $M = \mathbb{R} \times \Sigma$ is an Einstein (i.e., constant curvature) Lagrangian submanifold in $\mathbb{C}^3$, then M is flat and $\psi(\Sigma)$ is a flat Lagrangian surface in $\mathbb{C}^2$ (cf. [1]).

Remark 4.1. Let ψ be a Lagrangian isometric immersion from 2-dimensional Riemannian manifold Σ of constant curvature λ (cf. [1]) to $\mathbb{C}^2$. Then the induced metric g of the Lagrangian submanifold $M = \mathbb{R} \times \Sigma$ in $\mathbb{C}^3$ is a gradient Ricci soliton satisfying $\mathrm{Hess}_g f + \mathrm{Ric}_g = \lambda g$, where $f = \lambda t^2/2$ and t is the standard coordinate of the Euclidean factor $\mathbb{R}$ (cf. [9]).

5. Ruled Lagrangian submanifolds of $\mathbb{C}^3$ and Legendrian surfaces in S^5

In this section, we consider ruled Lagrangian submanifolds in $\mathbb{C}^3$ such that the map $\phi : \Sigma \to S^5$, which gives a direction of ruling, is an *immersion*. This case was also discussed in [6]. Let Σ be a real 2-dimensional manifold and let $\phi : \Sigma \to S^5$ be an immersion. Let $\psi : \Sigma \to \mathbb{C}^3$ be a map satisfying $\langle \phi(p), \psi(p) \rangle = 0$ for any $p \in \Sigma$. Let M be the image of the map $\Phi : \mathbb{R} \times \Sigma \to \mathbb{C}^3$ given by

$$\Phi(r, p) = r\phi(p) + \psi(p). \tag{5}$$

As ϕ is an immersion, Φ is an immersion almost every where in $\mathbb{R} \times \Sigma$. The images of points where Φ is not an immersion are generally singular points on M. Regarding M as an immersed copy of $\mathbb{R} \times \Sigma$ with (possibly singular) immersion Φ, we may define $\pi : M \to \Sigma$ by $\pi(r, p) = p$. Then (Σ, π) is a ruling on M.

Suppose Φ is an immersion at (r, p) in $\mathbb{R} \times \Sigma$. As $\phi : \Sigma \to S^5$ is an immersion, the pull-back of the round metric on S^5 makes Σ into a Riemannian 2-manifold. We choose isothermal coordinate (s, t) on Σ near p. Then we have

$$\rho = \rho(s, t) := \|\phi_s\| = \|\phi_t\| > 0, \quad \langle \phi_s, \phi_t \rangle = 0. \tag{6}$$

First we would like to find conditions for which Φ is a Lagrangian immersion. First partial derivatives of Φ are given as

$$\Phi_r = \frac{\partial \Phi}{\partial r}(r, p) = \phi(p), \quad \Phi_s = \frac{\partial \Phi}{\partial s}(r, p) = r\phi_s(p) + \psi_s(p), \tag{7}$$
$$\text{and} \quad \Phi_t = \frac{\partial \Phi}{\partial t}(r, p) = r\phi_t(p) + \psi_t(p).$$

Then Φ is a Lagrangian immersion if and only if

$$\langle \Phi_r, i\Phi_s \rangle = \langle \Phi_r, i\Phi_t \rangle = \langle \Phi_s, i\Phi_t \rangle = 0. \tag{8}$$

Substituting in for Φ_r, Φ_s, Φ_t using (5) gives equations upon ϕ and ψ and their derivatives, which are linear or quadratic polynomials in r. As the equations should hold for all $r \in \mathbb{R}$, the coefficient of each power of r should vanish. So we find that (8) holding for all r is equivalent to the equations

$$\langle \phi, i\phi_s \rangle = \langle \phi, i\phi_t \rangle = \langle \phi_s, i\phi_t \rangle = 0, \tag{9}$$
$$\langle \phi, i\psi_s \rangle = \langle \phi, i\psi_t \rangle = \langle \phi_s, i\psi_t \rangle - \langle \phi_t, i\psi_s \rangle = 0, \tag{10}$$
$$\langle \psi_s, i\psi_t \rangle = 0. \tag{11}$$

Note that $\phi : \Sigma \to S^5$ is a Legendrian immersion by (9).

(6) and (5) imply that the induced metric g on $\mathbb{R}\times\Sigma$ is expressed with respect to the basis $\{\partial/\partial r, \partial/\partial s, \partial/\partial t\}$ of the tangent space $T_{(r,p)}(\mathbb{R}\times\Sigma) = T_r\mathbb{R}\times T_p\Sigma$ as

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{12}$$

where

$$\begin{gathered} g_{11} = \|\Phi_r\|^2 = 1, \quad g_{12} = g_{21} = \langle \Phi_r, \Phi_s\rangle = c_1, \quad g_{13} = g_{31} = \langle \Phi_r, \Phi_t\rangle = c_2, \\ g_{22} = \|\Phi_s\|^2 = r^2\rho^2 + ra_1 + b_1, \quad g_{33} = \|\Phi_t\|^2 = r^2\rho^2 + ra_3 + b_3, \\ g_{23} = g_{32} = \langle \Phi_s, \Phi_t\rangle = ra_2 + b_2, \end{gathered} \tag{13}$$

with

$$\begin{gathered} a_1 = 2\langle \phi_s, \psi_s\rangle, \quad a_2 = \langle \phi_s, \psi_t\rangle + \langle \phi_t, \psi_s\rangle, \quad a_3 = 2\langle \phi_t, \psi_t\rangle, \\ b_1 = \|\psi_s\|^2, \quad b_2 = \langle \psi_s, \psi_t\rangle, \quad b_3 = \|\psi_t\|^2, \\ c_1 = \langle \phi, \psi_s\rangle, \quad c_2 = \langle \phi, \psi_t\rangle. \end{gathered} \tag{14}$$

We have

$$\begin{aligned} \det g = r^4\rho^4 &+ r^3\rho^2(a_1+a_3) + r^2(\rho^2\left(b_1+b_3-c_1^2-c_2^2\right) + a_1a_3 - a_2^2) \\ &+r(a_1b_3 - 2a_2b_2 + a_3b_1 - a_1c_2^2 + 2a_2c_1c_2 - a_3c_1^2) \\ &+b_1b_3 - b_2^2 - b_1c_2^2 + 2b_2c_1c_2 - b_3c_1^2. \end{aligned} \tag{15}$$

Next we calculate components of symmetric $(0,3)$ tensor field $T(X,Y,Z) = \langle \sigma(X,Y), iZ\rangle$ on ruled Lagrangian submanifold M^3 for $X,Y,Z \in TM$. $\Phi_{rr} = \partial^2\Phi/\partial r^2 = 0$ implies that

$$T_{111} = \langle \Phi_{rr}, i\Phi_r\rangle = 0, \quad T_{112} = \langle \Phi_{rr}, i\Phi_s\rangle = 0, \quad T_{113} = \langle \Phi_{rr}, i\Phi_t\rangle = 0. \tag{16}$$

(9) and (10) yield

$$T_{122} = \langle \Phi_{rs}, i\Phi_s\rangle = A_1, \quad T_{123} = \langle \Phi_{rs}, i\Phi_t\rangle = A_2, \quad T_{133} = \langle \Phi_{rt}, i\Phi_t\rangle = A_3, \tag{17}$$

where

$$A_1 = \langle \phi_s, i\psi_s\rangle, \quad A_2 = \langle \phi_s, i\psi_t\rangle = \langle \phi_t, i\psi_s\rangle, \quad A_3 = \langle \phi_t, i\psi_t\rangle. \tag{18}$$

Also we obtain

$$\begin{gathered} T_{222} = \langle \Phi_{ss}, i\Phi_s\rangle = r^2B_1 + rC_1 + D_1, \quad T_{223} = \langle \Phi_{ss}, i\Phi_t\rangle = r^2B_2 + rC_2 + D_2, \\ T_{233} = \langle \Phi_{st}, i\Phi_t\rangle = r^2B_3 + rC_3 + D_3, \quad T_{333} = \langle \Phi_{tt}, i\Phi_t\rangle = r^2B_4 + rC_4 + D_4, \end{gathered} \tag{19}$$

where

$$\begin{gathered} B_1 = \langle \phi_{ss}, i\phi_s\rangle, \quad B_2 = \langle \phi_{ss}, i\phi_t\rangle, \quad B_3 = \langle \phi_{st}, i\phi_t\rangle, \quad B_4 = \langle \phi_{tt}, i\phi_t\rangle, \\ C_1 = \langle \phi_{ss}, i\psi_s\rangle + \langle \psi_{ss}, i\phi_s\rangle, \quad C_2 = \langle \phi_{ss}, i\psi_t\rangle + \langle \psi_{ss}, i\phi_t\rangle, \\ C_3 = \langle \phi_{st}, i\psi_t\rangle + \langle \psi_{st}, i\phi_t\rangle, \quad C_4 = \langle \phi_{tt}, i\psi_t\rangle + \langle \psi_{tt}, i\phi_t\rangle, \\ D_1 = \langle \psi_{ss}, i\psi_s\rangle, \quad D_2 = \langle \psi_{ss}, i\psi_t\rangle, \quad D_3 = \langle \psi_{st}, i\psi_t\rangle, \quad D_4 = \langle \psi_{tt}, i\psi_t\rangle. \end{gathered} \tag{20}$$

We calculate Ricci tensor of the ruled Lagrangian submanifold M^3 in $\mathbb{C}^3$ by using the Gauss equation. We can express the components

$$R_{\alpha\beta} = \sum_{ijkl} g^{ij} g^{kl} (T_{\alpha\beta i} T_{jkl} - T_{\alpha ik} T_{\beta jl}) \tag{21}$$

$(\alpha, \beta, i, j, k, l = 1, 2, 3)$ of the Ricci tensor on M as the form $(\det g)^{-2} P(r)$, where $P(r)$ is a polynomial of r with which each coefficient is a function of (s, t) as:

$$\begin{aligned} R_{11} &= (\det g)^{-2} \left(-r^4\rho^4(A_1^2 + 2A_2^2 + A_3^2) + \text{lower term}\right), \\ R_{12} &= (\det g)^{-2} \left(r^6\rho^4(A_2(B_4 - B_2) + B_3(A_1 - A_3)) + \text{lower term}\right), \\ R_{13} &= (\det g)^{-2} \left(r^6\rho^4(A_2(B_1 - B_3) + B_2(A_3 - A_1)) + \text{lower term}\right), \\ R_{22} &= (\det g)^{-2} \left(r^8\rho^4(B_2(B_4 - B_2) + B_3(B_1 - B_3)) + \text{lower term}\right), \\ R_{23} &= (\det g)^{-2} \left(r^7\rho^2(B_2(B_4 - B_2) + B_3(B_1 - B_3)) + \text{lower term}\right), \\ R_{33} &= (\det g)^{-2} \left(r^8\rho^4(B_2(B_4 - B_2) + B_3(B_1 - B_3)) + \text{lower term}\right). \end{aligned} \tag{22}$$

Proposition 5.1. *Let (M, Σ, π) be an r-oriented ruled Lagrangian submanifold in $\mathbb{C}^3$. Let $\phi : \Sigma \to S^5$ and $\psi : \Sigma \to \mathbb{C}^3$ be the corresponding maps and suppose ϕ is an immersion. If M is Einstein (i.e., constant sectional curvature) with respect to the induced metric, then M is flat and ϕ, ψ satisfy the following: (i) $\phi : \Sigma \to S^5$ is a Legendrian immersion such that the metric on Σ induced by ϕ has constant Gaussian curvature 1, and (ii) $\psi_s, \psi_t \in \mathrm{span}_{\mathbb{R}}\{\phi, \phi_s, \phi_t\}$.*

Remark 5.2. Let $\phi : \Sigma \to S^5$ be a Legendrian immersion such that the metric on Σ induced by ϕ has constant Gaussian curvature 1. Then by taking a composition of ϕ and the Hopf fibration $S^5 \to \mathbb{CP}^2$, we have a Lagrangian surface Σ in $\mathbb{CP}^2$ with constant Gauss curvature. Such surfaces were classified in [2] and [3].

Also the scalar curvature R of M^3 is

$$R = \sum_{ij} g^{ij} R_{ij} = (\det g)^{-3} P_R(r), \tag{23}$$

where $P_R(r)$ is a polynomial of degree 8 with respect to r and each coefficient is a function of s and t. By comparing degrees of denominator and numerator of R, we obtain:

Proposition 5.3. *Let (M, Σ, π) be a r-oriented ruled Lagrangian submanifold in $\mathbb{C}^3$ Let $\phi : \Sigma \to S^5$ and $\psi : \Sigma \to \mathbb{C}^3$ be the corresponding maps and suppose ϕ is an immersion. If the scalar curvature R of M with respect to the induced metric is constant, then $R = 0$.*

6. Ruled Lagrangian submanifolds of $\mathbb{C}^3$ and horizontal curves in S^5

In this section, we consider ruled Lagrangian submanifolds in $\mathbb{C}^3$ such that the map $\phi : \Sigma \to S^5$, has *rank* 1. We use same notations as §5. Let Σ be a real 2-dimensional manifold and let $\phi : \Sigma \to S^5$ be a map such that the rank of its differential map $d\phi : T_p\Sigma \to T_{\phi(p)}S^5$ is equal to 1 at each point $p \in \Sigma$. Let $\psi : \Sigma \to$

$\mathbb{C}^3$ be a map satisfying $\langle \phi(p), \psi(p)\rangle = 0$ for any $p \in \Sigma$. The ruled submanifold M is the image of the map $\Phi : \mathbb{R} \times \Sigma \to \mathbb{C}^3$ given by $\Phi(r,p) = r\phi(p) + \psi(p)$ as (5). Suppose Φ is an immersion at (r,p) in $\mathbb{R} \times \Sigma$. Since the rank of $d\phi$ is equal to 1 at $p \in \Sigma$, we can take local coordinate (s,t) on Σ near p such that $\phi = \phi(t)$ with $|\phi'(t)| = 1$ and $\partial\phi/\partial s = 0$, by implicit function theorem.

We find conditions for which Φ is a Lagrangian immersion. First partial derivatives of Φ are given as

$$\Phi_r = \frac{\partial \Phi}{\partial r}(r,p) = \phi(p), \quad \Phi_s = \frac{\partial \Phi}{\partial s}(r,p) = \psi_s(p), \quad \text{and} \quad \Phi_t = \frac{\partial \Phi}{\partial t}(r,p) = r\phi'(p) + \psi_t(p). \tag{24}$$

Then by (8), Φ is a Lagrangian immersion if and only if

$$\langle \phi, i\phi'\rangle = 0, \tag{25}$$

$$\langle \phi, i\psi_s\rangle = \langle \phi, i\psi_t\rangle = \langle \phi', i\psi_s\rangle = 0, \tag{26}$$

$$\langle \psi_s, i\psi_t\rangle = 0. \tag{27}$$

(25) yields that the curve $t \mapsto \phi(t)$ in S^5 is *horizontal* with respect to the Hopf fibration $S^5 \to \mathbb{CP}^2$.

The induced metric g on $\mathbb{R} \times \Sigma$ is expressed with respect to the basis $\{\partial/\partial r, \partial/\partial s, \partial/\partial t\}$ of the tangent space $T_{(r,p)}(\mathbb{R} \times \Sigma) = T_r\mathbb{R} \times T_p\Sigma$ by (12). (6) implies that

$$\begin{gathered} g_{11} = \|\Phi_r\|^2 = 1, \quad g_{12} = g_{21} = \langle \Phi_r, \Phi_s\rangle = \langle \phi, \psi_s\rangle = 0, \\ g_{13} = g_{31} = \langle \Phi_r, \Phi_t\rangle = c_2, \quad g_{22} = \|\Phi_s\|^2 = b_1, \\ g_{23} = g_{32} = \langle \Phi_s, \Phi_t\rangle = ra_4 + b_2, \quad g_{33} = \|\Phi_t\|^2 = r^2 + ra_5 + b_3, \end{gathered} \tag{28}$$

with

$$\begin{gathered} a_4 = \langle \phi', \psi_s\rangle, \quad a_5 = \langle \phi', \psi_t\rangle, \quad b_1 = \|\psi_s\|^2, \\ b_2 = \langle \psi_s, \psi_t\rangle, \quad b_3 = \|\psi_t\|^2, \quad c_2 = \langle \phi, \psi_t\rangle. \end{gathered} \tag{29}$$

Note that $\langle \phi, \psi\rangle = 0$ implies $\langle \phi, \psi_s\rangle = 0$. We have

$$\det g = r^2(b_1 - a_4^2) + r(a_5 b_1 - 2a_4 b_2) + b_1(b_3 - c_2^2) - b_2^2. \tag{30}$$

Components of the tensor T are described as:

$$\begin{gathered} T_{111} = \langle \Phi_{rr}, i\Phi_r\rangle = 0, \quad T_{112} = \langle \Phi_{rr}, i\Phi_s\rangle = 0, \quad T_{113} = \langle \Phi_{rr}, i\Phi_t\rangle = 0, \\ T_{122} = \langle \Phi_{rs}, i\Phi_s\rangle = 0, \quad T_{123} = \langle \Phi_{rs}, i\Phi_t\rangle = 0, \quad T_{133} = \langle \Phi_{rt}, i\Phi_t\rangle = A_4, \\ T_{222} = \langle \Phi_{ss}, i\Phi_s\rangle = D_1, \quad T_{223} = \langle \Phi_{ss}, i\Phi_t\rangle = rC_5 + D_2, \\ T_{233} = \langle \Phi_{st}, i\Phi_t\rangle = rC_6 + D_3, \quad T_{333} = \langle \Phi_{tt}, i\Phi_t\rangle = r^2 B_5 + rC_7 + D_4, \end{gathered}$$

where

$$A_4 = \langle \phi', i\psi_t \rangle, \quad B_5 = \langle \phi'', i\phi' \rangle, \tag{31}$$
$$C_5 = \langle \psi_{ss}, i\phi' \rangle, \quad C_6 = \langle \psi_{st}, i\phi' \rangle, \quad C_7 = \langle \phi'', i\psi_t \rangle + \langle \psi_{tt}, i\phi' \rangle,$$
$$D_1 = \langle \psi_{ss}, i\psi_s \rangle, \quad D_2 = \langle \psi_{ss}, i\psi_t \rangle, \quad D_3 = \langle \psi_{st}, i\psi_t \rangle, \quad D_4 = \langle \psi_{tt}, i\psi_t \rangle.$$

We can express the components $R_{\alpha\beta}$ of the Ricci tensor on M as the form $(\det g)^{-2} P(r)$, where $P(r)$ is a polynomial of r with which each coefficient is a function of (s,t) as:

$$\begin{aligned} R_{11} &= -(\det g)^{-2} b_1^2 A_4^2, \\ R_{12} &= (\det g)^{-2} \left(r^2 a_4 A_4 (2b_1 C_5 - a_4 D_1) + \text{lower term} \right), \\ R_{13} &= (\det g)^{-2} \left(r^3 A_4 (C_5 (a_4^2 + b_1) - a_4 D_1) + \text{lower term} \right), \\ R_{22} &= (\det g)^{-2} \left(-r^4 b_1 C_5^2 + \text{lower term} \right), \\ R_{23} &= (\det g)^{-2} \left(-r^5 a_4 C_5^2 + \text{lower term} \right), \\ R_{33} &= (\det g)^{-2} \left(-r^6 C_5^2 + \text{lower term} \right). \end{aligned} \tag{32}$$

Also the scalar curvature R of M^3 is

$$R = \sum_{ij} g^{ij} R_{ij} = (\det g)^{-3} \left(r^4 a_4 A_4 C_5 (b_1 - a_4^2) + \text{lower term} \right) \tag{33}$$

If (M^3, g) is Einstein with $\mathrm{Ric} = \lambda g$ such that $\lambda \neq 0$, then

$$\det g \text{ is independent of } r. \tag{34}$$

7. An explicit construction of flat ruled Lagrangian submanifolds from some horizontal curves in S^5

So hereafter we consider this case (34). Then by (29) and (30), we have

$$0 = b_1 - a_4^2 = \|\psi_s\|^2 - \langle \phi', \psi_s \rangle^2. \tag{35}$$

Hence there are some functions $\nu(s,t)$ and $\mu(s,t)$ such that $\psi_s(s,t) = \nu(s,t)\phi'(t)$ and $\psi(s,t) = \mu(s,t)\phi(t)$ $(\partial\mu/\partial s = \nu)$. So the map Φ is written as

$$\Phi(r,s,t) = r\phi(t) + \mu(s,t)\phi'(t), \tag{36}$$

and the ruled submanifold M is given by a curve $\phi : I \to S^5$ (I is an interval) and a function $\mu(s,t)$. First partial derivatives of Φ are given as

$$\Phi_r = \phi, \quad \Phi_s = \mu_s \phi', \quad \Phi_t = (r + \mu_t)\phi' + \mu\phi''. \tag{37}$$

Then by (8), Φ is a Lagrangian immersion if and only if

$$\langle \phi, i\phi' \rangle = \langle \phi'', i\phi' \rangle = 0. \tag{38}$$

In particular, $\phi(t)$ is a horizontal curve in S^5.

The induced metric g on $\mathbb{R} \times \Sigma$ is

$$\begin{aligned} g_{11} = \|\Phi_r\|^2 = 1, \quad g_{12} = g_{21} = \langle \Phi_r, \Phi_s \rangle = \langle \phi, \psi_s \rangle = 0, \\ g_{13} = g_{31} = \langle \Phi_r, \Phi_t \rangle = -\mu, \quad g_{22} = \|\Phi_s\|^2 = \mu_s^2, \\ g_{23} = g_{32} = \langle \Phi_s, \Phi_t \rangle = (r + \mu_t)\mu_s, \quad g_{33} = \|\Phi_t\|^2 = (r + \mu_t)^2 + \mu^2 \|\phi''\|^2. \end{aligned} \tag{39}$$

and

$$\det g = \mu^2 \mu_s^2 \kappa_\phi^2, \tag{40}$$

where $\kappa_\phi = \sqrt{\|\phi''\|^2 - 1}$ is the curvature of $\phi(t)$ as a spherical curve in S^5. So in this case Φ is an immersion if and only if $\mu \neq 0$, $\mu_s \neq 0$ and $\kappa_\phi \neq 0$. By direct computation, we can see that components of the tensor T are all 0 except

$$T_{333} = \langle \Phi_{tt}, i\Phi_t \rangle = \mu^2 \langle \phi''', i\phi'' \rangle,$$

and we have $\mathrm{Ric} = 0$, i.e., M^3 is flat. Consequently we have:

Theorem 7.1. *Let I and J be open intervals in $\mathbb{R}$. Let $\phi : I \to S^5$ be a horizontal curve satisfying $\|\phi'\| = 1$, $\langle \phi'', i\phi' \rangle = 0$, $\kappa_\phi \neq 0$, $\langle \phi''', i\phi'' \rangle \neq 0$ and let $\mu : J \times I \to \mathbb{R}$ be a function satisfying $\mu \neq 0$ and $\mu_s \neq 0$. Then $\Phi(r, s, t) = r\phi(t) + \mu(s, t)\phi'(t)$ defines a flat ruled Lagrangian submanifold in $\mathbb{C}^3$, which is neither totally geodesic nor a Riemannian product.*

Proposition 7.2. *Let (M, Σ, π) be a r-oriented ruled Lagrangian submanifold in $\mathbb{C}^3$. Let $\phi : \Sigma \to S^5$ and $\psi : \Sigma \to \mathbb{C}^3$ be the corresponding maps and suppose the rank of ϕ is equal to 1. If M is Einstein (i.e., constant sectional curvature) with respect to the induced metric, then M is flat and ϕ, ψ satisfy the following: (i) $\phi(\Sigma)$ is a horizontal curve in S^5, and (ii) $\langle \psi_{ss}, i\phi' \rangle = 0$.*

Combining Propositions 5.1, 7.2 and §4, we get

Theorem 7.3. *Let (M, Σ, π) be a r-oriented ruled Lagrangian submanifold in $\mathbb{C}^3$. Let $\phi : \Sigma \to S^5$ and $\psi : \Sigma \to \mathbb{C}^3$ be the corresponding maps and suppose the rank of ϕ is constant. If M is Einstein (i.e., constant sectional curvature) with respect to the induced metric, then M is flat.*

Acknowledgment. J. T. Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0071643) and M. Kimura was partially supported by Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan, No. 21540083.

References

[1] B. Y. Chen, *Classification of Lagrangian surfaces of constant curvature in the complex Euclidean plane*, Proc. Edinb. Math. Soc. **48** (2005), 337–364.
[2] B. Y. Chen, *Classification of Lagrangian surfaces of constant curvature in the complex projective plane*, J. Geom. Phys. **53** (2005), 428–460.
[3] B. Y. Chen, *Three additional families of Lagrangian surfaces of constant curvature in the complex Euclidean plane*, J. Geom. Phys. **56** (2006), 666–669.

[4] G. Fischer and J. Piontkowski, *Ruled varieties: An introduction to algebraic differential geometry,* Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2001.
[5] Harvey and Lawson, *Calibrated Geometries*, Acta Math. **148** (1982), 47–157.
[6] D. Joyce, *Ruled special Lagrangian* 3*-folds in* $\mathbb{C}^3$, Proc. London Math. Soc. **85** (2002), 233–256.
[7] M. Kimura, *Minimal immersions of some circle bundles over holomorphic curves in complex quadric to sphere* , Osaka J. Math. **37** (2000), 883–903.
[8] M. Kimura, *Lagrangian submanifolds with codimension* 1 *totally geodesic foliation in complex projective space*, Kodai Math. J. **31** (2008), 38–45.
[9] P. Petersen and W. Wylie, *On Gradient Ricci solitons with symmetry*, Proc. Amer. Math. Soc. **137** (2009), 2085–2092.
[10] T. Sasaki, *Line congruence and transformation of projective surfaces*, Kyushu J. Math. **60** (2006), 101–243.

Jong Taek Cho: Department of Mathematics, Chonnam National University, Gwangju, 500-757, Korea
E-mail address: jtcho@chonnam.ac.kr

Makoto Kimura: Department of Mathematics, Faculty of Science and Engineering, Shimane University, Matsue, Shimane, 690-8504, Japan
E-mail address: mkimura@riko.shimane-u.ac.jp