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Fundamental Generalized Inverse $*$-Semigroups

Dedicated to Professor Michihiko Kikkawa on his 60th birthday

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Abstract

In [3] and [5], the authors gave a representation of a generalized inverse $*$-semigroup S, which is a generalization of the Preston-Vagner representation. If S is fundamental, we can obtain a more precise representation of S. The purpose of this paper is to give a generalization of the Munn representation (see [6]) for fundamental generalized inverse $*$-semigroups. This paper is the improvement of our earlier announcement [4].

By introducing a new concept of a *strong π-groupoid* $X(\pi; Y; \{\varphi_{e,f}\})$, we shall construct a fundamental generalized inverse $*$-semigroup $T_{X(\pi)}(\mathscr{M})$. Also, we shall show that a generalized inverse $*$-semigroup is fundamental if and only if it is $*$-isomorphic to a $\mathscr{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\pi)}(\mathscr{M})$ on a strong π-groupoid $X(\pi; Y; \{\varphi_{e,f}\})$.

1 Introduction

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^*x^*$,

(iii) $xx^*x = x$.

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if it satisfies $e^* = e$. For any subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively. If $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzw = xzyw$, S is called a *generalized inverse $*$-semigroup*.

Let S and T be regular $*$-semigroups. A homomorphism $\phi: S \to T$ is called a *$*$-homomorphism* if $(a\phi)^* = a^*\phi$. A congruence σ on S is called a *$*$-congruence* if $(a\sigma)^* = a^*\sigma$. A $*$-congruence σ on S is said to be *idempotent-separating* if $\sigma \subseteq \mathscr{H}$, where $\mathscr{H}$ is one of the Green's relations. Denote the maximum idempotent-separating $*$-congruence on S by μ_s or simply by μ. If μ_s is the identity relation on S, S is called *fundamental.* The following result is well-known, and we use it frequently throughout this paper.

RESULT 1.1 (see [2]). *Let S be a regular $*$-semigroup. Then we have the* *following*:

(1) $E(S) = P(S)^2$;

(2) *for any* $a \in S$ *and* $e \in P(S)$, $a^*ea \in P(S)$;

(3) *each* $\mathscr{L}$*-class and each* $\mathscr{R}$*-class have one and only one projection*;

(4) *for* $e, f \in P(S)$, *if* $ef \in P(S)$ *then* $fe \in P(S)$ *and* $ef = fe$;

(5) $\mu_s = \{(a, b) \in S \times S: a^*ea = b^*eb$ *and* $aea^* = beb^*$ *for all* $e \in P(S)\}$.

Let X be a set. By $|X|$, denote the cardinality of X. If $X = \cup\{X_i: i \in I\}$ is a partition of X, denote it by $X \sim \Sigma\{X_i: i \in I\}$. For a mapping $\alpha: A \to B$, denote the domain and the range of α by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset C of A, $\alpha|_C$ means the restriction of α to C. The notation and the terminology are those of [1], unless otherwise stated.

In §2, we shall first introduce a concept of a *strong* π*-groupoid* $X(\pi; Y; \{\varphi_{e,f}\})$ with mappings $\{\varphi_{e,f}: e \geq f, e, f \in Y\}$, where Y is a semilattice, π is a partition $X \sim \Sigma\{X_e: e \in Y\}$ of a partial groupoid X and each $\varphi_{e,f}(e \geq f)$ is a mapping of X_e to X_f. Next, for a strong π-groupoid $X(\pi; Y; \{\varphi_{e,f}\})$, we shall construct a generalized inverse $*$-semigroup $T_{X(\pi)}(\mathscr{M})$ such that $P(T_{X(\pi)}(\mathscr{M}))$ is partially isomorphic to X.

In §3, for a given generalized inverse $*$-semigroup S, we shall construct a strong π-groupoid $P(S)(\pi; I; \{\varphi_{i,j}\})$, where I is the structure semilattice of a normal band $E(S)$. For $a \in S$, define a mapping $\tau_a: P(Sa^*) \to P(Sa)$ by $e\tau_a = a^*ea$. Then we shall show that a mapping $\phi: S \to T_{P(S)(\pi)}(\mathscr{M})(a \mapsto \tau_a)$ is a $*$-homomorphism and that the kernel of ϕ is the maximum idempotent-separating $*$-congruence on S. Moreover, we shall show that a generalized inverse $*$-semigroup is fundamental if and only if it is $*$-isomorphic to a $\mathscr{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\pi)}(\mathscr{M})$ on a strong π-groupoid $X(\pi)$.

2. $T_{X(\pi)}(\mathscr{M})$

For a partial groupoid X, if there exist a semilattice Y, a partition $\pi: X \sim \Sigma\{X_e: e \in Y\}$ of X and mappings $\varphi_{e,f}: X_e \to X_f$ ($e \geq f$ in Y) such that

(1) for any $e \in Y$, $\varphi_{e,e} = 1_{X_e}$,

(2) if $e \geq f \geq g$, then $\varphi_{e,f}\varphi_{f,g} = \varphi_{e,g}$,

(3) for $x \in X_e$, $y \in X_f$, xy is defined in X if and only if $x\varphi_{e,ef} = y\varphi_{f,ef}$, and in this case $xy = x\varphi_{e,ef}$,

then X is called a *strong* π*-groupoid* with mappings $\{\varphi_{e,f}: e, f \in Y, e \geq f\}$, and it is denoted by $X(\pi; Y; \{\varphi_{e,f}\})$ or simply by $X(\pi)$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong π-groupoid. A subset A of X is called a π*-singleton subset* of $X(\pi; Y; \{\varphi_{e,f}\})$, if there exists $e \in Y$ such that

$$|A \cap X_f| = \begin{cases} 1 & \text{if } f \in Ye, \\ 0 & \text{otherwise,} \end{cases}$$

$$(A \cap X_f)\varphi_{f,g} = A \cap X_g \text{ for any } f, g \in Ye \text{ such that } f \geq g,$$

where Ye is the principal ideal of Y generated by e. In this case, we sometimes denote the π-singleton subset A by $A(e)$. If $A(e)$ is a π-singleton subset, then $|A \cap X_f| = 1$ for any $f \in Ye$. We denote the only one element of $A \cap X_f$ by a_f. We remark that for any π-singleton subset $A(e)$, $A(e) = \{a_e \varphi_{e,f} : f \in Ye\}$. Denote the set of all π-singleton subsets of $X(\pi; Y; \{\varphi_{e,f}\})$ by $\mathscr{X}$.

Two π-singleton subsets $A(e)$ and $B(f)$ are said to be *π-isomorphic* to each other, if there exists an isomorphism $\bar{\alpha}: Ye \to Yf$ as semilattices. In this case, the mapping $\alpha: A(e) \to B(f)$ defined by $a_g \alpha = b_{g\bar{\alpha}}$ $(g \in Ye)$ is called a *π-isomorphism* of $A(e)$ to $B(f)$. It is obvious that α is a bijection of $A(e)$ onto $B(f)$, and hence $\alpha \in \mathscr{I}_X$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong π-groupoid. Define an equivalence relation $\mathscr{U}$ on $\mathscr{X}$ by

$$\mathscr{U} = \{(A(e), B(f)) \in \mathscr{X} \times \mathscr{X} : Ye \cong Yf \text{ (as semilattices)}\}.$$

For $(A(e), B(f)) \in \mathscr{U}$, let $T_{A(e),B(f)}$ be the set of all π-isomorphisms of $A(e)$ onto $B(f)$, and let

$$T_{X(\pi)} = \bigcup_{(A(e),B(f)) \in \mathscr{U}} T_{A(e),B(f)}$$

REMARK. As we have seen in [2] and [5], the set $\mathscr{G}\mathscr{I}_{X(\pi')}$ of all partial one-to-one π-mappings on a π-set $X(\pi'; \{\sigma_{e,f}\})$ is an inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}_X$ on X. However, $T_{X(\pi)}$ is not generally an inverse subsemigroup of $\mathscr{I}_X$.

For any $\alpha, \beta \in T_{X(\pi)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$d(\theta_{\alpha,\beta}) = \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\},$$

$$r(\theta_{\alpha,\beta}) = \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\},$$

$$a\theta_{\alpha,\beta} = b \text{ if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}.$$

Then $\theta_{\alpha,\beta} \in T_{X(\pi)}$. For, let $r(\alpha) = \{a_g : g \in Ye\}$ and $d(\beta) = \{b_h : h \in Yf\}$. Since $Ye \cap Yf = Yef$, $\theta_{\alpha,\beta}$ is a bijection of $\{a_g : g \in Yef\}$ onto $\{b_g : g \in Yef\}$ which maps a_g to b_g. Also, we can easily obtain that $\theta_{\alpha,\beta}^{-1} = \theta_{\beta^{-1},\alpha^{-1}}$.

Let $\mathscr{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{X(\pi)}\}$, and define a multiplication $\circ$ and a unary operation $*$ on $T_{X(\pi)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$

$$\alpha^* = \alpha^{-1}.$$

Let $\alpha: A(e) \to B(f)$ and $\beta: C(g) \to D(h)$ be any elements of $T_{X(\pi)}$. Then it is obvious that $\alpha \circ \beta$ is a bijection of $\{a_i : i \in Y((fg)\bar{\alpha}^{-1})\}$ onto $\{d_j : j \in Y((fg)\bar{\beta})\}$ which maps

a_i to $d_{i\bar{\alpha}\bar{\beta}}$. Thus the multiplication is closed. It is clear that $T_{X(\pi)}(\circ, *)$ is a regular $*$-semigroup. We denote it by $T_{X(\pi)}(\mathscr{M})$.

Next, we shall show that $E(T_{X(\pi)}) = \mathscr{M}$. It is obvious that $\mathscr{M} \subseteq E(T_{X(\pi)})$. Let $\alpha \in E(T_{X(\pi)})$. Then

$$\alpha \circ \alpha = \alpha\theta_{\alpha,\alpha}\alpha = \alpha.$$

Since $d(\theta_{\alpha,\alpha}) \subseteq r(\alpha)$ and $r(\theta_{\alpha,\alpha}) \subseteq d(\alpha)$, we have $\alpha = \theta_{\alpha,\alpha}^{-1} = \theta_{\alpha^{-1},\alpha^{-1}} \in \mathscr{M}$, and hence $E(T_{X(\pi)}) = \mathscr{M}$.

It is clear that $P(T_{X(\pi)}) = \{1_{A(e)}: A(e) \in \mathscr{X}\}$. It follows immediately from the definition of the multiplication of $T_{X(\pi)}(\mathscr{M})$ that $T_{X(\pi)}(\mathscr{M})$ is a generalized inverse $*$-semigroup. Define a mapping $\psi: X \to P(T_{X(\pi)}(\mathscr{M}))$ as follows: for $x \in X_e$,

$$x\psi = 1_{A(e)},$$

where $A(e) = \{x\varphi_{e,f}: f \in Ye\}$. Then it is clear that ψ is a bijection. To show that ψ is a partial isomorphism, for $x \in X_e$ and $y \in X_f$, assume that xy is defined in X. By (3) above, $xy = x\varphi_{e,ef} = y\varphi_{f,ef} \in X_{ef}$. Let $A(e) = \{x\varphi_{e,g}: g \in Ye\}$, $B(f) = \{y\varphi_{f,g}: g \in Yf\}$ and $C(ef) = \{(xy)\varphi_{ef,g}: g \in Yef\}$. It follows from (2) above that $1_{A(e)} \circ 1_{B(f)} = 1_{C(ef)} \in P(T_{X(\pi)})$ and that $(xy)\psi = (x\psi) \circ (y\psi)$. Conversely, if $1_{A(e)} \circ 1_{B(f)} \in P(T_{X(\pi)})$, then $a_e\varphi_{e,ef} = b_f\varphi_{f,ef}$. By (3) above, a_eb_f is defined in X, and hence $(a_eb_f)\psi = 1_{A(e)} \circ 1_{B(f)}$. Thus ψ is a partial isomorphism of X onto $P(T_{X(\pi)})$. Now, we have the following theorem.

THEOREM 2.1 *A regular $*$-semigroup $T_{X(\pi)}(\mathscr{M})$ is a generalized inverse $*$-semigroup whose set of projections is partially isomorphic to X.*

COROLLARY 2.2 *A partial groupoid X is partially isomorphic to the set of projections of a generalized inverse $*$-semigroup if and only if it is a strong π-groupoid.*

3 Representations

Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Let $E \sim \Sigma\{E_i: i \in I\}$ be the structure decomposition of E, and let $P_i = P(E_i)$. Then $\pi: P \sim \Sigma\{P_i: i \in I\}$ is a partition of P. For any $i, j \in I$ $(i \geq j)$, define a mapping $\varphi_{i,j}: P_i \to P_j$ by

$$e\varphi_{i,j} = efe \qquad \text{for some (any) } f \in P_j.$$

It follows from [7] that each $\varphi_{i,j}$ is a mapping, and it is easy to see that $P(\pi; I; \{\varphi_{i,j}\})$ is a strong π-groupoid. Now, we can consider the generalized inverse $*$-semigroup $T_{P(\pi)}(\mathscr{M})$, where $\mathscr{M} = \{\theta_{\alpha,\beta}: \alpha$ and β are π-isomorphisms among π-singleton subsets of $P(\pi)\}$.

LEMMA 3.1 *For any $a \in S$, $P(Sa)$ $(= P(Sa^*a))$ is a π-singleton subset of $P(\pi)$.*

PROOF. Let $a \in S$ and let $a^*a \in P_i$. To show the lemma, it is sufficient to prove that $P(Sa)$ is equal to $\{(a^*a)\varphi_{i,j} : j \in Ii\} = A$, say. Let $x \in P(Sa)$. Then there exist $y \in S$ and $j \in I$ such that $x = a^*y^*ya \in P_j$, whence $x = (a^*a)(a^*y^*ya)(a^*a) = (a^*a)\varphi_{i,j} \in A$. Thus $P(Sa) \subseteq A$. The opposite inclusion is obvious.

LEMMA 3.2 *For any $a \in S$, define a mapping $\tau_a: P(Sa^*) \to P(Sa)$ by*

$$e\tau_a = a^*ea.$$

Then $\tau_a \in T_{P(\pi)}(\mathscr{M})$ and $\tau_a^* = \tau_{a^*}$.

PROOF. It is obvious that τ_a is a bijection of a π-singleton subset $P(Sa^*)$ onto a π-singleton subset $P(Sa)$ with inverse mapping τ_{a^*}. Let γ be the smallest inverse semigroup congruence on S. Then it is easy to see that $E(S/\gamma) = I$ and $\{e\gamma : e \in P(Sa^*)\} = I((aa^*)\gamma)$. It follows from Theorem 4.9 of [1] that a mapping $\bar{\tau}_a : e\gamma \mapsto (a^*ea)\gamma$ is a (semilattice) isomorphism of $I((aa^*)\gamma)$ onto $I((a^*a)\gamma)$ and that for any $e \in P(Sa^*)$, $e\tau_a = (aa^*)\varphi_{(aa^*)\gamma,(e\gamma)\bar{\tau}_a}$. Hence $\tau_a \in T_{X(\pi)}(\mathscr{M})$.

LEMMA 3.3 *For any $a, b \in S$, $\theta_{\tau_a,\tau_b} = \tau_{a^*abb^*}$.*

PROOF. Let e be any element of $d(\theta_{\tau_a,\tau_b})$. Then $e \in P(Sa)$ and there exist $i \in I$ and $f \in P(Sb^*)$ such that $e, f \in P_i$, whence there exist $x, y \in S$ such that $e = a^*x^*xa$ and $f = by^*yb^*$. Since E is a normal band, $e = efe = (a^*x^*xa)(by^*yb^*)(a^*x^*xa) = (bb^*a^*a)^*(yb^*a^*x^*xa)^*(yb^*a^*x^*xa)(bb^*a^*a) \in P(Sbb^*a^*a) = d(\tau_{a^*abb^*})$.

Conversely, let e be any element of $d(\tau_{a^*abb^*})$. Then there exists $x \in S$ such that $e = a^*abb^*x^*xbb^*a^*a$, whence $e \in P(Sa)$. Put $f = bb^*a^*ax^*xa^*abb^*$. Then $f \in P(Sb^*)$ and $e, f \in P_i$ for some $i \in I$, since E is a band. Thus $e \in d(\theta_{\tau_a,\tau_b})$ and

$$e\tau_{a^*abb^*} = bb^*a^*aea^*abb^* = bb^*a^*ax^*xa^*abb^* = f = e\theta_{\tau_a,\tau_b}.$$

Hence we have $\theta_{\tau_a,\tau_b} = \tau_{a^*abb^*}$.

THEOREM 3.4 *Let S be a generalized inverse $*$-semigroup such that $E(S) = E$ and $P(S) = P$. Let $E \sim \Sigma\{E_i : i \in I\}$ be the structure decomposition of E and $P_i = P(E_i)$. Denote the partition $P \sim \Sigma\{P_i : i \in I\}$ of P by π, and, for any $i, j \in I$ $(i \geq j)$, define a mappig $\varphi_{i,j}: P_i \to P_j$ by $e\varphi_{i,j} = efe$ for some $f \in P_j$. Then $P(\pi; I; \{\varphi_{i,j}\})$ is a strong π-groupoid and $T_{P(\pi)}(\mathscr{M})$ is a generalized inverse $*$-semigroup.*

Moreover, for any $a \in S$, define a mapping $\tau_a: P(Sa^*) \to P(Sa)$ by $e\tau_a = a^*ea$. Then a mapping $\phi: S \to T_{P(\pi)}(\mathscr{M})$ $(a \mapsto \tau_a)$ is a $*$-homomorphism and the kernel of ϕ is the maximum idempotent-separating $*$-congruence on S.

PROOF. It remains to prove the last statement. To see that ϕ is a homomorphism, it is sufficient to show that $d(\tau_a \circ \tau_b) = d(\tau_{ab})$ for all $a, b \in S$. For any $a, b \in S$,

$$d(\tau_a \circ \tau_b) = d(\tau_a \tau_{a^*abb^*} \tau_b)$$

$$= (P(Sa) \cap (P(Sa^*abb^*) \cap P(Sb^*))\tau^*_{a^*abb^*})\tau^*_a$$
$$= (P(Sa) \cap P(Sbb^*a^*a))\tau^*_a$$
$$= (P(Sbb^*a^*a))\tau^*_a$$
$$= P(Sbb^*a^*aa^*)$$
$$= P(S(ab)^*)$$
$$= d(\tau_{ab}).$$

It is clear that $(a\phi)^* = a^*\phi$. Therefore, ϕ is a $*$-homomorphism.

Denote the maximum idempotent-separating $*$-congruence on S by μ. Assume that $(a, b) \in \mu$. Since $\mu \subseteq \mathscr{H}$, $aa^* = bb^*$ and $a^*a = b^*b$, whence τ_a and τ_b have the same domain and range. Let e be any element of $d(\tau_a)$. Since $e \in P$, $e\tau_a = a^*ea = b^*eb = e\tau_b$, whence $\tau_a = \tau_b$, and hence $\mu \subseteq ker\ \phi$. Conversely, let $(a, b) \in ker\ \phi$. Then τ_a and τ_b have the same domain and range, whence $aa^* = bb^*$ and $a^*a = b^*b$. For any $e \in P$,

$$a^*ea = a^*(aa^*eaa^*)a = (aa^*eaa^*)\tau_a = (bb^*ebb^*)\tau_b$$
$$= b^*(bb^*ebb^*)b = b^*eb.$$

On the othe hand, since ϕ is a $*$-homomorphism, $\tau_{a^*} = (\tau_a)^* = (\tau_b)^* = \tau_{b^*}$. By the similar calculation, we have $aea^* = beb^*$ for any $e \in P$, and hence $(a, b) \in \mu$. Thus we have the lemma.

A regular $*$-subsemigroup T of a regular $*$-semigroup S is said to be *$\mathscr{P}$-full* if $P(T) = P(S)$.

THEOREM 3.5. *A generalized inverse $*$-semigroup S is fundamental if and only if it is $*$-isomorphic to a $\mathscr{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\pi)}(\mathscr{M})$ on a strong π-groupoid $X(\pi; I; \{\varphi_{i,j}\})$ such that $P(T_{X(\pi)}(\mathscr{M}))$ is partially isomorphic to $P(S)$.*

PROOF. If S is fundamental, it follows from Theorem 3.4 that $ker\ \phi = \mu = 1_S$, and hence ϕ embeds S in $T_{P(\pi)}(\mathscr{M})$ on the strong π-groupoid $P(\pi; I; \{\varphi_{i,j}\})$, constructed above. By Lemma 2.1, P is partially isomorphic to $P(T_{P(\pi)}(\mathscr{M}))$, and hence $S\phi$ $(\cong S)$ is a $\mathscr{P}$-full generalized inverse $*$-subsemigroup of $T_{P(\pi)}(\mathscr{M})$.

Conversely, if S is a $*$-isomorphic to a $\mathscr{P}$-full generalized inverse $*$-semigroup S' of $T_{X(\pi)}(\mathscr{M})$, then $1_{A(e)} \in S'$ for all π-singleton subsets $A(e)$, where $A(e) = \{a_e\varphi_{e,f} : f \in Ie\}$. Let $\alpha: A(e) \to B(f)$ and $\beta: C(g) \to D(h)$ be elements of S' such that $(\alpha, \beta) \in \mu$. Then $(\alpha, \beta) \in \mathscr{H}$, and so $\alpha \circ \alpha^* = \beta \circ \beta^*$ and $\alpha^* \circ \alpha = \beta^* \circ \beta$. Since $A(e) = d(\alpha) = d(\alpha \circ \alpha^*) = d(\beta \circ \beta^*) = C(g)$, we have $A(e) = C(g)$, whence $e = g$ and $a_e = c_g$. Similarly, we have that $B(f) = D(h)$, $f = h$ and $b_f = d_h$.

Let x be any element of $d(\alpha)$ and set $x \in P_i$. Then $x = a_e\varphi_{e,i} = c_g\varphi_{g,i} \in A(e)$.

Since $(\alpha, \beta) \in \mu$,

$$\alpha^* \circ 1_{X(i)} \circ \alpha = \beta^* \circ 1_{X(i)} \circ \beta.$$

Since these two mappings must have the same range, $X(i\bar{\alpha}) = X(i\bar{\beta})$, whence $i\bar{\alpha} = i\bar{\beta}$ and

$$x\alpha = (a_e\varphi_{e,i})\alpha = b_f\varphi_{f,i\bar{\alpha}} = d_h\varphi_{h,i\bar{\beta}} = (c_g\varphi_{g,i})\beta = x\beta.$$

Thus $\alpha = \beta$, and hence S', and so S, is fundamental.

COROLLARY 3.6 *For any strong π-groupoid $X(\pi; I; \{\varphi_{i,j}\})$, the generalized inverse $*$-semigroup $T_{X(\pi;I;\{\varphi_{i,j}\})}$ is fundamental.*

References

[1] J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

[2] T. Imaoka, *On fundamental regular $*$-semigroups*, Mem. Fac. Sci. Shimane Univ. **14** (1980), 19–23.

[3] T. Imaoka, *Representations of generalized inverse $*$-semigroups*, Acta Sci. Math. (Szeged), to appear.

[4] T. Imaoka, I. Inata and H. Yokoyama, *Some remarks on representations of fundamental generalized inverse $*$-semigroups*, RIMS Kokyuroku **910**; *Semigroups, Formal Languages and Combinatorics on Words* (Proceedings), Kyoto Univ., 1995, 14–18.

[5] T. Imaoka, I. Inata and H. Yokoyama, *Representations of locally inverse $*$-semigroups*, Internat. J. Algebra Comput., to appear.

[6] W. D. Munn, *Uniform semilatties and bisimple inverse semigroups*, Quart. J. Math. Oxford (2) **17** (1966), 151–159.

[7] M. Yamada, *Regular semigroups whose idempotents satisfy permutation identities*, Pacific J. Math. **21** (1967), 371–392.