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Remarks on Akivis Left Loops

Michihiko KIKKAWA

Department of Mathematics, Faculty of Science,
Shimane University, Matsue, 690 JAPAN
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After introducing the concept of algebraic projective relation of abstract homogeneous systems, we consider the projectivity of normal left subloops of homogeneous left loops. We introduce the concept of Akivis left loops on abstract groups, and show that any normal subgroup H of an abstract group G induces a normal left subloop of each Akivis left loop on G, which is in projective relation with H, and that Akivis left loops of the quotient group G/H are equal to quotient homogeneous left loops of Akivis left loops.

Introduction

The concept of homogeneous Lie loops, introduced in 1975 [3] as non-associative generalization of Lie groups, has brought out a new ternary algebraic multiplication called homogeneous system (cf. [4]), which has been studied extensively in [4], [5], [6], [7], [8], [9], [10], [11], [12],[16],[17],[18]. In these works, it has been found that any homogeneous left loop is characterized by the associated homogeneous system, and vice versa. The concept of normal subloops in the theory of loops is rather complicated to consider. However, for homogeneous left loops, the concept of normal left subloops has been introduced lucidly by using the associated homogeneous systems (cf. [7]).

On the other hand, in studying geodesic homogeneous left Lie loops as an generalized theory of Lie groups which grew out from purely geometric theory of geodesic loops (cf. [2], [19]), the concept of projectivity between two geodesic homogeneous left Lie loops has been introduced by differential geometric methods, by using homogeneous systems, too (cf. [13], [15], [16]).

As canonical examples, we have found in [16] and [20] that local Lie loops on Lie groups introduced by Akivis in 1972 [1] are in projective relation with the group multiplication, and shown that only such local Lie loops given by Akivis are in projective relation with the group multiplication for any simple Lie groups of odd dimension (cf. [20]).

In this paper, we introduce the new concept of Akivis left loops on any non-abelian abstract groups, from an abstract viewpoint of Akivis local loops. We introduce also the purely algebraic concept of projectivity for homogeneous systems and homogeneous left loops revised from that of [14]. Then, we investigate the projectivity of Akivis left loops on groups and its normal subgroups. The key principle laying these theory is to learn the results of homogeneous left Lie loops obtained by means of the differential geometric methods, and to reconstruct them algebraically.

§1. Algebraic Characterization of Projectivity of Homogeneous Systems

Let (G, η) be a *homogeneous system* on a non-empty set G, that is, η is a ternary operation $\eta\colon G \times G \times G \to G$ satisfying;

$$\eta(x, x, y) = y, \tag{1.1}$$

$$\eta(x, y, x) = y, \tag{1.2}$$

$$\eta(x, y, \eta(y, x, z)) = z, \tag{1.3}$$

$$\eta(u, v, \eta(x, y, z) = \eta(\eta(u, v, x), \eta(u, v, y), \eta(u, v, z)) \tag{1.4}$$

for all x, y, z, u, v in G.

Let H be a subset of G and assume that it forms a subsystem $(H, \eta|_H)$ of (G, η). Then, it is clear that $(H, \eta|_H)$ is a homogeneous subsystem of (G, η). We use the same terminology and notations as in [5], [7] and [8]: Denote by

$$xH = \eta(H, x, H)$$

for any x in G. Then, $(H, \eta|_H)$ is *invariant* if it satisfies

$$\eta(x, y, xH) = yH \qquad \text{for any } x, y \text{ in } G.$$

A *normal subsystem* $(H, \eta|_H)$ of (G, η) is a subsystem satisfying

$$\eta(xH, yH, zH) = \eta(x, y, z)H \qquad \text{for any } x, y, z \text{ in } G.$$

We know that any normal subsystem is invariant. Also, we know that if H is a normal subsystem of (G, η), the *quotient homogeneous system* $(G/H, \eta)$ are well-defined by the natural projection $\pi\colon G \to G/H = \{xH\colon x \in G\}$ (cf. [7]).

DEFINITION 1.1. A homogeneous system $(G, \tilde{\eta})$ on the same underlying set G as η will be said to be *in projective relation* with (G, η) if it satisfies the following;

$$\eta(x, y, y) = \tilde{\eta}(x, y, y), \tag{1.5}$$

$$\eta(u, v, \tilde{\eta}(x, y, z)) = \tilde{\eta}(\eta(u, v, x), \eta(u, v, y), \eta(u, v, z)), \tag{1.6}$$

$$\tilde{\eta}(u, v, \eta(x, y, z)) = \eta(\tilde{\eta}(u, v, x), \tilde{\eta}(u, v, y), \tilde{\eta}(u, v, z)) \tag{1.7}$$

for all x, y, z, u, v in G.

Assume that a homogeneous system $(G, \tilde{\eta})$ is in projective relation with

(G, η). If $(H, \eta|_H)$ and $(H, \tilde{\eta}|_H)$ are subsystems of (G, η) and $(G, \tilde{\eta})$, respectively, then, it is easy to see that $\tilde{\eta}|_H$ is in projective relation with $\eta|_H$. Moreover, we have

PROPOSITION 1.1. *Assume that both of the subsystems $(H, \eta|_H)$ and $(H, \tilde{\eta}|_H)$ are normal. Then, the quotient homogeneous systems $(G/H, \eta)$ and $(G/H, \tilde{\eta})$ are in projective relation.*

Let (G, μ) be a *homogeneous left loop*, that is, G is a non-empty set equipped with a multiplication $\mu: G \times G \to G$ which satisfies the following;

i) There exists a two-sided unit element e, i.e.,

$$\mu(e, x) = \mu(x, e) = x \qquad \text{for all } x \in G. \tag{1.8}$$

ii) Any left translation $L_x: G \to G$; $L_x y := \mu(x, y)$ is a bijection.

iii) The *left inverse property* is provided; i.e.,

$$L_x^{-1} = L_{x^{-1}}, \quad \text{where} \quad x^{-1} := L_x^{-1} e. \tag{1.9}$$

iv) Any *left inner mapping* $L_{x,y}: G \to G$ is an automorphism of (G, μ), where

$$L_{x,y} := L_{\mu(x,y)}^{-1} L_x L_y. \tag{1.10}$$

By setting $\eta: G \times G \times G \to G$ as;

$$\eta(x, y, z) := L_x \mu(L_x^{-1} y, L_x^{-1} z), \tag{1.11}$$

we have a *homogeneous system* (G, η) *associated with* the homogeneous left loop (G, μ). Conversely, any homogeneous system (G, η) with some fixed element $e \in G$ is a homogeneous left loop (G, μ) by setting

$$\mu(x, y) := \eta(e, x, y), \tag{1.12}$$

whose unit element is e and the associated homogeneous system is (G, η). Thus, we see that to give a homogeneous system with any fixed element (origin) is equivalent to give a homogeneous left loop with the origin as the unit element:

DEFINITION 1.2. Let (G, μ) be a homogeneous left loop with the associated homogeneous system (G, η). A left subloop $(H, \mu|_H)$ of (G, μ) is said to be a *normal* (resp. *invariant*) left subloop if the homogeneous system $(H, \eta|_H)$ associated with $(H, \mu|_H)$ is a normal (resp. invariant) subsystem of (G, η). If $(H, \mu|_H)$ is a normal left subloop of (G, μ), then the *quotient homogeneous left loop* $(G/H, \mu)$ is defined by a natural manner.

Let (G, μ) be a homogeneous left loop with the associated homogeneous system (G, η). For any element x of G, we set the powers of x by;

$$x^m := (L_x)^m e \qquad \text{for any integer } m.$$

The homogeneous left loop (G, μ) is said to be *left diassociative* if it satisfies the following relations for any $x \in G$:

$$L_{x^m, x^n} = \mathrm{id} \qquad \text{for any integers } m \text{ and } n. \tag{1.13}$$

It is easy to see that any left diassociative left loop is power associative.

REMARK. A homogeneous left Lie loop (G, μ) is said to be *geodesic* if, in some neighborhood of the unit element e, it is coincident with the geodesic local loop at e with respect to the canonical connection. It can be seen that (G, μ) is geodesic if and only if any geodesic curve $x(t)$ through the point $e = x(0)$ satisfies the relation

$$L_{x(t), x(s)} = \mathrm{id} \qquad \text{for any real numbers } t \text{ and } s. \tag{1.13)$'$}$$

In this case, the geodesic curve $x(t)$ forms a one-parameter subgroup of the left loop (G, μ). The last relation (1.13)$'$ corresponds to algebraic relation (1.13).

DEFINITION 1.3. (Cf. [14]) Two left diassociative homogeneous left loops μ and $\tilde{\mu}$ on the same set G are *in projective relation* if they have the same unit element and the associated homogeneous systems (G, η) and $(G, \tilde{\eta})$ are in projective relation.

Assume that two left diassociative homogeneous left loops (G, μ) and $(G, \tilde{\mu})$ with the same unit element e are in projective relation. Then, any power x^k of any element x with respect to the multiplications μ and $\tilde{\mu}$ are always coincident. In fact, x^{-1} and $x^{2k} = (x^k)^2$ are coincident since

$$x^{-1} = \eta(x, e, e) = \tilde{\eta}(x, e, e) \tag{1.14}$$

$$x^{2k} = \eta(e, x^k, x^k) = \tilde{\eta}(e, x^k, x^k) \tag{1.15}$$

for any integer k. Assume that $x^k (0 < k < m+1)$ is coincident for μ and $\tilde{\mu}$. Since the power x^{2k} is coincident for μ and $\tilde{\mu}$, we have

$$\begin{aligned} x^{2m+1} &= \mu(x, x^{2m}) \\ &= \eta(e, x, \tilde{\eta}(e, x^m, x^m)) \\ &= \tilde{\eta}(\eta(e, x, e), \eta(e, x, x^m), \eta(e, x, x^m)) \\ &= \tilde{\eta}(x,, x^{m+1}, x^{m+1}) \\ &= \tilde{\eta}(e, x, \tilde{\eta}(e, x^m, x^m)) \\ &= \tilde{\mu}(x, x^{2m}), \end{aligned}$$

which show that x^{2m+1} is coincident for μ and $\tilde{\mu}$.

We have introduced in [13] and [15] the concept of projectivity for geodesic (local) left Lie loops, by means of the canonical connections, that is; two geodesic homogeneous (local) left Lie loops (G, μ) and $(G, \tilde{\mu})$ on the same manifold G with the same unit element, say e, is said to be in projective relation if (i) they have the same system of geodesic curves with respect to the canonical connection, and (ii) their associated homogeneous systems satisfy (1.6) and (1.7). Since any geodesic curve through e is a one-parameter subgroup of the respective left loop, by (1.13)′, any geodesic homogeneous left Lie loop is left diassociative in the exponential image of the tangent Lie triple algebra, so the condition (i) corresponds to the algebraic condition (1.5) for left diassociative homogeneous left loops. Conversely, if two geodesic homogeneous left Lie loops (G, μ) and $(G, \tilde{\mu})$ are in projective relation in the algebraic sense of Def. 1.2, it can be shown that any one-parameter subgroup is coincident for two kinds of multiplications μ and $\tilde{\mu}$. This implies that the condition (i) above holds, and the geodesic homogeneous left Lie loops are in projective relation in the sense of [15]. Hence we have;

PROPOSITION 1.2. *Assume that two geodesic homogeneous left Lie loops* (G, μ) *and* $(G, \tilde{\mu})$ *are globally left diassociative. Then, they are in projective relation* (in the sense of [15]) *if they are in projective relation in algebraic sense of* Def. 1.2 *as left diassociative homogeneous left loops.*

§2. Akivis left loops

Let (G, μ^0) be a non-abelian abstract group with the multiplication $xy = \mu^0(x, y)$. For each integer p, set

$$\mu^p(x, y) := x^{p+1} y x^{-p}. \tag{2.1}$$

Then we have a left loop (G, μ^p). By some general consideration of projectivity of abstract group, we have shown in [14] that it is a homogeneous left loop. In what follows, we will call (G, μ^p) *Akivis left loop of degree* p on the group (G, μ^0). It is evident that the original group multiplication is given by $p = 0$. In the sequel, we check the projectivity of Akivis left loops directly from the viewpoint of the revised definition of projectivity in §1.

REMARK. In 1978, M. Akivis ([1]) has introduced a one-parameter family of local loops $\omega^\alpha (\alpha \in \mathbf{R})$ on a Lie group G given by

$$\omega^\alpha(x, y) := x^\alpha y x^{1-\alpha},$$

and shown that this local loops are geodesic local loops with respect to some linear connections on G. Later, in 1991, it was found that this mamily of local loops, named *Akivis local loops*, plays an important role in the theory of projectivity of Lie groups (cf. [16], [20]).

Now we consider the homogeneous system η^p associated with Akivis left loop μ^p of degree p on a group (G, μ^0), where the group multiplication is denoted by juxtaposition as $\mu^0(x, y) = xy$. Denote by L_x the left translation of the group G by an element x, and by δ_x the inner automorphism: $\delta_x y = xyx^{-1}$. Then, we can show that left translations $L_x^{(p)}$ and left inner mappings $L_{x,y}^{(p)}$ of μ^p are expressed by

$$L_x^{(p)} = L_x(\delta_x)^p \tag{2.2}$$

and

$$L_{x,y}^{(p)} = (\delta_x)^p(\delta_{xy})^{-p}(\delta_y)^p. \tag{2.3}$$

The homogeneous system η^p associated with μ^p is given by

$$\eta^p(x, y, z) = L_y(\delta_{x^{-1}y})^p L_x^{-1} z. \tag{2.4}$$

Especially, by setting $p = 0$, we get the homogeneous system of the group μ^0:

$$\eta^0(x, y, z) = yx^{-1}z. \tag{2.5}$$

PROPOSITION 2.1. *Akivis left loop μ^p is left diassociative.*

PROOF. Since $(L_x^{(p)})^m e = (L_x)^m e = x^m$ hold by (2.2), we see that any power of elements with respect to μ^p are coincident with those of the original group μ^0. Then, for any integers m and n, we get $L_{x^m,x^n}^{(p)} = \mathrm{id}$ from (2.3).

PROPOSITION 2.2. *Akivis left loop μ^p is homogeneous.*

PROOF. By (2.3) we see that any left inner mapping $L_{x,y}^{(p)}$ of μ^p is an automorphism of the original group μ^0. Since the multiplication μ^p is given by (2.1), which is expressed in terms of the multiplication of μ^0, we see that $L_{x,y}^{(p)}$ is an automorphism of μ^p. The remaining relations can be checked easily.

From this result we see that the associated ternary system η^p is a homogeneous system.

PROPOSITION 2.3. *For any integers p and q, the Akivis left loops μ^p and μ^q are in projective relation in the sense of* Def. 1.3.

PROOF. It is sufficient to show that the homogeneous systems η^p and η^q, associated with μ^p and μ^q, respectively, are in projective relation. By (2.4) we get

$$\begin{aligned}\eta^p(x, y, y) &= L_y(\delta_{x^{-1}y})^p L_x^{-1} y \\ &= yx^{-1}y \\ &= \eta^0(x, y, y),\end{aligned}$$

which show that η^p and η^q satisfy the relation (1.5). On the other hand, we get

$$(2.6)\qquad \begin{aligned}&\eta^p(u, v, x)^{-1}\eta^p(u, v, y)\\ &= (v(\delta_{u^{-1}v})^p(u^{-1}x))^{-1}(v(\delta_{u^{-1}v})^p(u^{-1}y))\\ &= ((\delta_{u^{-1}v})^p(u^{-1}x)^{-1})((\delta_{u^{-1}v})^p(u^{-1}y))\\ &= (\delta_{u^{-1}v})^p(x^{-1}y).\end{aligned}$$

Then, we have

$$\begin{aligned}&\eta^q(\eta^p(u, v, x), \eta^p(u, v, y), \eta^p(u, v, z))\\ &= L_{\eta^p(u,v,y)}(\delta_{(\eta^p(u,v,x))^{-1}\eta^p(u,v,y)})^q L^{-1}_{\eta^p(u,v,x)}\eta^p(u, v, z)\\ &= L_{v(\delta_{u^{-1}v})^p(u^{-1}y)}(\delta_{(\delta_{u^{-1}v})^p(x^{-1}y)})^q(\delta_{u^{-1}v})^p(x^{-1}z)\\ &= L_v(\delta_{u^{-1}v})^p L_u^{-1}L_y(\delta_{x^{-1}y})^q L_{x^{-1}}z\\ &= \eta^p(u, v, \eta^q(x, y, z)).\end{aligned}$$

Thus, the relation (1.6) (and hence (1.7)) is shown for η^p and η^q.

§3. Normal left subloops of Akivis left loops

Let μ^p be an Akivis left loop on a non-abelian group (G, μ^0). It is evident that any subgroup H of (G, μ^0) forms a homogeneous left subloop of μ^p, and that $(H, \mu^p|_H)$ is the Akivis left loop on the subgroup $(H, \mu^0|_H)$.

THEOREM 3.1. *Let G be an abstract group, (G, μ^p) the Akivis left loop of degree p on G for any integer p. Assume that H is a normal subgroup of G. Then, $(H, \mu^p|_H)$ is a normal left subloop of μ^p. Moreover, the Akivis left loop of any degree on the quotient group G/H is the quotient homogeneous left loop $(G/H, \mu^p)$ of (G, μ^p) modulo $(H, \mu^p|_H)$.*

PROOF. By (2.4) we get

$$\eta^p(f, g, h) = L_g[\delta_{f^{-1}g}]^p L_f^{-1}h$$

for any $f, g, h \in H$, and we see that $(H, \eta^p|_H)$ is a subsystem of (G, η^p). Moreover, for any element x in G, (2.4) assures that $\eta^p(h, x, H) = L_xH$ holds for $h \in H$, that is;

$$(3.1)\qquad xH = \eta^p(H, x, H) = L_xH.$$

Then, we have

$$\begin{aligned}\eta^p(xH, yH, zH) &= L_{yH}(\delta_{x^{-1}yH})^p L^{-1}_{xH}zH\\ &= L_y(\delta_{x^{-1}y})^p L_x^{-1}zH\\ &= \eta^p(x, y, z)H,\end{aligned}$$

which shows that $(H, \eta^p|_H)$ is a normal subsystem of (G, η^p). Denote by η the homogeneous system associated with the quotient group G/H. Then, Prop. 1.1 implies that the quotient homogeneous systems $(G/H, \eta)$ and $(G/H, \eta^p)$ are in projective relation. Since the homogeneous left loop $(G/H, \mu)$ given by

$$\mu(L_x H, L_y H) := \eta(H, L_x H, L_y H)$$

is equal to the quotient group G/H, we see that the Akivis left loop of any degree q on the group G/H is given by

$$\begin{aligned}\mu^q(xH, yH) &= L_{xH}(\delta_{xH})^q yH \\ &= \eta^q(e, x, y)H \\ &= \mu^q(x, y)H,\end{aligned}$$

which completes the proof.

COROLLARY 3.2. *If an Akivis left loop of some degree on an abstract group G is simple, then the group G is itself simple.*

In [20] it is shown that, for any simple Lie group G of odd dimension, only the Akivis local loops μ^p (p is any *real number*) given by

$$\mu^p(x, y) := x^{p+1} y x^{-p},$$

where x and y are in some neighborhood of the unit element e of G, are geodesic homogeneous local Lie loops in projective relation with G. By this fact, we are suggested the following;

OPEN PROBLEM. *For any given abstract simple group G, is there any diassociative homogeneous left loop except for Akivis left loops on G which is in projective relation with the group G?*

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