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GEOMETRY OF GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. A geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ of a complex projecive space $\mathbb{C}P^n(c)$ is one of the most interesting objects in differential geometry. This expository paper consists of two parts. In the first half, we study curve theory on $G(r)$ (see [4, 9]). In the latter half, we investigate $G(r)$ from the viewpoint of submanifold theory ([2, 11]).

1. INTRODUCTION

A geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in an n-dimensional complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ of constant holomorphic sectional curvature $c(> 0)$ is impotant in intrinsic geometry as well as extrinsic geometry (i.e., submanifold theory).

In intrinsic geometry, for example it is known that $G(r)$ is a naturally reductive Riemannian homogeneous manifold, so that every geodesic of $G(r)$ is a homogeneous curve, namely it is an orbit of a one-parameter subgroup of the isometry group $\mathrm{I}(G(r))$ of $G(r)$ (cf. [12]). Moreover, when $\tan^2(\sqrt{c}\,r/2) > 2$, $G(r)$ is a Berger sphere (see [15]). Inspired by these facts, we are interested in geodesics on $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. We first investigate the length spectrum of $G(r)$ in detail (see [4]). We next study non-geodesic homogeneous curves on $G(r)$. We construct a family of closed non-geodesic homogeneous curves on $G(r)$ with the same length by using an isometric embedding

$$f \circ \iota_{G(r)} : G(r) \xrightarrow{\iota_{G(r)}} \mathbb{C}P^n(c) \xrightarrow{f} S^{n(n+2)-1}\left(\frac{n+1}{2n}c\right), \tag{1.1}$$

where $\iota_{G(r)}$ is a natural inclusion mapping of $G(r)$ into $\mathbb{C}P^n(c)$ and f is so-called the first standard minimal embedding of $\mathbb{C}P^n(c)$ into an $(n(n+2)-1)$-dimensional

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sphere $S^{n(n+2)-1}((n+1)c/(2n))$ of constant sectional curvature $(n+1)c/(2n)$ (see [9]).

In extrinsic geometry, again by using the above minimal embedding f we immerse each real hypersurface M^{2n-1} of $\mathbb{C}P^n(c)$ into the ambient sphere $S^{n(n+2)-1}((n+1)c/(2n))$ as follows:

$$f \circ \iota_M : M \xrightarrow{\iota_M} \mathbb{C}P^n(c) \xrightarrow{f} S^{n(n+2)-1}\left(\frac{n+1}{2n}c\right), \tag{1.2}$$

where ι_M is an isometric immersion of M^{2n-1} into $\mathbb{C}P^n(c)$. Note that the isometric immersion $f \circ \iota_M$ does not have parallel second fundamental form for each real hypersurface M of $\mathbb{C}P^n(c)$. On the other hand, by direct computation we can see that $f \circ \iota_M$ has parallel mean curvature vector in this sphere if and only if M is locally congruent to the geodesic sphere $G(r)$ with $\tan^2(\sqrt{c}\, r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ (cf. [11]). Needless to say, this geodesic sphere is a Berger sphere. Furthermore, it has an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. In particular, when $c = 8n+4$, this geodesic sphere is a Sasakian space form of constant ϕ-sectional curvature $8n+5$ (see [2]). These facts imply that for each of $c(>0)$ and $n(\geqq 2)$, every N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c}$ with $(n+1)c/(2n) \geqq \tilde{c}$ and $N > n(n+2)-1$ admits a $(2n-1)$-dimensional Riemannian submanifold M^{2n-1} satisfying the following properties.

(1) M is diffeomorphic but not isometric to a Euclidean sphere.
(2) M is a homogeneous submanifold of the ambient sphere $S^N(\tilde{c})$, i.e., M is an orbit of some subgroup of the isometry group $\mathrm{SO}(N+1)$ of $S^N(\tilde{c})$. However, M is not a Riemannian symmetric space.
(3) The mean curvature vector of M in $S^N(\tilde{c})$ is nonzero-parallel with respect the normal connection of M.
(4) M has an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. Especially, in the case of $c = 8n+4$, M is a Sasakian space form of constant ϕ-sectional curvature $8n+5$.

In the latter half of this paper, we clarify these fundamental properties of a certain geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$.

2. Length spectrum of geodesic spheres $G(r)$ in $\mathbb{C}P^n(c)$

Let M^{2n-1} $(n \geqq 2)$ be a real hypersurface with a unit nomal local vector field $\mathcal{N}$ in $\mathbb{C}P^n(c)$. We denote by $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ the almost contact metric structure of M induced from the Kähler structure J of the ambient space $\mathbb{C}P^n(c)$. That is, this structure is defined by

$$\xi = -J\mathcal{N},\ \eta(X) = \langle X, \xi\rangle \text{ and } \phi X = JX - \eta(JX)\xi \text{ for all vectors } X \text{ on } M,$$

so that it satisfies

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1 \text{ and } \eta(\phi X) = 0 \text{ for arbitrary } X \text{ on } TM.$$

We here recall the following fundamental equations for M, which are so-called Gauss formula and Weingarten formula, respectively.

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX, \tag{2.1}$$

where $\widetilde{\nabla}$ and ∇ are the Riemannian connections of $\mathbb{C}P^n(c)$ and M, respectively, and A is the shape operator of M in $\mathbb{C}P^n(c)$. Then it follows from the fact that $\widetilde{\nabla} J = 0$ and (2.1) that

$$\nabla_X \xi = \phi AX \tag{2.2}$$

and

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle \xi, \tag{2.3}$$

where X and Y are any vectors on M.

In the following, we consider a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. So we use the commutative condition $\phi A = A\phi$ without explanation. We recall the invariance ρ_γ for a geodesic $\gamma = \gamma(s)$ on $G(r)$, which is defined by $\rho_\gamma = \langle \dot\gamma(s), \xi_{\gamma(s)}\rangle$ for $-\infty < s < \infty$. Equation (2.2) guarantees the constancy of ρ_γ with $-1 \leqq \rho_\gamma \leqq 1$.

$$\begin{aligned}\nabla_{\dot\gamma}\rho_\gamma &= \nabla_{\dot\gamma}\langle \dot\gamma, \xi\rangle = \langle \dot\gamma, \nabla_{\dot\gamma}\xi\rangle \\ &= \langle \dot\gamma, \phi A\dot\gamma\rangle = \langle \dot\gamma, A\phi\dot\gamma\rangle = \langle A\dot\gamma, \phi\dot\gamma\rangle \\ &= -\langle \phi A\dot\gamma, \dot\gamma\rangle = 0.\end{aligned}$$

This invariance ρ_γ is said to be the *structure torsion* of a geodesic γ on $G(r)$. We shall state the congruence theorem on geodesics of $G(r)$ in terms of their structure torsions. For this purpose we review fundamental notions on congruency for curves in Riemannian manifolds. Two curves γ_1, γ_2 on a Riemannian manifold N are said to be congruent to each other in the usual sense if there exist an isometry of φ of N and a constant s_0 satisfying $\gamma_2(s) = (\varphi \circ \gamma_1)(s + s_0)$ for all s. In the case we can take $s_0 = 0$, they are said to be *strongly congruent* to each other. That is, we call two curves γ_1, γ_2 on N strongly congruent to each other if there is an isometry φ of N with $\gamma_2(s) = (\varphi \circ \gamma_1)(s)$ for all s. Trivially, a Riemannian manifold N is either a Euclidean space or a Riemannian symmetric space of rank one if and only if for every pair of geodesics γ_1, γ_2 on N they are strongly congruent to each other. In this paper, we treat a curve on a Riemannian manifold N is a mapping of the real line $\mathbb{R}$ into N.

Lemma 1 ([4])**.** *On a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ of $\mathbb{C}P^n(c)$, two geodesics γ_1, γ_2 are strongly congruent to each other if and only if their structure torsions ρ_{γ_1}, ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.*

We are now in a position to study lengths of closed geodesics of $G(r)$. It suffices to consider the case of $c = 4$. Let $\Pi : S^{2n+1}(1) \to CP^n(4)$ denote the Hopf fibration of a unit sphere. For a smooth curve γ on $\mathbb{C}P^n(4)$ a smooth curve $\tilde\gamma$ on $S^{2n+1}(1)$ is called a horizontal lift of γ if $\dot{\tilde\gamma}(s)$ is a horizontal vector and $d\Pi(\dot{\tilde\gamma}(s)) = \dot\gamma(s)$ for all s. We note that a curve γ on $G(r)$ is closed if and only if its horizontal lift $\tilde\gamma$ on

$S^{2n+1}(1)$ satisfies $\tilde{\gamma}(s) = e^{i\theta}\tilde{\gamma}(s+s_0)$ with some cnstants $\theta \in [0, 2\pi)$ and $s_0(>0)$ for each $s \in (-\infty, \infty)$. The following elementary lemma is a key in our argument.

Lemma 2 ([4]). *Let σ be a smooth simple curve on $\mathbb{C}P^n(4)$. Suppose that a horizontal lift $\tilde{\sigma}$ of σ on $S^{2n+1}(1)$ is represented as*

$$\tilde{\sigma}(s) = Ae^{\sqrt{-1}\,as} + Be^{\sqrt{-1}\,bs} + Ce^{\sqrt{-1}\,cs} + De^{\sqrt{-1}\,ds},$$

which is a curve in $\mathbb{C}^{n+1}$ with nonzero vectors $A, B, C, D \in \mathbb{C}^{n+1}$ and mutually distinct real numbers a, b, c, d satisfying $a+b+c+d=0$ and $a \neq 0$. Then σ is closed if and only if all the ratios $b/a, c/a, d/a$ are rational. In this case, its length is

$$\text{length}(\sigma) = 2\pi \times \text{L.C.M.}\left\{\frac{1}{|b-a|}, \frac{1}{|c-a|}, \frac{1}{|d-a|}\right\}.$$

Here, for positive numbers $\alpha_1, \alpha_2, \alpha_3$, we denote by L.C.M.$\{\alpha_1, \alpha_2, \alpha_3\}$ *the minimum value of the set $\{j\alpha_1 | j = 1, 2, \dots\} \cap \{j\alpha_2 | j = 1, 2, \dots\} \cap \{j\alpha_3 | j = 1, 2, \dots\}$.*

We remember that every geodesic of $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is a homogeneous curve (see [12]), so that it is a simple curve. Then by Lemma 2 we obtain the following sufficient condition for a geodesic $\gamma = \gamma(s)$ on $G(r)$ to be closed, which can be written by its structure torsion ρ_γ:

Theorem 1 ([4]). *For a geodesic γ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ we have the following properties according to their structure torsions*:

(1) *When $\rho_\gamma = \pm 1$, it is closed and its length is $\pi \sin(\sqrt{c}\,r)$;*
(2) *When $\rho_\gamma = 0$, it is also closed and its length is $2\pi \sin(\sqrt{c}\,r/2)$;*
(3) *When $0 < |\rho_\gamma| < 1$, it is closed if and only if its structure torsion ρ_γ is given by*

$$\rho_\gamma = \frac{\pm q}{\sin(\sqrt{c}\,r/2)\sqrt{p^2\tan^2(\sqrt{c}\,r/2) + q^2}}$$

with some relatively prime positive integers p and q with $q < p\tan^2(\sqrt{c}\,r/2)$. In this case, its length is

$$\text{length}(r) = 2\delta(p,q)\pi\sqrt{\left(p^2\sin^2(\sqrt{c}\,r/2) + q^2\cos^2(\sqrt{c}\,r/2)\right)/c}\,.$$

Here, $\delta(p,q)$ takes the value 2 when pq is even and takes the value 1 when pq is odd.

In consideration of Lemma 1 and Theorem 1 we find the following:

Theorem 2 ([4]). *On a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, there exist countably infinite congruence classes of closed geodesics. The length spectrum* LSpec$(G(r))$ *of $G(r)$ is a discrete unbounded subset in the real line $\mathbb{R}$.*

We here investigate the first length spectrum λ_1, the second length spectrum λ_2 and the third length spectrum λ_3 of $G(r)$ in detail.

Proposition 1 ([4]). *A geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ has the following properties on the lengths of closed geodesics.*

(1) λ_1, λ_2 *and* λ_3 *are simple, that is, each of their multiplicities is one.*
(2) $\lambda_1 = (2\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r)$*, which is the length of the geodesics of structure torsion* ± 1.
(3) *When* $0 < r \leqq \pi/(2\sqrt{c}\,)$*, we have* $\lambda_2 = (4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)$*, which is the length of the geodesics with null structure torsion.*
When $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$*, we have* $\lambda_2 = 2\pi/\sqrt{c}$*, which is the length of the geodesics with structure torsion* $\pm\cot(\sqrt{c}\,r/2)$.
(4) *When* $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$ *we have* $\lambda_3 = (4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)$*, which is the length of the geodesics with null structure torsion.*
When $0 < r \leqq \pi/(2\sqrt{c}\,)$ *and it satisfies* $\sqrt{2k-1} \leqq \cot r < \sqrt{2k+1}$ $(k = 1, 2, \ldots)$*, we have* $\lambda_3 = 2\pi\sqrt{\{4k(k+1)\sin^2(\sqrt{c}\,r/2)+1\}/c}$*, which is the length of the geodesics with structure torsion*
$\pm 1/\Big(\sin(\sqrt{c}\,r/2)\sqrt{(2k+1)^2\tan^2(\sqrt{c}\,r/2)+1}\,\Big)$.

We remark that the sectional curvature K of $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ lies in the closed interval $[(c/4)\cot^2(\sqrt{c}\,r/2), c + (c/4)\cot^2(\sqrt{c}\,r/2)]$. Hence, as mentioned in Introduction, in the case of $\tan^2(\sqrt{c}\,r/2) > 2$ we find that it is an example of a so-called Berger sphere. But for all lengths except the bottom λ_1 of $\mathrm{LSpec}(G(r))$, we find that the following inequality of Klingerberg's type holds.

Corollary 1. *Except geodesics with structure torsion* ± 1*, every geodesic* γ *of a geodesic sphere* $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$ *satisfies*
$\mathrm{length}(r) > 4\pi/\sqrt{c\,(4+\cot^2(\sqrt{c}\,r/2))}$.

Each element of $\mathrm{LSpec}(G(r))$ is not necessarily simple. For example, for $G(\pi/4)$ in $\mathbb{C}P^n(4)$ we have

$$\mathrm{LSpec}(G(\pi/4)) = \Big\{\pi, \sqrt{2}\,\pi, \sqrt{5}\,\pi, \sqrt{10}\,\pi, \sqrt{13}\,\pi, \sqrt{17}\,\pi, 5\pi, \sqrt{26}\,\pi, \sqrt{29}\,\pi, \sqrt{34}\,\pi, \sqrt{37}\,\pi, \sqrt{41}\,\pi, \sqrt{50}\,\pi, \sqrt{53}\,\pi, \sqrt{58}\,\pi, \sqrt{61}\,\pi, \sqrt{65}\,\pi, \sqrt{73}\,\pi, \ldots\Big\}.$$

Though each element from $\lambda_1 = \pi$ to $\lambda_{16} = \sqrt{61}\,\pi$ is simple, we find that the multiplicity of $\lambda_{17} = \sqrt{65}\,\pi$ is two. It is the common length of the geodesics with structure torsions $3/\sqrt{65}$ and $7/\sqrt{65}$.

Theorem 3 ([4])**.** *For a geodesic sphere* $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$ *we obtain the following*:

(1) *If* $\tan^2(\sqrt{c}\,r/2)$ *is irrational, every element of* $\mathrm{LSpec}(G(r))$ *is simple.*
(2) *If* $\tan^2(\sqrt{c}\,r/2)$ *is rational, the multiplicity of each element of* $\mathrm{LSpec}(G(r))$ *is finite, but not uniformly bounded and satisfies* $\limsup_{\lambda\to\infty} m(\lambda) = \infty$*. Its growth is less than polynomial growth. It satisfies* $\lim_{\lambda\to\infty} \lambda^{-\delta} m(\lambda) = 0$ *for arbitrary positive* δ.
(3) *We denote by* $n(\lambda)$ *the number of congruency classes of closed geodesics whose length is not longer than* λ*. Its growth is polynomial order of the second degree and satisfies* $\lim_{\lambda\to\infty} \lambda^{-2} n(\lambda) = 3cr/\left(4\pi^4\sin(\sqrt{c}\,r)\right)$.

This theorem guarantees that on a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ with irrational $\tan^2(\sqrt{c}\,r/2)$ in $\mathbb{C}P^n(c)$ two closed geodesics are congruent if and only if they have the same length. On the other hand, if $\tan^2(\sqrt{c}\,r/2)$ is rational, we cannot classify congruency classes of closed geodesics only by their lengths.

3. Non-geodesic homogeneous curves of $G(r)$ in $\mathbb{C}P^n(c)$

We are interested in finding a nice family of curves including all geodesics of $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$. To do this, we recall the notion of Sasakian curves.

On a real hypersurface N in a Kähler manifold $(\widetilde{M}, J)$ a smooth curve γ is said to be a Sasakian curve if it satisfies

$$\nabla_{\dot\gamma}\dot\gamma = k\phi\dot\gamma \tag{3.1}$$

with some constant k, where ϕ is the structure tensor of N induced by J. Needless to say, for an arbitrary constant k and a unit vector v at each point $x \in N$, there exists the unique Sasakian curve γ satisfying (3.1) with initial condition that $\gamma(0) = x$ and $\dot\gamma(0) = v$. Sasakian curves on a manifold admitting an almost contact metric structure can be considered as correspondences of Kähler circles on Kähler manifolds.

We here recall two invariances for a Sasakian curve $\gamma = \gamma(s)$ on $G(r)$ in $\mathbb{C}P^n(c)$. One is the structure torsion $\rho_\gamma = \langle \dot\gamma(s), \xi_{\gamma(s)}\rangle$. The other is the normal curvatue $\kappa_\gamma = \langle A\dot\gamma(s), \dot\gamma(s)\rangle$. By the same computation as above and (3.1) we find the constancy of ρ_γ:

$$\begin{aligned}\nabla_{\dot\gamma}\rho_\gamma &= \langle \nabla_{\dot\gamma}\dot\gamma, \xi\rangle + \langle \dot\gamma, \nabla_{\dot\gamma}\xi\rangle = k\langle \phi\dot\gamma, \xi\rangle + \langle \dot\gamma, \phi A\dot\gamma\rangle \\ &= \langle \dot\gamma, \phi A\dot\gamma\rangle = \langle \dot\gamma, A\phi\dot\gamma\rangle = -\langle \phi A\dot\gamma, \dot\gamma\rangle = 0.\end{aligned}$$

Furthermore, by the fact $\langle (\nabla_X A)X, X\rangle = 0$ for all vectors X on $G(r)$, we see the constancy of κ_γ. We remark that for a Sasakian curve γ satisfying (3.1) on $G(r)$ the first curvature $\kappa_1 = \|\nabla_{\dot\gamma}\dot\gamma\|$ of γ is given by $\kappa_1 = |k|\sqrt{1-\rho_\gamma^2}$, so that it is constant along γ. Hence, in the following we say the constant k to be the *coefficient* of a Sasakian curve γ satisfying (3.1). As a matter of course we treat geodesics as Sasakian curves in a trivial sense.

For about Sasakian curves on $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, the following is obtained in [3].

Lemma 3. *Let γ_i $(i = 1, 2)$ be Sasakian curves of coefficients κ_i and structure torsions ρ_{γ_i} on a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. They are strongly congruent to each other if and only if one of the following conditions holds*:

i) $|\rho_{\gamma_1}| = |\rho_{\gamma_2}| = 1$;
ii) $\rho_{\gamma_1} = \rho_{\gamma_2} = 0$ *and* $|\kappa_1| = |\kappa_2|$;
iii) $0 < |\rho_{\gamma_1}| = |\rho_{\gamma_2}| < 1$ *and* $\kappa_1\rho_{\gamma_1} = \kappa_2\rho_{\gamma_2}$.

As immediate consequences of this lemma we obtain the following corollary on the homogeneity of Sasakian curves on $G(r)$.

Corollary 2. *Every Sasakian curve on $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is homogeneous. That is, it is an orbit of a one-parameter subgroup of the isometry group $\mathrm{I}(G(r))$ of $G(r)$.*

In order to get geometric properties of Sasakian curves on a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, we study them through the isometric embedding $f \circ \iota_{G(r)}$ given by (1.1). As mentioned in Introduction, this embedding $f \circ \iota_{G(r)}$ does not have parallel second fundamental form but is equivariant. So we can hence treat our geodesic sphere $G(r)$ as a homogeneous submanifold in this sphere through this embedding $f \circ \iota_{G(r)}$.

Proposition 1 tells us that every integral curve γ of the characteristic vector field ξ on $G(r)$ is the shortest closed geodesic for each radius $r \in (0, \pi/\sqrt{c}\,)$. Furthermore, its shape $\iota_{G(r)} \circ \gamma$ through the inclusion $\iota_{G(r)} : G(r) \to \mathbb{C}P^n(c)$ is a Kähler circle of curvature $\sqrt{c}\,\cot(\sqrt{c}\,r)$ in $\mathbb{C}P^n(c)$, namely this integral curve γ satisfies either $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \sqrt{c}\,\cot(\sqrt{c}\,r)J\dot\gamma$ or $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\sqrt{c}\,\cot(\sqrt{c}\,r)J\dot\gamma$. If we see this curve through the embedding $f \circ \iota_{G(r)}$, we find the curve $f \circ \iota_{G(r)} \circ \gamma$ is a small circle on a sphere $S^{n(n+2)-1}((n+1)c/(2n))$ by the following lemma.

Lemma 4 ([6])**.** *A smooth curve μ on $\mathbb{C}P^n(c)$ is a Kähler circle of curvature κ if and only if the curve $f \circ \mu$ on $S^{n(n+2)-1}((n+1)c/(2n))$ is a circle of positive curvature $\sqrt{\kappa^2 + ((n-1)c/(2n))}$.*

In this context, we naturally come to the position to pose the following problem:

Problem 1. Find and classify smooth curves γ on $G(r)$ whose shape $f \circ \iota_{G(r)} \circ \gamma$ through the equivariant isometric embedding $f \circ \iota_{G(r)}$ are circles in $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$.

By virtue of Lemma 4 this problem is equivalent to the problem to find and to classify curves on $G(r)$ which are mapped to Kähler circles in $\mathbb{C}P^n(c)$ through the inclusion $\iota_{G(r)}$. For a smooth curve γ on $G(r)$ we get by the Gauss formula that $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N}$ and $J\dot\gamma = \phi\dot\gamma + \rho_\gamma\mathcal{N}$. We can hence obtain the following.

Lemma 5 ([10])**.** *A smooth curve γ on $G(r)$ can be seen as a Kähler circle of curvature κ on $\mathbb{C}P^n(c)$ through the inclusion ι if and only if it satisfies both of the equations $\nabla_{\dot\gamma}\dot\gamma = \pm\kappa\phi\dot\gamma$ and $\langle A\dot\gamma, \dot\gamma\rangle = \pm\kappa\rho_\gamma$, where double signs take the same signatures.*

By use of this lemma we can get the following answer to our problem. The answer depends on the radius of a geodesic sphere.

Theorem 4. *Let $G(r)$ be a geodesic sphere of radius $0<r \leqq \pi/(2\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$.*

(1) *For $0 \leqq k < c\big\{\cot^2(\sqrt{c}\,r)+(n-1)/(2n)\big\}$, there are no curves on $G(r)$ whose shape through $f \circ \iota_{G(r)}$ is a circle of curvature k on $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$.*

(2) *When $k^2 = c\big\{\cot^2(\sqrt{c}\,r) + (n-1)/(2n)\big\}$, the shape of a curve γ on $G(r)$ through $f \circ \iota_{G(r)}$ is a circle of curvature k if and only if it is a geodesic with structure torsion $\rho_\gamma = \pm 1$, which is an integral curve of ξ on $G(r)$.*

(3) *When* $k^2 > c\{\cot^2(\sqrt{c}\,r) + (n-1)/(2n)\}$, *the shape of a curve* γ *on* $G(r)$ *through* $f \circ \iota_{G(r)}$ *is a circle of curvature* k *if and only if it is a Sasakian curve of coefficient* $\pm\sqrt{k^2 - (n-1)c/(2n)}$ *whose structure torsion is*

$$\rho_\gamma = \pm c^{-1/2}\{\sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)}\}\cot(\sqrt{c}\,r/2),$$

where double signs take the same signatures.

Trivially these curves in (2), (3) *are closed with length* $2\pi/\sqrt{k^2 + (n+1)c/(2n)}$.

Theorem 5. *Let* $G(r)$ *be a geodesic sphere of radius* r *with* $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$ *in* $\mathbb{C}P^n(c)$.

(1) *For* $0 \leqq k < \sqrt{(n-1)c/(2n)}$, *there are no curves on* $G(r)$ *whose shape thorough* $f \circ \iota_{G(r)}$ *is a circle of curvature* k *on* $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$.

(2) *When* $k = \sqrt{(n-1)c/(2n)}$, *the shape of a curve on* $G(r)$ *through* $f \circ \iota_{G(r)}$ *is a circle of curvature* k *if and only if it is a geodesic with structure torsion* $\rho_\gamma = \pm\cot(\sqrt{c}\,r/2)$.

(3) *When* $\sqrt{(n-1)c/(2n)} < k < c\{\cot^2(\sqrt{c}\,r) + (n-1)/(2n)\}$, *the shape of a curve* γ *on* $G(r)$ *through* $f \circ \iota_{G(r)}$ *is a circle of curvature* k *if and only if it is a Sasakian curve of coefficient* $\pm\sqrt{k^2 - (n-1)c/(2n)}$ *whose structure torsion is*

$$\rho_\gamma = \pm c^{-1/2}\left\{\sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)}\right\}\cot(\sqrt{c}\,r/2)$$

or

$$\rho_\gamma = \pm c^{-1/2}\left\{-\sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)}\right\}\cot(\sqrt{c}\,r/2),$$

where double signs take the same signatures.

(4) *When* $k = c\{\cot^2(\sqrt{c}\,r) + (n-1)/(2n)\}$, *the shape of a curve on* $G(r)$ *through* $f \circ \iota_{G(r)}$ *is a circle of curvature* k *if and only if it is a geodesic with structure torsion* $\rho_\gamma = \pm 1$.

(5) *When* $k > c\{\cot^2(\sqrt{c}\,r) + (n-1)/(2n)\}$, *the shape of a curve* γ *on* $G(r)$ *through* $f \circ \iota_{G(r)}$ *is a circle of curvature* k *if and only if it is a Sasakian curve of coefficient* $\pm\sqrt{k^2 - (n-1)c/(2n)}$ *whose structure torsion is*

$$\rho_\gamma = \pm c^{-1/2}\left\{\sqrt{k^2 + (n+1)c/(2n)} - \sqrt{k^2 - (n+1)c/(2n)}\right\}\cot(\sqrt{c}\,r/2),$$

where double signs take the same signatures.

Trivially these curves in (2), (3), (4), (5) *are closed with length* $2\pi/\sqrt{k^2+(n+1)c/(2n)}$.

Remark 1. (1) For each curve γ in Theorems 4 and 5, the curve $\iota_{G(r)} \circ \gamma$ is a homogeneous curve on totally geodesic $\mathbb{C}P^1(c)(= S^2(c))$ of $\mathbb{C}P^n(c)$ (see [3]). This fact shows that each curve in Theorems 4 and 5 is an orbit of a one-parameter subgroup of SO(3).

(2) For each Sasakian curve γ on $G(r)$ the curve $\iota_{G(r)} \circ \gamma$ is a homogeneous curve on totally geodesic $\mathbb{C}P^2(c)$ of $\mathbb{C}P^n(c)$ (see [1]). Hence every Sasakian curve on $G(r)$ is an orbit of a one-parameter subgroup of SU(3).

(3) Curves in Theorem 4(3), Theorem 5(3) and Theorem 5(5) are non-geodesic Sasakian curves.
(4) There exist two non-geodesic Sasakian curves in Theorem 5(3) which are not congruent to each other with respect to $\mathrm{I}(G(r))$, but they are mapped to a circle of the same curvature k on $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$. Hence these curves, considered as curves on this sphere, are congruent to each other with respect to the isometry group $\mathrm{SO}(n(n+2))$ of the sphere.

4. Properties of certain geodesic spheres $G(r)$ in $\mathbb{C}P^n(c)$ in submanifold theory

We first investigate the minimal embedding $f : \mathbb{C}P^n(c) \to S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ which is defined by eigenfunctions of the first eigenvalue of the Laplacian Δ on $\mathbb{C}P^n(c)$. The inner product of the first normal space of f is given by

$$\begin{aligned}(4.1)\qquad \langle \sigma_1(X,Y), \sigma_1(Z,W)\rangle = -(c/(2n))\langle X,Y\rangle\langle Z,W\rangle + (c/4)\big(\langle X,W\rangle\langle Y,Z\rangle \\ + \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle\big).\end{aligned}$$

Here, σ_1 denotes the second fundamental form of f. Equation (4.1) shows the following properties of f:

i) f_1 is minimal;
ii) $\sigma_1(JX,JY) = \sigma_1(X,Y)$ for $\forall X, Y \in T\mathbb{C}P^n(c)$ (namely, σ is J-invariant);
iii) $\|\sigma_1(X,X)\| = \sqrt{(n-1)c/(2n)}$ for each unit vector X on $\mathbb{C}P^n(c)$ (that is, f is $\sqrt{(n-1)c/(2n)}$-isotropic (cf. [14])).

We remark that σ_1 is J-invariant is equivalent to saying that the second fundamental form σ_1 of our embedding f is parallel. As mentioned in Introduction, the embedding f is usually called the first standard minimal embedding.

In this section, we immerse all real hypersurfaces M of $\mathbb{C}P^n(c)$ into the sphere $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ (see (1.2)). Note that for every real hypersurface M, the second fundamental form of the isometric immersion $f \circ \iota_M : M \to S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ is not parallel. However, in this class $\{(M, f\circ\iota_M) | \iota_M : M \to \mathbb{C}P^n(c)$ is an isometric immersion$\}$ of all submanifolds in the sphere $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$, there exist nonzero-constant mean curvature submanifolds. For example, direct calculation tells us that the mean curvature H_r $(0 < r < \pi/\sqrt{c}\,)$ defined by the length of the mean curvature vector of the embedding $f\circ\iota_{G(r)} : G(r) \to S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ given by (1.1) is expressed as

$$H_r^2 = \frac{c}{4(2n-1)^2}\left\{(2n-1)^2\cot^2\left(\frac{\sqrt{c}}{2}r\right) + \tan^2\left(\frac{\sqrt{c}}{2}r\right) + \frac{-4n^2+4n-2}{n}\right\} \neq 0.$$

In this context, it is natural to pose the following problem:

Problem 2. Classify submanifolds $(M, f\circ\iota_M)$ given by (1.2) having parallel mean curvature vector with respect to the normal connection in the sphere $S^{n(n+2)-1}\big((n+1)c/(2n)\big)$.

The following proposition plays as a key in this section.

Proposition 2. *Let M^{2n-1} be a real hypersurface of $\mathbb{C}P^n(c)$ through an isometric immersion ι_M and $f : \mathbb{C}P^n(c) \to S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ the first standard minimal embedding. Then M is locally congruent to the geodesic sphere $G(r)$ with $\tan^2(\sqrt{c}\, r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ if and only if the immersion $f \circ \iota_M : M \to S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ has parallel mean curvature vector with respect to the normal connection. Moreover, this submanifold $(M, f \circ \iota_M)$ is homogeneous in this ambient sphere.*

Remark 2. The geodesic sphere $G(r)$ in Proposition 2 is a Berger sphere, since $\tan^2(\sqrt{c}\, r/2) = 2n+1 > 2$.

We next study the almost contact structure of our geodesic sphere in Propsition 2. For this purpose we review fundamental notions in contact geometry. Let M^{2m+1} be an almost contact metric manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. That is, this structure satisfies the following identities:

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y)$$

for all vectors X, Y on M. M is called a *Sasakian manifold* if the structure tensor ϕ of M satisfies the following differential equation:

$$(\nabla_X \phi)Y = \langle X, Y\rangle\xi - \eta(Y)X \quad \text{for } \forall X, Y \in TM, \tag{4.2}$$

where ∇ denotes the Riemannian connection of the Riemannian metric $\langle\ ,\ \rangle$ of M. A Sasakian manifold is called a *Sasakian space form* of constant ϕ-sectional curvature c if the sectional curvature $K(u, \phi u) := \langle R(u, \phi u)\phi u, u\rangle = c$ holds for every unit vector u orthogonal to ξ, where R is its curvature tensor. For construction of Sasakian space forms, see pp. 99-100 in [5].

In the following, we shall consider case that a real hypersurface M of $\mathbb{C}P^n(c)$ is a Sasakian manifold with respect to the alomost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of the ambient space $\mathbb{C}P^n(c)$. Then it follows from (4.2) and (2.3) that ξ is principal. Hence, again by using (4.2) and (2.3) we find that $Au = -u$ for each vector u orthogonal to ξ, so that our real hypersurface M is a member of totally η-umbilic hypersurfaces in $\mathbb{C}P^n(c)$. Hence, using the classification theorem of totally η-umbilic hypersurfaces in $\mathbb{C}P^n(c)$ (see [13]), we see that the shape operator A of our Sasakian manifold M in $\mathbb{C}P^n(c)$ is written as

$$AX = -X + (c/4)\eta(X)\xi \quad \text{for each vector } X \in TM. \tag{4.3}$$

Conversely, it follows from (2.3) and (4.3) that Equation (4.2) holds. Thus we know that M is a Sasakian manifold if and only if M has the shape operator A satisfying (4.3). Furthermore, M has constant ϕ-sectional curvature $c+1$.

Therefore, from the discussion here and Proposition 2 we obtain the following:

Proposition 3 ([2])**.** *The geodesic sphere $G(r)$ with $\tan^2(\sqrt{c}\, r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ is a Sasakian manifold with respect to the almost contact metric structure induced from the ambient space $\mathbb{C}P^n(c)$ if and only if $c = 8n+4$. Moreover, this geodesic sphere is a Sasakian space form of constant ϕ-sectional curvature $c+1$.*

By virtue of our discussion we establish the following:

Theorem 6 ([11]). *For each of $c > 0$, $n(\geqq 2)$ and $N > n(n+2)-1$, there exists a $(2n-1)$-dimensional Riemannian submanifold M^{2n-1} in an N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c}$ with $(n+1)c/(2n) \geqq \tilde{c}$ satisfying the following three conditions*:

(1) *M^{2n-1} is a homogeneous submanifold with nonzero parallel mean curvature vector with respect to the normal connection in the ambient sphere $S^N(\tilde{c})$*;
(2) *M^{2n-1} is a Bereger sphere*;
(3) *M^{2n-1} has an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. In particular, when $c = 8n+4$, this submanifold M is a Sasakian space form of constant ϕ-sectional curvature $8n+5$.*

Moreover, for each of $c > 0$ and $n(\geqq 2)$, when $N = n(n+2)-1$, there exists a $(2n-1)$-dimensional Riemannian submanifold M^{2n-1} in an N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c} = (n+1)c/(2n)$ satisfying the above conditions (1), (2), (3).

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