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# PARAHYPERBOLIC NETWORKS

SUJITH SIVAN AND MADHU VENKATARAMAN

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Abstract. In the study of potential theory on an infinite network $X$ or an infinite tree $T$ with terminal vertices, if the Laplacian is not defined at the terminal vertices considering them as boundary points of $X$, then $X$ always has positive potentials and the constants are harmonic on $X$. Consequently, the harmonic classification theory of $X$ has to be studied differently from an earlier study which treats $T$ as a Brelot harmonic space.

## 1. Introduction

A Brelot harmonic space is a connected, locally compact, but not compact, space provided with a harmonic sheaf satisfying the axioms $1, 2, 3$ of Brelot [4]. The paper "Trees as Brelot Spaces" [3] intends to show that an infinite tree $T$ along with its vertices and edges and an associated Laplacian operator serves as a model for Brelot spaces. Specifically, by a tree, we mean here (Cartier [5]) an infinite (countable) graph which is connected (that is, there is a path connecting any two vertices), locally finite (that is, every vertex has only a finite number of adjacent vertices, called neighbours) and without cycles (that is, there is a unique path connecting any two vertices) or self loops; and provided with a nearest neighbour transition probability $p(x,y) \geq 0$ such that $p(x,y) > 0$ if and only if $x$ and $y$ are neighbours and $\sum_{y \in T} p(x,y) = 1$ for any vertex $x$ in $T$; we do not suppose $p(x,y) = p(y,x)$. A vertex $z \in T$ is terminal if and only if $z$ has only one neighbour in $T$. For any real-valued function $u$ defined on the vertices of $T$, the Laplacian $\Delta u(x)$ is defined in [3] for any non-terminal vertex $x \in T$,

$$\Delta u(x) = \sum_{y \in T} p(x,y)[u(y) - u(x)].$$

$u$ is said to be superharmonic (respectively harmonic) if $\Delta u \leq 0$ (respectively $\Delta u = 0$). It is then proved that when the harmonic functions are linearly extended on the

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edges, one arrives at a situation where the topologically connected space $T$ along with this extended harmonic structure can be considered as a Brelot Harmonic Space. Then taking the restrictions of the relevant functions on the vertices of $T$, [3] establishes potential-theoretically significant discrete case results in $T$.

This note exhibits certain restrictions that are needed in the paper [3] where the results are good only if the trees do not have any terminal vertices. The source of the inconsistencies is that when the Laplacian is calculated, the terminal vertices are ignored. For example, as in the Brelot axiomatic potential theory, it is asserted that if $u \geq 0$ is superharmonic on $T$ and $u = 0$ at a vertex, then $u \equiv 0$. Now consider the following example: Let $T = \{0, 1, 2...\}$. Let $p(n, n+1) = \frac{1}{2}$ and $p(n, n-1) = \frac{1}{2}$ if $n \geq 1$ and $p(0, 1) = 1$. Let $u$ be the function on $T$ such that $u(n) = n$ for all $n$. Then $u \geq 0$ and $\Delta u(n) = 0$ if $n \geq 1$. Hence according to the definition given in that paper, $u \geq 0$ is harmonic on $T$, since the Laplacian is taken only at the non-terminal vertices, that is $n \geq 1$. However this contradicts the Minimum Principle in a Brelot Space, since $u(0) = 0$.

Note that the constant function 1 is harmonic on any $T$; if $T$ has terminal vertices, take $u(x) = 0$ if $x$ is terminal and $u(x) = 1$ if $x$ is not terminal. Then $u$ is a non-negative superharmonic function on $T$, that is not harmonic. Hence, $T$ considered as a Brelot Space should have potentials on $T$. In other words, if we do not calculate the Laplacian at the terminal vertices of $T$, then every $T$ with some terminal vertices should be a hyperbolic tree (that is, a tree on which positive potentials exist). Of course, if there are no terminal vertices in $T$, then $T$ can be either hyperbolic or parabolic in which case 0 is the only non-negative potential on $T$.

Another definition [3, Definition 1.1] that disturbs the rhythm concerns the definitions of the interior and the boundary of a set $S$. The interior $\overset{\circ}{S}$ consists of all vertices $v \in S$ such that every vertex of $T$ which is a neighbour of $v$ belongs to $S$; the boundary $\partial S$ is defined as the set of all vertices $v \in S$ such that exactly one neighbour of $v$ is in $\overset{\circ}{S}$. Now suppose $E$ is a set which contains a terminal vertex $z$ and its only neighbour. Then $z \in \overset{\circ}{E}$. Also $z$ has exactly one neighbour which is in $E$ so that $z \in \partial E$. Thus, for this set $E$, $\overset{\circ}{E} \cap \partial E \neq \emptyset$. This causes concern when the Dirichlet problem is involved [3, p.722].

## 2. Terminal Vertices as boundary points

By a network $X$, we mean an infinite graph which is connected and locally finite. We do not place any restrictions on cycles or self loops in $X$. There is a collection of numbers $t(x, y) \geq 0$, called conductance, such that $t(x, y) > 0$ if and only of $x \sim y$ (the symbol $x \sim y$ denotes that $x$ and $y$ are neighbours in $X$). For any vertex $x \in X$, we write $t(x) = \sum_{y \in X} t(x, y)$. Since $X$ is locally finite, $t(x)$ is finite; since $X$ is connected, $t(x) > 0$. Note that we have not placed the restriction $t(x, y) = t(y, x)$ for any pair $x, y \in X$ as in Yamasaki [6]. Hence any tree $T$ with the nearest neighbour transition probability structure can be considered as an infinite network without self-loops or cycles. However if necessary, then we can define a symmetric conductance on $T$ as follows. Fix any $x \in T$, there is a unique path $\{e = x_0, x_1, \dots, x_n = x\}$ connecting $e$ and $x$. Define

$$\phi(x) = \frac{p(e, x_1)p(x_1, x_2)\dots p(x_{n-1}, x_n)}{p(x_n, x_{n-1})p(x_{n-1}, x_{n-2})\dots p(x_1, e)}$$

write $t(x, y) = \phi(x)p(x, y)$. If $x \sim y$, then $\phi(y) = \phi(x)\frac{p(x, y)}{p(y, x)}$ so that $t(x, y) = \phi(x)p(x, y) = \phi(y)p(y, x) = t(y, x)$. Consequently, $\{t(x, y)\}$ is a set of symmetric conductance on $T$.

In a network $X$, a vertex $z$ is known as a terminal vertex if $z$ has only one neighbour in $X$. That is, $z$ is the end of a path in $X$ and it is natural to imagine $z$ as a boundary point of $X$. As such, the Laplacian at $z$ is not calculated in a tree $T$ in [3]. However, this choice of terminal vertices as boundary points of $X$ necessitates a careful study of superharmonic functions on $X$, that is not analogous to the usual potential theory on the Euclidean spaces. Since the results in [3] are irreprochable when the infinite tree $T$ does not have any terminal vertices (as in the case of homogeneous trees), we assume that $T$ has terminal vertices. In fact, in this article we prove our results more generally on an infinte network $X$ with conductance $t(x, y)$ which may or may not be symmetric, but $X$ has at least one terminal vertex. Let $\overset{\circ}{X}$ consist of all non-terminal vertices of $X$ and $\partial X = X \setminus \overset{\circ}{X}$ the non-empty set consisting of all terminal vertices of $X$. Then $\overset{\circ}{X}$ is connected. For if $a$ and $b$ are in $\overset{\circ}{X}$, (since $X$ is connected ) there is a path $\{a, x_1, \dots, x_n, b\}$ in $X$ connecting $a$ and $b$. Note that none of the vertices $x_i$ is a terminal vertex. Hence the path is entirely in $\overset{\circ}{X}$. Hence $\overset{\circ}{X}$ is connected. Moreover $\overset{\circ}{X}$ is locally finite. Now $X$ is infinite and the neighbour of a terminal vertex should be a vertex in $\overset{\circ}{X}$; consequently, $\overset{\circ}{X}$ cannot be a finite set. In the special case of $X$ being a tree, remark that $\overset{\circ}{X}$ also is a tree.

**Definition 2.1.** *For a subset $E$ of $X$, the interior $\overset{\circ}{E}$ consists of all vertices $v \in E \cap \overset{\circ}{X}$ such that every vertex of $X$ which is a neighbour of $v$ is in $E$. The boundary of $E$ is $\partial E = E \setminus \overset{\circ}{E}$; thus every terminal vertex in $E$ is placed in $\partial E$.*

**Definition 2.2.** *Let $u$ be a real-valued function on a subset $E$ of $X$, $\overset{\circ}{E} \neq \emptyset$. Then $u$ is superharmonic (respectively harmonic, subharmonic) on $E$ if $\Delta u(x) = \sum_{y \in E} t(x, y)[u(y) - u(x)] \leq 0$ (respectively $\Delta u(x) = 0, \Delta u(x) \geq 0$), for every $x \in \overset{\circ}{E}$.*

**Theorem 2.3** (Minimum Principle)**.** *Let $s(x)$ be a superharmonic function on a finite set in $E$ in $X$. Then $s(x) \geq \min_{z \in \partial E} s(z)$ for all $x \in E$. In particular, if $h$ is harmonic on $E$ and if $h = 0$ on $\partial E$, then $h = 0$ on $E$.*

*Proof.* Let $\beta = \min_{x\in E} s(x)$. Then $\beta \leq \alpha = \min_{z\in\partial E} s(z)$. Suppose $\beta < \alpha$. Then there exists some $x_0 \in \overset{\circ}{E}$ such that $s(x_0) = \beta$. Take some $y \in \overset{\circ}{X} \setminus E$. Let $\{x_0, x_1, \ldots, x_n = y\}$ be a path connecting $x_0$ to $y$. Let $i$ be the largest index such that $x_0, x_1, \ldots, x_i$ are all in $\overset{\circ}{E}$. Then $x_{i+1} \notin \overset{\circ}{E}$. Since $x_{i+1} \sim x_i$, $x_{i+1} \in E$ and hence $x_{i+1} \in \partial E$ and $s(x_{i+1}) \leq \alpha < \beta$. Since $\Delta s \leq 0$ on $\overset{\circ}{E}$ and $\beta$ is the minimum value, the fact that $s(x_0) = \beta$ implies that $s(x) = \beta$ for all $x \sim x_0$. In particular $s(x_1) = \beta$. This leads to the result $s(x_1) = s(x_2) = \ldots = s(x_i) = s(x_{i+1}) = \beta$. Thus $\beta = s(x_{i+1}) \geq \alpha > \beta$, a contradiction, which shows that $\beta \geq \alpha$. Hence $\min_{x\in E} s(x) = \min_{z\in\partial E} s(z)$. □

An arbitrary subset $E$ in $X$ is said to be *circled* if any $z \in \partial E$ has at least one neighbour in $\overset{\circ}{E}$. In this sense, $X$ is circled and its interior $\overset{\circ}{X}$ is connected. Let $A$ be a finite set of non-terminal vertices. Let $E_1 = V(A)$ be the set consisting of $A$ and also all vertices $x$ in $X$ such that $x$ has a neighbour in $A$. Then $E_1$ is a finite set, $A \subset \overset{\circ}{E}_1$ and $E_1$ is circled. Define by recurrence $E_{i+1} = V(E_i)$ for $i \geq 1$. Since $X$ is connected, any $x$ should be in some $E_i$. Thus $\{E_i\}$ an increasing sequence of finite circled sets such that $A \subset \overset{\circ}{E}_i \subset \overset{\circ}{E}_{i+1}$ for $i \geq 1$ and $X = \cup E_i$ and $\overset{\circ}{X} = \cup \overset{\circ}{E}_i$. We shall refer to $\{E_i\}$ as an exhaustion of $X$ by finite circled sets.

**Theorem 2.4.** *Suppose $E$ is a circled set such that $\overset{\circ}{E}$ is connected. Let $s$ be a superharmonic function on $E$ attaining its minimum value $\alpha$ at a vertex in $\overset{\circ}{E}$. Then $s(x) = \alpha$ for $x \in E$.*

*Proof.* For some $x_0 \in \overset{\circ}{E}$, let $s(x_0) = \alpha \leq s(x)$ for all $x \in E$. Then since $\overset{\circ}{E}$ is connected, as in the case of Theorem 2.3, we prove that $s(x) = \alpha$ for all $x \in \overset{\circ}{E}$. Let $z \in \partial E$. Then $z$ has a neighbour $a$ in $\overset{\circ}{E}$. Since $s(a) = \alpha$,

$$\Delta s(a) = \sum_y t(a,y)[s(y) - s(a)] + t(a,z)[s(z) - s(a)] \leq 0$$

will lead to a contradiction if $s(z) > s(a)$. Hence $s(z) \leq \alpha$ which implies that $s(z) = \alpha$. Hence $s = \alpha$ on $\partial E$ also, so that $s(x) = \alpha$ for every $x$ in $E$. □

**Theorem 2.5** (Generalised Dirichlet Problem)**.** *Let $F$ be an arbitrary subset of $X$. Let $E \subset \overset{\circ}{F}$. Let $f$ be a real-valued function on $F \setminus E$ such that there exists a superharmonic function $u$ and a subharmonic function $v$ on $F$ such that $v \leq f \leq u$ on $F \setminus E$ and $v \leq u$ on $F$. Then there exists a function $h$ on $F$ such that $h = f$ on $F \setminus E$, $v \leq h \leq u$ on $F$ and $\Delta h = 0$ at every vertex in $E$. Moreover $h$ can be so chosen that if $h'$ is another function on $F$ such that $h' = f$ on $F \setminus E$ and $\Delta h' = 0$ on $E$, then $h' \leq h$ on $F$.*

*Proof.* Let $v_1$ be the function on $F$ such that $v_1 = f$ on $F \setminus E$ and $v_1 = v$ on $E$. Similarly let $u_1$ be the function on $F$ such that $u_1 = f$ on $F \setminus E$ and $u_1 = u$ on $E$.

Then $v_1 \leq u_1$ on $F$ by hypothesis; also $\Delta u_1(x) \leq 0$ and $\Delta v_1(x) \geq 0$ for $x \in E$. Let $\mathcal{F}$ be the family of functions $s$ on $F$ such that $s \leq u_1$ on $F$, $s = f$ on $F \setminus E$ and $\Delta s \geq 0$ on $E$. Then $v_1 \in \mathcal{F}$ and $\mathcal{F}$ is a Perron family of subharmonic functions on $E$ (as shown in [1]). If $h(x) = \sup_{s \in \mathcal{F}} s(x)$, then $h$ has the properties stated in the theorem. Moreover if $h^{'}$ is another such function, then $h^{'} \in \mathcal{F}$ so that $h^{'} \leq h$ on $F$. □

**Corollary 2.6.** *Let $F$ be an arbitrary set in $X$. Let $E \subset \overset{\circ}{F}$. Let $f$ be a real-valued function on $F \setminus E$ such that $|f| \leq u$ on $F \setminus E$ where $u$ is a non-negative superharmonic function on $F$. Then there exists a (largest) function $h$ on $F$ such that $h = f$ on $F \setminus E$, $|h| \leq u$ on $F$ and $\Delta h = 0$ at every vertex in $E$. Also,*

(i) *If $u \geq 0$ is a superharmonic function on an arbitrary subset $F$ of $X$, then $u$ has the greatest harmonic minorant $h \geq 0$ on $F$ such that $h = u$ on $\partial F$.*
(ii) *If $F$ is a finite subset of $X$ and if $f$ is a real-valued function on $\partial F$, then there exists a unique harmonic function $h$ on $F$ such that $h = f$ on $\partial F$. (Here the uniqueness of the solution $h$ is a consequence of the Minimum Principle Theorem 2.3).*

**Definition 2.7.** *A superharmonic function $p \geq 0$ on a set $E$ is said to be a potential on $E$ if and only if the greatest harmonic minorant of $p$ on $E$ is 0. Equivalently, $p \geq 0$ is a potential on $E$ if and only if*

i) *$p$ is superharmonic on $E$, and*
ii) *if $u$ is a subharmonic function on $E$ such that $u \leq p$, then $u \leq 0$.*

*Remark* 2.8.

a) If $s \geq 0$ is a superharmonic function on $E$, then $s$ is the unique sum of a potential $p$ and a non-negative harmonic function $h$ on $E$. Here $h$ is the greatest harmonic minorant of $s$ on $E$.
b) According to the above definition $p \equiv 0$ is also considered a potential on $X$. Hence, we shall use the term non-zero potential on $X$ to refer to a potential $p$ on $X$ that is not identically 0.

*Notation* 2.9. Let $f$ be a real-valued function on a subset $E$ of $X$. Denote by $\hat{f}$ the function on $E$ such that $\hat{f}(x) = f(x)$ if $x$ is a non-terminal vertex and $\hat{f}(x) = 0$ if $x$ is a terminal vertex.

**Theorem 2.10.** *There always exist non-zero potentials on $X$.*

*Proof.* The constant function 1 is a positive harmonic function on $X$. Then $v(x) = \hat{1}(x)$ is a non-negative superharmonic, but not harmonic, function on $X$. For, if $x \in \overset{\circ}{X}$, then $v(x) = 1$, which is the maximum value of $v$ and hence $\Delta v(x) \leq 0$. Hence $v$ is superharmonic on $X$.

To see that $v$ is not harmonic, take some $z \in \partial X$. Since $z$ is terminal, if $a \sim z$, then $a \in \overset{\circ}{X}$. Consequently,

$$\begin{aligned}
\Delta v(a) &= \sum_{y} t(a,y)[v(y)-v(a)] \\
&= \sum_{y\neq z} t(a,y)[v(y)-v(a)] + t(a,z)[v(z)-v(a)] \\
&= \sum_{y\neq z} t(a,y)[v(y)-v(a)] + t(a,z)[0-1] \\
&< 0.
\end{aligned}$$

That is, $v(x)$ is not harmonic at $x = a$. Now if $h$ is the greatest harmonic minorant of $v$ on $X$, then $p = v - h$ is a non-zero potential on $X$. □

**Proposition 2.11.** *If $p \geq 0$ is a potential on a subset $E$ of $X$, then $p = 0$ on $\partial E$.*

*Proof.* Let $h \geq 0$ be a harmonic function on $E$ such that $h = p$ on $\partial E$ and $h \leq p$ on $E$ (Corollary 2.6(i)). Since $p$ is a potential on $E$, its greatest harmonic minorant on $E$ is 0. In particular $p = h = 0$ on $\partial E$. □

**Theorem 2.12** (Green's function)**.** *Let $e$ be a non-terminal vertex in $X$. Then there exists a unique potential $G_e(x)$ on $X$ such that $\Delta G_e(x) = -\delta_e(x)$ for every $x \in \overset{\circ}{X}$.*

*Proof.* Let $p$ be a non-zero potential on $X$. Take $F = X$ in Theorem 2.5 so that $\overset{\circ}{F} = \overset{\circ}{X}$; let $E = \overset{\circ}{X} \setminus e$ and $f = p$ on $F \setminus E$. Then there exists a function $g$ on $X$ such that $g = p$ on $X \setminus E$, $0 \leq g \leq p$ on $X$ and $\Delta g = 0$ at every vertex in $E$. Note that

$$\begin{aligned}
\Delta g(e) &= \sum_{y} t(e,y)[g(y)-g(e)] \\
&= \sum_{y} t(e,y)[g(y)-p(e)] \\
&\leq \sum_{y} t(e,y)[p(y)-p(e)] \\
&= \Delta p(e) \\
&\leq 0.
\end{aligned}$$

But $\Delta g(e) \neq 0$. For, if $\Delta g(e) = 0$, then $g$ is a non-negative harmonic function on $F = X$ and is majorized by a potential, hence $g \equiv 0$ which is not the case since $g(e) = p(e) > 0$. Let $\Delta g(e) = -\alpha$ where $\alpha > 0$. Then $G_e(x) = \frac{g(x)}{\alpha}$ is a potential on $X$ with point harmonic support at $e$ and $\Delta G_e(x) = -\delta_e(x)$ for every $x \in \overset{\circ}{X}$.

To prove the uniqueness, suppose $v$ is another potential on $X$ such that $\Delta v(x) = -\delta_e(x)$ for every $x \in \overset{\circ}{X}$. Then $u(x) = G_e(x) - v(x)$ is such that $\Delta u(x) = 0$ for every $x$ in $\overset{\circ}{X}$, that is $u$ is harmonic on $X$. Since $u(x) \leq G_e(x)$ on $X$, then $u \leq 0$; since $-u \leq v$ on $X$, then $-u \leq 0$. Hence $u \equiv 0$. $\square$

*Remark* 2.13. Cartier [5, p.226] introduces the Green's kernal $G(x, y)$ on a tree $T$, probabilistically; $G(x, y)$ is the expected number of times the associated random walks starting at $x$ visits $y$. Then, assuming that $0 < G(x, y) < \infty$ for all vertices [5, p.222], for a real-valued function $v \geq 0$ on $T$, $Gv(x) = \sum_{y \in T} G(x, y)v(y)$ is called the potential of $v$. If $Gv(x)$ is finite for one (and hence any) $x$ in $T$, then $\Delta Gv(x) = -v(x)$ for every $x$ in $T$.

In [3, Definition 1.2], potentials on a tree $T$ are defined as in the above Definition 2.7 (as usual, the Laplacians being calculated at the non-terminal vertices only). Then, making a reference to [5], it is wrongly claimed [3, p.721] that every potential $p$ on $T$ is of the form $Gf$ for a unique non-negative function $f$ with support equal to the harmonic support of $p$ and this identification $p = Gf$ is often invoked later.

We say that a superharmonic function $s$ on $X$ is said to have the *harmonic support* in $A$ if $\Delta s(x) = 0$ for every $x \in \overset{\circ}{X} \setminus A$. If $A$ is a finite set and if $\Delta s(x) = 0$ for every $x \in \overset{\circ}{X} \setminus A$, then we say that $s$ has finite harmonic support.

**Theorem 2.14.** *Let $u$ be a superharmonic function defined outside a finite set in $X$. Then there exist on $X$ a superharmonic function $v$ and two potentials $p_1$ and $p_2$ with finite harmonic support such that $u = p_1 - p_2 + v$ outside a finite set. If $u$ is harmonic, then $v$ can be chosen to be harmonic on $X$.*

*Proof.* Let $u$ be defined outside a finite set $A$ such that $\Delta u(x) \leq 0$ at every non-terminal vertex $x$ outside $A$. Assume $u$ is defined on $X$ by giving values 0 at each vertex of $A$. Let $B$ be the set of all non-terminal vertices on $\partial A$. Write $p_1(x) = \sum_{b \in B} [\Delta u(b)]^- G_b(x)$ and $p_2(x) = \sum_{b \in B} [\Delta u(b)]^+ G_b(x)$. Then $p_1$ and $p_2$ are potentials on $X$ with finite harmonic support in $B$. Define

$$v(x) = u(x) - p_1(x) + p_2(x) \text{ on X.}$$

If $x \in \overset{\circ}{A}$, then $\Delta v(x) = 0$; if a non-terminal vertex $x \notin A$, then $\Delta v(x) = \Delta u(x) \leq 0$; and if $x = b \in B$, then

$$\Delta v(b) = \Delta u(b) + [\Delta u(b)]^- - [\Delta u(b)]^+ = 0.$$

Hence $v$ is superharmonic function on $X$ and $u(x) = p_1(x) - p_2(x) + v(x)$ outside A.

Suppose $u$ is harmonic outside a finite set in $X$. Then we can take $\Delta u(x) = 0$ at every non-terminal vertex $x \notin A$. Then $\Delta v(x) = 0$ at every $x \in \overset{\circ}{X}$. Hence $v$ is harmonic on $X$. $\square$

**Theorem 2.15** (Domination principle)**.** *Let $p$ be a potential on $X$ with harmonic support in $A$. Let $s \geq 0$ be a superharmonic function on $X$ such that $s \geq p$ on $A$. Then $s \geq p$ on $X$.*

*Proof.* Let $u = \inf(s, p)$. Then $u \geq 0$ is superharmonic on $X$ and $u = p$ on $A$. Let $v = p - u$. Then at every $a \in \overset{\circ}{X} \setminus A$, $\Delta v(a) = \Delta p(a) - \Delta u(a) \geq 0$ since $\Delta p(a) = 0$ and $\Delta u(a) \leq 0$; and at every $a \in A \cap \overset{\circ}{X}$,

$$\begin{aligned}\Delta v(a) &= \sum_y t(a,y)[v(y) - v(a)] \\ &\geq 0, \text{ since } v(y) \geq 0 \text{ and } v(a) = 0.\end{aligned}$$

Hence $\Delta v(a) \geq 0$ if $a \in \overset{\circ}{X}$, that is $v$ is subharmonic on $X$. Since $0 \leq v \leq p$ on $X$, $v \equiv 0$. That is, $s \geq p$ on $X$. □

## 3. Parahyperbolic Networks

In the proof of Theorem 2.10, it was showed that $\hat{1}$ is a non-negative superharmonic, but not harmonic, function on $X$. It may turn out in some cases that $\hat{1}$ is a potential on $X$.

*Example* 3.1. Let $X = \{a, b, x_1, x_2, \ldots\}$ be an arrow-shaped infinite tree with only two terminal vertices $a$ and $b$, each having $x_1$ as the neighbour. Let $p(x_1, a) = p(x_1, b) = \frac{1}{4}$ and $p(x, y) = \frac{1}{2}$ for any pair of neighbours $x$ and $y$ that are non-terminal vertices. Then $p(x) = \hat{1}(x)$ is a potential on $X$. For, let $h$, $0 \leq h \leq p$, be a harmonic function on $X$. Then $h(a) = 0 = h(b)$. Let $h(x_1) = \alpha$ and $h(x_2) = \beta$. Then

$$0 = \Delta h(x_1) = \frac{1}{2}(\beta - \alpha) + \frac{1}{4}(0 - \alpha) + \frac{1}{4}(0 - \alpha)$$

so that $\beta = 2\alpha$. Since $h(x_1) = \alpha$ and $h(x_2) = 2\alpha$, the fact that

$$0 = \Delta h(x_2) = \frac{1}{2}(\alpha - 2\alpha) + \frac{1}{2}[h(x_3) - 2\alpha]$$

implies that $h(x_3) = 3\alpha$. It is clear now that if $h(x)$ is harmonic on $X$, then $h(x_n) = n\alpha$. But $0 \leq h(x) \leq 1$ for all $x \in X$. Hence $\alpha = 0$, that is $h \equiv 0$ and $p(x) = \hat{1}(x)$ is a potential on $X$.

**Definition 3.2.** *A network $X$ is said to be parahyperbolic if and only if the function $\hat{1}$ is a potential on $X$.*

*Remark* 3.3. The term "parahyperbolic" is found in Anandam [2] in the context of the potential theory associated with a second order elliptic differential operator defined on a domain $\Omega$ in $\mathbb{R}^n, n \geq 2$. If $\hat{1}$ is a potential on $X$, then the superharmonic functions on $X$ have many properties similar to those of the superharmonic functions defined on a parabolic Riemann Surface, hence the term "parahyperbolic".

**Proposition 3.4.** *In a network $X$, the following statements are equivalent.*

i) *$X$ is parahyperbolic.*
ii) *If $s$ is a superharmonic function on $X$ such that $|s| \leq \hat{1}$ on $X$, then $s$ is a potential on $X$.*
iii) *If $h$ is a harmonic function on $X$ such that $|h| \leq \hat{1}$, then $h \equiv 0$.*

*Proof.* i) $\Rightarrow$ ii). Let $s$ be a superharmonic function on $X$ such that $|s| \leq \hat{1}$. Since $\hat{1}$ is a potential by hypothesis and $-\hat{1} \leq s$, we conclude that $-s \leq 0$ and hence $s \geq 0$. Moreover $s$ is majorized by the potential $\hat{1}$. Hence $s$ is a potential on $X$.

ii) $\Rightarrow$ iii). If $h$ is a harmonic function such that $|h| \leq \hat{1}$, then $h$ is a potential on $X$ by the assumption. This is possible only if $h \equiv 0$.

iii) $\Rightarrow$ i). We know that $\hat{1}$ is superharmonic on $X$. Let $h$ be the greatest harmonic minorant of $\hat{1}$. Then by assumption $h \equiv 0$, so that $\hat{1}$ is a potential on $X$, that is $X$ is parahyperbolic. □

**Theorem 3.5.** *$X$ is parahyperbolic if and only if the following Minimum Principle is satisfied: Let $F$ be any subset of $X$. Let $u$ be a lower bounded superharmonic function on $F$ such that $u \geq 0$ on $\partial F$. Then $u \geq 0$ on $F$.*

*Proof.* Let $X$ be parahyperbolic. Let $v$ be the function on $X$ such that $v = \inf(u, 0)$ on $F$ and $v = 0$ outside $F$. Note that $v$ is a lower bounded superharmonic function on $X$. Since all the terminal vertices of $F$ are in $\partial F$ and the other terminal vertices are outside $F$, $v = 0$ at each terminal vertex of $X$. Moreover, since $v$ is lower bounded by assumption, $v \geq -m$ on $X$ for some $m \geq 0$. Hence $-v \leq m\hat{1}$ on $X$. Since $X$ is parahyperbolic, $\hat{1}$ is a potential; consequently, $-v \leq 0$, that is $v \geq 0$. In particular $u \geq 0$ on $F$.

Conversely, if the Minimum Principle is satisfied, then $X$ should be parahyperbolic. For otherwise there is a bounded harmonic function $h$ on $X$ such that $0 < h < 1$ on $\overset{\circ}{X}$ and $h = 0$ on $\partial X$. We shall consider the two cases $F = X$ and $F \neq X$ separately.

i) Suppose $F = X$. Then $u = -h$ is a lower bounded harmonic function on $F$ such that $u = 0$ on $\partial F$. If the Maximum Principle is satisfied, then $u \geq 0$ on $F$, a contradiction.

ii) Suppose $F \neq X$. Take a vertex $e \in \overset{\circ}{X}$ and let $A = v(e)$ be the set consisting of $e$ and all its neighbours. Let $F = X \setminus \{e\}$. Then $\overset{\circ}{F} = \overset{\circ}{X} \setminus A$ and $\partial F = (A \setminus \{e\}) \cup \partial X$. Let $R_h^A$ stand for the infimum of all non-negative superharmonic functions on $X$ that majorize $h$ on $A$; note that $R_h^A = h$ on $A \cup \partial X \supset \partial F$ and $\Delta R_h^A(x) = 0$ for every $x \in \overset{\circ}{X} \setminus A$. Since potentials exist on $X$, and since $A$ is a finite set, then we can find a potential $p$ on $X$ that majorizes $h$ on $A$. Hence the infimum $R_h^A$ which is a non-negative superharmonic function is majorized by $p$ so that $R_h^A$ is a non-zero potential on $X$. Hence $u = h - R_h^A$ is a bounded harmonic function on $F$ and $u = 0$ on $\partial F$. Since the Minimum Principle is assumed to be valid, then $u$ should be identically 0 on $F = X \setminus \{e\}$; and since $u = 0$ on $A \cup \partial X \supset \{e\}$, $u \equiv 0$ on $X$. But this is not possible since $h$ is harmonic on $X$ and $R_h^A$ is a potential on $X$. This shows that $X$ should be parahyperbolic. □

**Harmonic measure of the point at infinity**: Let $\{E_n\}$ be an exhaustion of $X$ by an increasing sequence of finite sets. Let $h_n$ be the Dirichlet solution in $E_n \setminus \overset{\circ}{E}_1$, with boundary values $\hat{1}$ on $\partial E_n$ and 0 on $\partial E_1$. Extend $h_n$ by 1 outside $E_n$. Denote this extension by $s_n$. Then $s_n$ is superharmonic on $X \setminus E_1$ and $s_n = 0$ on $\partial E_1$. Note that $s_n$ is a decreasing sequence of non-negative superharmonic functions. Hence $s = \lim s_n$ is superharmonic on $X \setminus E_1$, and $s = 0$ on $\partial E_1$. If $x$ is a non-terminal vertex in $X \setminus E_1$, then for some integer $m$, $s_n$ is harmonic at the vertex $x$ for $n \geq m$. Hence $s$ is harmonic at $x$. Since $x$ is an arbitrary vertex in $\overset{\circ}{X} \setminus E_1$, $s$ is harmonic, non-negative and $s = 0$ at each terminal vertex in $X \setminus \overset{\circ}{E}_1$. This function $s$ is called the harmonic measure of the point at infinity of $X$. If $s \equiv 0$, the harmonic measure of the point at infinity is said to be 0. The property that the harmonic measure of the point at infinity is 0 is independent of the choice of the exhaustion of $X$ will be clear from the following result.

**Theorem 3.6.** *$X$ is parahyperbolic if and only if the harmonic measure of the point at infinity is* 0.

*Proof.* Let $s$ be the harmonic measure of the point at infinity. Extend $s$ by 0 on $E_1$. Then this extended function, also denoted by $s$, is a non-negative subharmonic function on $X$ and $0 \leq s \leq \hat{1}$ on $X$. If $X$ is parahyperbolic, then $\hat{1}$ is a potential and hence $s \equiv 0$.

Conversely, suppose the harmonic measure of the point at infinity is 0. Then $X$ should be parahyperbolic. For, assume the contrary. Then there exists a harmonic function $h$ on $X$ such that $0 < h < 1$ on $\overset{\circ}{X}$ and $h = 0$ on $\partial X$. Let $v = h - R_h^{E_1}$ on $X$. Since $h$ is harmonic and $R_h^{E_1}$ is a potential, $v$ is not identically 0. Now $v \leq s_n$ on $E_n \setminus E_1$. Consequently, $v \leq s$ on $X \setminus E_1$ which implies that the harmonic measure of the point at infinity is not 0, a contradiction. Hence $X$ is parahyperbolic. $\square$

We say that a function $f$ on $X$ tends to $\infty$ at the point at infinity, if for any $\alpha > 0$, there exists a finite set $E$ such that $f(x) > \alpha$ for any vertex $x \in \overset{\circ}{X} \setminus E$.

*Example* 3.7. Let $X$ consist of non-terminal vertices $x_1, x_2, \ldots$ and terminal vertices $x_{i1}$ and $x_{i2}$ for $i \geq 1$. $x_{i1}$ and $x_{i2}$ have $x_i$ as neighbour for $i \geq 1$; $x_j$ has $x_{j+1}$ and $x_{j-1}$ as neighbours for $j \geq 2$; $p(x_1, x_2) = \frac{1}{2}$ and each $x_j$ has four neighbours for $j \geq 2$ with the same transition probability $\frac{1}{4}$; $p(x_1, x_{11}) = p(x_1, x_{12}) = \frac{1}{4}$. Let $h \geq 0$ be the function defined on $X$ such that $h(x_{11}) = h(x_{12}) = 0$, $h(x_1) = 1$ and $h(x_j) = h(x_{j1}) = h(x_{j2}) = j$ for $j \geq 2$. Then $h(x)$ is a harmonic function on $X$ tending to infinity.

**Theorem 3.8.** *If there exists a superharmonic function $u \geq 0$ outside a finite set tending to $\infty$ at the point at infinity, then $X$ is parahyperbolic.*

*Proof.* Choose an exhaustion of $X$ by an increasing sequence of finite sets $E_n$, such that $\Delta \hat{u}(x) \leq 0$ if $x \in \overset{\circ}{X} \setminus \overset{\circ}{E}_1$ and $\hat{u}(x) > n$ if $x$ is any non-terminal vertex outside $E_{n-1}$. Let $z$ be any non-terminal vertex outside $E_1$. Then for some $n$, $z \in \overset{\circ}{E}_n$. Now $\frac{\hat{u}(x)}{n} \geq \hat{1}(x)$ if $x \in \partial E_n$. Since $\frac{\hat{u}(x)}{n} \geq 0$ on $\partial E_1$, by the Minimum Principle, $\frac{\hat{u}(x)}{n} \geq s_n(x)$ on $E_n \setminus \overset{\circ}{E}_1$, since $\partial(E_n \setminus \overset{\circ}{E}_1) \subset \partial E_n \cup \partial E_1$. Recall that $s_n$ is the Dirichlet solution in $E_n \setminus \overset{\circ}{E}_1$ with boundary values $\hat{1}$ on $\partial E_n$ and 0 on $\partial E_1$(as in the definition of the harmonic measure of the point at infinity).

In particular, $\frac{\hat{u}(z)}{n} \geq s_n(z)$. Since the harmonic measure of the point at infinity $s(x) = \lim s_n(x)$, and since $\hat{u}(z)$ is finite, we conclude that $s(z) \leq 0$. Since $z$ is an arbitrary non-terminal vertex outside $E_1$, and since $s = 0$ at each terminal vertex in $X \setminus \overset{\circ}{E}_1$, we can conclude that the harmonic measure of the point at infinity is 0. Hence by Theorem 3.6, $X$ is parahyperbolic. □

## 4. $\overset{\circ}{X}$-superharmonic functions

The properties of potentials obtained above in a network with terminal vertices can be examined from a different perspective. We have mentioned earlier that $\overset{\circ}{X}$, the interior of $X$, is itself a connected network, with the conductance system $\{t(x,y)\}$ in $X$ restricted to $\overset{\circ}{X}$. Recall $t(x) = \sum_{y \in X} t(x,y)$ for every $x \in X$.

**Definition 4.1.** *A real-valued function $f$ on $\overset{\circ}{X}$ is said to be $\overset{\circ}{X}$- superharmonic at a vertex $a \in \overset{\circ}{X}$ if and only if*

$$t(a)f(a) \geq \sum_{y \in \overset{\circ}{X}} t(a,y)f(y)$$

*If $f$ is $\overset{\circ}{X}$-superharmonic at each vertex in $\overset{\circ}{X}$, then we say that $f$ is $\overset{\circ}{X}$-superharmonic. The definitions of subharmonic and harmonic at a vertex are given accordingly.*

*Note* 4.2. In conformity with the above nomenclature, we shall say that $f$ is $X$-superharmonic if

$$t(a)f(a) \geq \sum_{y \in X} t(a,y)f(y), \text{ for every } a \in \overset{\circ}{X}.$$

*Notation* 4.3. Earlier, if $f$ is defined on $X$, then we have denoted by $\hat{f}$ the function on $X$, equal to $f$ on $\overset{\circ}{X}$ and to 0 on $\partial X$. Now, if $g$ is defined on $\overset{\circ}{X}$, then we denote by $\check{g}$ the function on $X$, equal to $g$ on $\overset{\circ}{X}$ and to 0 on $\partial X$.

**Proposition 4.4.** *Let $f$ be a real-valued function on $\overset{\circ}{X}$. Then $f$ is $\overset{\circ}{X}$-superharmonic (respectively $\overset{\circ}{X}$-subharmonic) if and only if $\check{f}$ is $X$-superharmonic (respectively $X$-subharmonic).*

*Proof.* Let $f$ be $\overset{\circ}{X}$-superharmonic. Then for every $a \in \overset{\circ}{X}$,

$$\begin{aligned} t(a)\check{f}(a) = t(a)f(a) &\geq \sum_{y \in \overset{\circ}{X}} t(x,y)f(y) \\ &= \sum_{y \in X} t(a,y)\check{f}(y), \text{ since } \check{f}(y) = 0 \text{ if } y \in \partial X. \end{aligned}$$

Hence $\check{f}$ is $X$-superharmonic. Conversely, let $\check{f}$ be $X$-superharmonic. Then for every $a \in \overset{\circ}{X}$,

$$\begin{aligned} t(a)f(a) = t(a)\check{f}(a) &\geq \sum_{y\in X} t(a,y)\check{f}(y) \\ &= \sum_{y\in \overset{\circ}{X}} t(a,y)\check{f}(y) \\ &= \sum_{y\in \overset{\circ}{X}} t(a,y)f(y) \end{aligned}$$

Hence $f$ is $\overset{\circ}{X}$-superharmonic.

Similar property for $X$-subharmonic function also holds good by changing $\geq$ into $\leq$ in the above proof. □

**Corollary 4.5.** *The constant function* 1 *is a* $\overset{\circ}{X}$*-superharmonic function, that is not* $\overset{\circ}{X}$*-harmonic.*

*Proof.* $\check{1} = \hat{1}$ is known to be $X$-superharmonic (the proof of Theorem 2.10). Hence 1 is $\overset{\circ}{X}$-superharmonic.

Now, let $a \in \overset{\circ}{X}$ be such that there is a vertex $z \in \partial X$ as a neighbour of $a$. Then,

$$\begin{aligned} t(a) &= \sum_{y\in X} t(a,y) \\ &= \sum_{y\in X, y\neq z} t(a,y) + t(a,z) \\ &> \sum_{y\in X, y\neq z} t(a,y) \\ &\geq \sum_{y\in \overset{\circ}{X}} t(a,y) \end{aligned}$$

Hence the constant function 1 is not $\overset{\circ}{X}$-harmonic at the vertex $a$. □

**Lemma 4.6.** *If* $s$ *is a* $X$*-potential, then* $s$ *is a* $\overset{\circ}{X}$*-potential. If* $s$ *is a* $\overset{\circ}{X}$*-potential, then* $\check{s}$ *is a* $X$*-potential.*

*Proof.* Let $s$ be a $X$-potential. Then $s = \check{s}$ (Proposition 2.11) and by Proposition 4.4, $s$ is a $\overset{\circ}{X}$-superharmonic function. To show that $s$ is a $\overset{\circ}{X}$-potential, take a $\overset{\circ}{X}$-subharmonic function $u$ such that $0 \leq u \leq s$ on $\overset{\circ}{X}$. Then, by Proposition 4.4, $\check{u}(x)$ is $X$-subharmonic and $0 \leq \check{u} \leq \check{s} = s$. Hence $\check{u} \equiv 0$, that is $u \equiv 0$ on $\overset{\circ}{X}$ and hence $s$ is a $\overset{\circ}{X}$-potential.

On the other hand, if $s$ is a $\overset{\circ}{X}$-potential, then we show that $\check{s}$ is a $X$-potential. For that, let $h$ be a $X$-subharmonic function on $X$, such that $0 \leq h \leq \check{s}$. Then, for any $x \in \overset{\circ}{X}$,

$$\begin{aligned} t(x)h(x) &\leq \sum_{y \in X} t(x,y)h(y) \\ &= \sum_{y \in \overset{\circ}{X}} t(x,y)h(y), \text{ since } h(y) = 0 \text{ if } y \in \partial X. \end{aligned}$$

Hence on $\overset{\circ}{X}$, $h$ is $\overset{\circ}{X}$-subharmonic majorized by the $\overset{\circ}{X}$-potential $s$. Hence $h \equiv 0$. This implies that $\check{s}$ is a $X$-potential. □

**Theorem 4.7.** *The constant function* 1 *is a* $\overset{\circ}{X}$*-potential if and only if* $X$ *is parahyperbolic.*

*Proof.* Let $X$ be parahyperbolic. That is $\check{1} = \hat{1}$ is a $X$-potential. Hence by Lemma 4.6, 1 is a $\overset{\circ}{X}$-potential. Conversely, let 1 be a $\overset{\circ}{X}$-potential. Then $\check{1} = \hat{1}$ is a $X$-potential (Lemma 4.6), that is $X$ is parahyperbolic. □

For a real-valued function $f$ on $X$, let us write for $x \in \overset{\circ}{X}$,

$$\begin{aligned} (-\Delta)f(x) &= t(x)f(x) - \sum_{y \in X} t(x,y)f(y), \\ (-\Delta^{'})f(x) &= t(x)f(x) - \sum_{y \in \overset{\circ}{X}} t(x,y)f(y) \end{aligned}$$

*Note* 4.8. i) For any real-valued function $g$ on $\overset{\circ}{X}$, $(-\Delta^{'})g(x)$ is defined for every vertex $x \in \overset{\circ}{X}$. The function $g$ is $\overset{\circ}{X}$-superharmonic (respectively $\overset{\circ}{X}$-harmonic or $\overset{\circ}{X}$ − *subharmonic*) if and only if $(-\Delta^{'})g(x) \geq 0$ (respectively $(-\Delta^{'})g(x) = 0$ or $(-\Delta^{'})g(x) \leq 0$) for every $x \in \overset{\circ}{X}$.
ii) If $f \geq 0$ is a real-valued function on $X$, then for any $x \in \overset{\circ}{X}$,

$$\begin{aligned} (-\Delta)f(x) &= t(x)f(x) - \sum_{y \in X} t(x,y)f(y) \\ &\leq t(x)f(x) - \sum_{y \in \overset{\circ}{X}} t(x,y)f(y) \\ &= (-\Delta^{'})f(x) \end{aligned}$$

In fact, the problems related to potentials on a network $X$ with a Laplacian not defined at terminal vertices can be transformed into problems related to potentials on the network $\overset{\circ}{X}$ with a Laplacian defined at each vertex of $\overset{\circ}{X}$. Recall that an infinite network $X$, in Yamasaki [6], is any connected infinite graph, with countable vertices and countable edges, and without self-loops and provided with a conductance $t(x,y) \geq 0$ associated with any pair of vertices $x$ and $y$ in $X$ such that $t(x,y) = t(y,x)$ for every pair and $t(x,y) > 0$ if and only if $x$ and $y$ are neighbours. As shown in [1], most of the results in a network $X$ can be proved potential-theoretically without the assumption that $t(x,y)$ is symmetric, except possibly the Green's formulas. Thus, we can prove the following. (Note that $\overset{\circ}{X}$ as a network may have its own terminal vertices which are not accorded a special status when the Laplacian $\Delta^{'}$ is applied in $\overset{\circ}{X}$.)

(1) If $s_1$ and $s_2$ are $\overset{\circ}{X}$-superharmonic and if $\alpha_1$, $\alpha_2$ are two non-negative numbers, then $\alpha_1 s_1 + \alpha_2 s_2$ and $\inf(s_1, s_2)$ are $\overset{\circ}{X}$-superharmonic.
(2) If $s \geq u$ on $\overset{\circ}{X}$ where $s$ is $\overset{\circ}{X}$-superharmonic and $u$ is $\overset{\circ}{X}$-subharmonic, then there exists the greatest harmonic function $h$ on $\overset{\circ}{X}$ such that $s \geq h \geq u$ on $\overset{\circ}{X}$.
(3) For any $e \in \overset{\circ}{X}$, there exists a unique potential $p_e(x)$ on $\overset{\circ}{X}$ such that $(-\Delta^{'})p_e(x) = \delta_e(x)$ for every $x \in \overset{\circ}{X}$.
(4) The Minimum Principle and the solution to the Dirichlet problem on a finite set in $\overset{\circ}{X}$ are available for $\overset{\circ}{X}$-superharmonic functions.
(5) If $s_n$ is a sequence of $\overset{\circ}{X}$-superharmonic functions (respectively $\overset{\circ}{X}$-harmonic) and if $s(x) = \lim s_n(x)$ is finite at each $x \in \overset{\circ}{X}$, then $s$ is $\overset{\circ}{X}$-superharmonic (respectively $\overset{\circ}{X}$-harmonic).

**Proposition 4.9.** *For a vertex $e \in \overset{\circ}{X}$, if $G_e(x)$ is the Green's function on $X$ with point harmonic support at $e$, then $G_e(x)$ is the $\overset{\circ}{X}$-Green's function with point harmonic support at $e$. Conversely, if $p_e(x)$ is the $\overset{\circ}{X}$-Green's function, then $\check{p}_(x)$ is the $X$-Green's function with point harmonic support at $e$.*

*Proof.* For any $x \in \overset{\circ}{X}$,

$$
\begin{aligned}
\delta_e(x) = (-\Delta)G_e(x) &= t(x)G_e(x) - \sum_{y \in X} t(e,y)G_e(y) \\
&= t(x)G_e(x) - \sum_{y \in \overset{\circ}{X}} t(e,y)G_e(y), \text{ since } G_e(y) = 0 \text{ if } y \in \partial X \\
&= (-\Delta^{'})G_e(x).
\end{aligned}
$$

Conversely, if $(-\Delta)' p_e(x) = \delta_e(x)$ on $\overset{\circ}{X}$, then by Lemma 4.6, $\check{p}_e(x)$ is a $X$-potential. For $x \in \overset{\circ}{X}$,

$$\begin{aligned}(-\Delta)\check{p}_e(x) &= t(x)\check{p}_e(x) - \sum_{y\in X} t(x,y)\check{p}_e(y)\\ &= t(x)p_e(x) - \sum_{y\in \overset{\circ}{X}} t(x,y)\check{p}_e(y), \text{ since } \check{p}_e(y) = 0 \text{ if } y \in \partial X\\ &= t(x)p_e(x) - \sum_{y\in \overset{\circ}{X}} t(x,y)p_e(y), \text{ since } \check{p}_e(y) = p_e(y) \text{ if } y \in \overset{\circ}{X}\\ &= (-\Delta)p_e(x)\\ &= \delta_e(x).\end{aligned}$$

Hence the proposition follows. □

**Theorem 4.10.** *$X$ is a parahyperbolic network if and only if any $X$-superharmonic function $u$ defined outside a finite set in $X$ such that $|u| \leq \hat{1}$, is of the form $u = p_1 - p_2$ outside a finite set where $p_1$ and $p_2$ are bounded $X$-potentials on $X$.*

*Proof.* Let $X$ be parahyperbolic. Without loss of generality, assume that $u$ is defined on the whole $X$ by giving the value 0 at the undefined vertices. Then $|u| \leq \hat{1}$ and $u$ is superharmonic outside a finite set $A$. Then, by Theorem 2.14, there exist two $X$-potentials $q_1$ and $q_2$ with finite harmonic support and a $X$-superharmonic function $v$ such that $u = q_1 - q_2 + v$ outside a finite set. Since $q_1$ and $q_2$ are $X$-potentials with finite harmonic support, $|q_i| \leq M_i\hat{1}$, $i = 1, 2$. Consequently, since $|v| \leq |u| + q_1 + q_2$ outside a finite set in $X$, $|v| \leq M\hat{1}$ on $X$ for some constant $M > 0$. Then by Proposition 3.2, $v$ is a potential on $X$. Write $p_1 = q_1 + v$ and $p_2 = q_2$. Then $u = p_1 - p_2$ outside a finite set in $X$, where $p_1$ and $p_2$ are bounded $X$-potentials on $X$.

Conversely, suppose the representation is valid for $u$ defined outside a finite set in $X$ such that $|u| \leq \hat{1}$. Then $X$ should be parahyperbolic. For otherwise there exists a non-zero harmonic function $h$ on $X$ such that $0 \leq h \leq \hat{1}$ on $X$. Now by hypothesis $h = p_1 - p_2$ outside a finite set in $X$. Since $h \leq p_1$ outside a finite set in $X$, then $h \leq 0$ on $X$. Similarly, we show that $-h \leq 0$ and consequently $h \equiv 0$, a contradiction. This shows that $X$ should be parahyperbolic. □

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Sujith Sivan: Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai- 600 005, India.
*E-mail address*: suji_svn@yahoo.com

Madhu Venkataraman: Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai- 600 005, India.
*E-mail address*: madhu.riasm@gmail.com