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Completeness theorem of Kleene logic

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Abstract

In this paper we shall define a Kleene logic whose Lindenbaum algebra is the Kleene algebra with implication (simply called i-Kleene algebra) and show that

(1) Every i-Kleene algebra can be embedded to the simplest i-Kleene algebra $3 = \{0, 1/2, 1\}$;

(2) The Lindenbaum algebra of the Kleene logic is an i-Kleene algebra;

(3) The completeness theorem of the Kleene logic is established;

(4) The Kleene logic is decidable.

Key words: i-Kleene algebras, Lindenbaum algebra, Kleene logic, Completeness theorem, Decidability

1 Introduction

It is familiar that there is close relationships between logics and their Lindenbaum algebras. For example, the Lindenbaum algebra of the classical propositional logic is a Boolean algebra, that of the intuitionistic propositional logic is a Heyting algebra and so on. Now we have a natural question.

What is a logic whose Lindenbaum algebra is a Kleene algebra?

While many logics have implications and hence do the corresponding algebras, the Kleene algebra does not have one. We have to consider the Kleene algebras with implication to correspond the logics.

In the following we define Kleene algebras with implication (or simply called i-Kleene algebra) and a Kleene logic KL, whose Lindenbaum algebra becomes the i-Kleene algebra. We show that

(1) Every i-Kleene algebra can be embedded to the simplest i-Kleene algebra $3 = \{0, 1/2, 1\}$;

(2) The Lindenbaum algebra of KL is the i-Kleene algebra;

(3) For every formula A, A is probable in KL if and only if $\tau(A) = 1$ for any valuation function τ;

(4) The Kleene logic KL is decidable.

2 i-Kleene algebras

In this section we define an i-Kleene algebra. By an i-Kleene algebra, we mean the algebra $K = (K, \wedge, \vee, \to, N, 0, 1)$ of type (2, 2, 2, 1, 0, 0) such that

(1) $(K, \wedge, \vee, 0, 1)$ is a bounded distributive lattice;

(2) $N: K \to K$ is a map satisfying the following conditions

(N1) $N0 = 1$, $N1 = 0$

(N2) $x \leq y$ implies $Ny \leq Nx$

(N3) $N^2x = x$, where $N^2x = N(Nx)$

(N4) $x \wedge Nx \leq y \vee Ny$ (Kleene's law)

(3) the implication $\to$ satisfies

(I1) $x \to x = 1$

(I2) $Nx \vee y \leq x \to y$

(I3) $x \wedge (x \to y) \leq Nx \vee y$

(I4) $x \wedge Nx \wedge y \wedge Ny \wedge N(x \to y) = 0$

(I5) $x \wedge Ny \leq N(x \to y) \vee y$

Example: As a model of the i-Kleene algebra we have $3 = \{0, 1/2, 1\}$ defined by

$$x \wedge y = \min\{x, y\}$$
$$x \vee y = \max\{x, y\}$$
$$Nx = 1 - x$$
$$x \to y = \min\{1 - x + y, 1\} \text{ for every } x \text{ and } y \text{ in } 3.$$

By simple calculation it turns out that the structure $3 = (3, \wedge, \vee, \to, N, 0, 1)$ is the i-Kleene algebra. In the following we denote it simply by 3.

A non-empty subset F of K is called a filter when it satisfies the conditions:

(f1) $x, y \in F$ imply $x \wedge y \in F$;

(f2) $x \in F$ and $x \leq y$ imply $y \in F$.

A filter F is called proper when it is a proper subset of K. We define three kinds of filters of K. By a maximal filter F, we mean the proper filter F such that $F \subseteq G$ implies $F = G$ for any proper filter G. A proper filter F is called prime if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for every $x, y \in K$. Lastly, we say a proper filter F an ultrafilter when $x \in F$ or $Nx \in F$ for every x in K.

LEMMA 1 *Let K be any i-Kleene algebra and $x \in K$. If $x \neq 0$, then there is a maximal filter F of K such that $x \in F$.*

PROOF. $\Gamma = \{H \mid x \in H$ and H is a proper filter of $K\}$ is not an empty set, because the principal filter $[x)$ generated by x is in Γ. It is clear that Γ is an inductive set. Thus, by Zorn's lemma, there is a maximal element F in Γ. The filter F is clearly a maxima filter containing x.

LEMMA 2 *Let F be a maximal filter of K. If $x \notin F$, then there exists an element $u \in F$ such that $u \wedge x = 0$. Hence $x \notin F$ implies $Nx \in F$.*

PROOF. Let Δ be a set $\{y \in K \mid x \wedge u \leq y, u \in F\}$. It is easy to show that Δ is a filter properly contining F and x. Since F is maximal, this implies that $\Delta = K$ and hence that $0 \in \Delta$. We then have that $x \wedge u = 0$ for some u in F. For that element $u \in F$ we have

$$\begin{aligned} u &= u \wedge 1 \\ &= u \wedge N(x \wedge u) \\ &= u \wedge (Nx \vee Nu) \\ &= (u \wedge Nx) \vee (u \wedge Nu) \leq (u \wedge Nx) \vee (x \vee Nx) \\ &= x \vee Nx \text{ by Kleene's law.} \end{aligned}$$

This yields that $u \leq (x \vee Nx) \wedge u = u \wedge Nx \leq Nx$. Hence we have $Nx \in F$.
It is easy to show the following lemma. Hence we omit the proof.

LEMMA 3 *If F is a maximal filter, then it is the prime filter.*

THEOREM 1 *The implication $\rightarrow$ cannot be defined by other operations $\wedge$, $\vee$, and N.*

PROOF. It is evident from that we have $1/2 \rightarrow 1/2 = 1$ in 3.
Let M be an arbitrary maximal filter of K. We define the following subsets M_j $(j \in 3)$ of K:

$$\begin{aligned} M_0 &= \{x \in K \mid x \notin M, Nx \in M\} \\ M_{1/2} &= \{x \in K \mid x \in M, Nx \in M\} \\ M_1 &= \{x \in K \mid x \in M, Nx \notin M\} \end{aligned}$$

Now we define an equivalence relation $\sim$ as follows: For $x, y \in K$,

$$x \sim y \Leftrightarrow \text{there exists } M_j \text{ such that } x, y \in M_j.$$

LEMMA 4 *$x \sim y$ iff $x \in M \Leftrightarrow y \in M$ and $Nx \in M \Leftrightarrow Ny \in M$*

LEMMA 5 *The relation $\sim$ is a congruent relation on K.*

PROOF. It sufficies to show that
(1) $x \sim a$ and $y \sim b$ implies $x \wedge y \sim a \wedge b$;
(2) $x \sim a$ and $y \sim b$ implies $x \rightarrow y \sim a \rightarrow b$;
(3) $x \sim a$ implies $Nx \sim Na$.

It is clear by lemma 4 that the condition (3) holds so we consider the cases of (1) and of (2).

Case (1): Suppose that $x \sim a$ and $y \sim b$. It is sufficient to show that

$$x \wedge y \in M \text{ iff } a \wedge b \in M \text{ and}$$
$$Nx \wedge Ny \in M \text{ iff } Na \wedge Nb \in M.$$

Suppose that $x \wedge y \in M$. Since x and y are in M, we have that both a and b are in M by lemma 4. Thus, we obtain that $a \wedge b$ is in M. The converse is similiar.

Next if we assume that $N(x \wedge y) \in M$, then we get that $Nx \vee Ny \in M$ and hence that $Nx \in M$ or $Ny \in M$ by lemma 3. This means that $Na \in M$ or $Nb \in M$ and so that $Na \vee Nb = N(a \wedge b) \in M$. The converse also holds.

Case (2): We have to show that

$$x \to y \in M \text{ iff } a \to b \in M \text{ and}$$
$$N(x \to y) \in M \text{ iff } N(a \to b) \in M.$$

We suppose that $x \to y \in M$. We have two cases $y \in M$ and $y \notin M$. If $y \in M$ (hence $b \in M$), then we have $a \to b \in M$ by $b \leq Na \vee b \leq a \to b$. In the case of $y \notin M$ (hence $b \notin M$), if $x \in M$, since x and $x \to y$ are in M, then we have $Nx \in M$ because of $y \notin M$ and $x \wedge (x \to y) \leq Nx \vee y \in M$. Hence we obtain that $Na \in M$. Well, umless $a \to b$ is in M, this means that neither Na nor b are in M by $Na \vee b \leq a \to b$. This is a contradiction. Thus we have $a \to b \in M$. On the other hand, if $x \notin M$ then $a \notin M$ and hence $Na \in M$. Since $Na \leq Na \vee b \leq a \to b$, we get $a \to b \in M$.

In the sequel we have that if $x \to y \in M$ then $a \to b \in M$ in each case. The converse is similar.

Next we assume that $N(x \to y) \in M$. It is sufficient to prove $N(a \to b) \in M$. By $N(x \to y) \leq x \wedge Ny$, we have x, $Ny \in M$ and hence a, $Nb \in M$.

If $b \in M$ (hence $y \in M$), then we have $Nx \notin M$. For otherwise that x, Nx, y, Ny, $N(x \to y) \in M$ means $x \wedge Nx \wedge y \wedge Ny \wedge N(x \to y) = 0 \in M$. But this is a contradiction. Thus we have $Na \notin M$. In this case, since $a \wedge Nb \leq Na \vee N(a \in b) \in M$ and $Na \notin M$, we get that $N(a \to b) \in M$.

If $b \notin M$, then we obtain that $N(a \to b) \in M$ by $a \wedge Nb \leq N(a \to b) \vee b \in M$.

In each case this means that $N(x \to y) \in M$ implies $N(a \to b) \in M$. It is clear that the converse holds. Hence $N(x \to y) \in M$ iff $N(a \to b) \in M$.

These indicate that the relation $\sim$ is the congruence relation.

Let $[x] = \{y \in K \mid x \sim y\}$ be the equivalence class of x and $K/\sim\, = \{[x] \mid x \in K\}$ be the set of all equivalence classes. The relation $\sim$ in congruent means that we can consistently define operations $\wedge$, $\vee$, N, and $\to$ on $K/\sim$ as follows.

$$[x] \wedge [y] = [x \wedge y]$$
$$[x] \vee [y] = [x \vee y]$$
$$N[x] = [Nx]$$
$$[x] \to [y] = [x \to y]$$

The general theory of universal algebras shows the next theorem.

THEOREM 2 *Let K be an i-Kleene algebra and M be a maximal filter of K. Then an algebraic system $K/\sim\, = (K/\sim, \wedge, \vee, N, \to, [0], [1])$ is the i-Kleene algebra and the map $\xi: K \to K/\sim$ defined by $\xi(x) = [x]$ is a homomorphism.*

Remarking the maximality of the filter M of K, we have a stronger result as to $K/\sim$. We define a map $V: K/\sim \to 3$ by $V([x]) = j$ if $x \in M_j$, where $j \in 3 = \{0, 1/2, 1\}$. We have an important lemma.

LEMMA 6 *The map V is injective and homomorphic. Moreover if $M_{1/2}$ is not empty then V is surjective and hence $K/\sim$ is isomorphic to 3.*

PROOF. It is evident that V is injective. We shall show that V is homomorphic, that is,

(a) $V([x] \wedge [y]) = \min\{V([x]), V([y])\}$
(b) $V(N[x]) = 1 - V([x])$
(c) $V([x] \to [y]) = \min\{1 - V([x]) + V([y]), 1\}$

For the sake of simplicity, we only show two cases (a) and (c).

Case (a):

(1) $V([x]) = V([y]) = 0$; It sufficies to show that $V([x \wedge y]) = 0$, that is, $x \wedge y \in M_0$. By assumption we have $x, y \notin M$ and $Nx, Ny \in M$. Clearly we get that $x \wedge y \notin M$. On the other hand, since M is maximal and $x \wedge y \notin M$, we have $N(x \wedge y) \in M$. This means that $x \wedge y \in M_0$.

(2) $V([x]) = 0$, $V([y]) = 1/2$; It is suffifient to prove that $x \wedge y \in M_0$, that is $x \wedge y \notin M$ and $N(x \wedge y) \in M$. By assumption we have $x \notin M$ and $Nx, y, Ny \notin M$. Since $x \notin M$, it is clear that $x \wedge y \notin M$. It follows that $N(x \wedge y) \in M$ by similar argument above. Thus we get that $x \wedge y \in M_0$.

(3) $V([x]) = 0$, $V([y]) = 1$; We can show that $x \wedge y \in M_0$ as above.

(4) $V([x]) = V([y]) = 1/2$; It should be the case that $x \wedge y \in M_{1/2}$, that is, $x \wedge y$ and $N(x \wedge y)$ are in M. By assumtion we get that $x, Nx, y, Ny \in M$. It is obvious that $x \wedge y$ and $N(x \wedge y) = Nx \vee Ny$ are in M.

(5) $V([x]) = V([y]) = 1$; It sufficies to show that $x \wedge y \in M_1$, that is, $x \wedge y \in M$ and $N(x \wedge y) \notin M$. It is that $x, y \in M$ and $Nx, Ny \notin M$ by assmuption. Now we have $x \wedge y \in M$. We also have that $N(x \wedge y) \notin M$. Otherwise, since $N(x \wedge y) = Nx \vee Ny \in M$, we get that $Nx \in M$ or $Ny \in M$. But this is a contradiction.

Hence $x \wedge y \in M_1$.

All the other cases are proved similarly.

Case (c):

(6) $V([x]) = 1$, $V([y]) = 0$; It is sufficient to show that $x \to y \in M_0$ that is, $x \to y \notin M$ and $N(x \to y) \in M$. By supposition, it holds that $x \in M$, $Nx \notin M$, $y \notin M$, and $Ny \in M$. That $x \to y \in M$ implies $Nx \vee y \in M$, because of $x \wedge (x \to y) \leq Nx \vee y$. Since M is the maximal filter, we get that $Nx \in M$ or $y \in M$ but this is a contradiction. Thus we have $x \to y \notin M$. In this case it is clear that $N(x \to y) \in M$. These mean that $x \to y \in M_0$.

(7) $V([x]) = 1$, $V([y]) = 1/2$; We show $x \to y \in M_{1/2}$, that is, $x \to y$ and $N(x \to y)$ are in M. By assumption we get that $x \in M$, $Nx \notin M$, and y, $Ny \in M$. Since $y \leq Nx \vee y \leq x \to y$, it follows $x \to y \in M$ by $y \in M$. On the other hand, since $x \wedge (x \to y) \leq Nx \vee y$, we obtain that $x \wedge Ny = N(Nx \vee y) \leq N(x \wedge (x \to y)) = Nx \vee N(x \to y) \in M$. That x, $Ny \in M$ yields that $Nx \vee N(x \to y) \in M$ and hence that $N(x \to y) \in M$ by $Nx \notin M$. So we have $x \to y \in M_{1/2}$.

(8) $V([x]) = V([y]) = 1$; In this case we shall show $x \to y \in M_1$. By assumption it follows that x, $y \in M$ and Nx, $Ny \notin M$. Since $y \leq Nx \vee y \leq x \to y$, we get $x \to y \in M$. If $N(x \to y) \in M$, then we have $Ny \in M$ by $N(x \to y) \leq Ny$. This contradicts to the assumption. Thus it follows that $N(x \to y) \notin M$ and hence that $x \to y \in M_1$.

(9) $V([x]) = 1/2$, $V([y]) = 0$; It sufficies to show that $x \to y \in M_{1/2}$, that is, $x \to y$ and $N(x \to y)$ are in M. By assumption we have that x, $Nx \in M$, $y \notin M$, and $Ny \in M$. The inequality $Nx \leq Nx \vee y \leq x \to y$ gives the result $x \to y \in M$. Since $x \wedge Ny \leq N(x \to y) \vee y$, we have that $N(x \to y) \vee y \in M$. That $y \notin M$ yields $N(x \to y) \in M$. Hence we get that $x \to y \in M_{1/2}$.

We can prove the other cases similarly, so we omit their proofs.

It follows the next theorem without difficulty by this lemma.

THEOREM 3 *Any i-Kleene algebra can be embedded to the simplest i-Kleene algebra* 3.

PROOF. The result is given by the composition map $V \circ \xi$ of $\xi: K \to K/\sim$ and $V: K/\sim \to 3$.

3 Kleene logic

In this section we shall define a Kleene logic KL. First of all, we use a countable set of propositional variables $p_1, p_2, \ldots, p_n, \cdots$ and logical symbols $\wedge$, $\vee$, $\neg$, and $\to$. We denote the set of propositional variables by Π, that is, $\Pi = \{p_1, p_2, \ldots, p_n, \ldots\}$. The formulas of KL are defined as usual. Let $A, B, C, \ldots$ be arbitrary formulas of KL. In the following we list an axiom system of KL.

Axioms;

(A1) $A \wedge B \rightarrow A$

(A2) $A \rightarrow A \vee B$

(A3) $A \wedge B \rightarrow B \wedge A$

(A4) $A \vee B \rightarrow B \vee A$

(A5) $A \rightarrow A$

(A6) $A \rightarrow A \wedge A$

(A7) $A \rightarrow A \vee A$

(A8) $(A \rightarrow B) \rightarrow (A \wedge C \rightarrow B \wedge C)$

(A9) $(A \rightarrow B) \rightarrow (A \vee C \rightarrow B \vee C)$

(A10) $\neg\neg A \rightarrow A$

(A11) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

(A12) $A \rightarrow (B \rightarrow A)$

(A13) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$

(A14) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$

(A15) $A \wedge \neg A \rightarrow B \vee \neg B$

(A16) $\neg A \vee B \rightarrow (A \rightarrow B)$

(A17) $A \wedge \neg A \wedge B \wedge \neg B \wedge \neg (A \rightarrow B) \rightarrow \neg (C \rightarrow C)$

(A18) $A \wedge \neg B \rightarrow \neg (A \rightarrow B) \vee B$

(A19) $A \wedge (A \rightarrow B) \rightarrow \neg A \vee B$

Rule of inference; B is deduced by A and $A \rightarrow B$ (modus ponens).

Let KL be a Kleene logic and A be a formula of KL. By $\vdash_{KL} A$ we mean that there is a sequence of formulas $A_1, A_2, \ldots, A_n$ of KL such that

(1) $A = A_n$

(2) For every A_i, it is an axiom or it is deduced by A_j and A_k $(j, k < i)$ by the rule of inference.

We say that A is provable in KL when $\vdash_{KL} A$. If no confusion arises, we simply denote it by $\vdash A$.

Remark: We abbriviate $p_0 \rightarrow p_0$ by t and $\neg t$ by f. In this case, it follows that $\vdash (A \rightarrow A) \rightarrow t$ and $\vdash t \rightarrow (A \rightarrow A)$ for every formula A. Now the axiom (A17) can be descibed by $A \wedge \neg A \wedge B \wedge \neg B \wedge \neg (A \wedge B) \rightarrow f$. We employ that formula as the axiom instead of (A17).

It is easy to show the next lemmas, so we omit their proofs.

Lemma 7 *If* $\vdash A \rightarrow B$ *and* $\vdash B \rightarrow C$, *then we have* $\vdash A \rightarrow C$.

Lemma 8 *For ervery formula* A *of* KL, *we have that* $\vdash A \rightarrow t$ *and* $\vdash f \rightarrow A$.

A function $\tau: \Pi \rightarrow 3$ is called a valuation function. The domain of the valuation function can be extended uniquely to the set of all formulas as follows:

$$\tau(A \wedge B) = \min\{\tau(A), \tau(B)\}$$

$$\tau(A \vee B) = \max\{\tau(A), \tau(B)\}$$

$$\tau(\neg A) = 1 - \tau(A)$$

$$\tau(A \rightarrow B) = \tau(A) \rightarrow \tau(B)$$

Henceforth we use the same symbol τ for the extended valuation function.

We can show that the Kleene logic KL is sound for i-Kleene algebras, that is, if $\vdash_{KL} A$ then $\tau(A) = 1$ for any valuation function τ.

THEOREM 4 *Let A be an arbitrary formula of KL. If $\vdash_{KL} A$ then $\tau(A) = 1$ for every valuation function τ.*

PROOF. By induction on the construction of a proof. It sufficies to show that $\tau(X) = 1$ for every axiom X and that $\tau(X) = \tau(X \rightarrow Y) = 1$ implies $\tau(Y) = 1$. We only show the latter. Suppose that $\tau(X) = \tau(X \rightarrow Y) = 1$. Since $\tau(X \rightarrow Y) = \min\{1 - \tau(X) + \tau(Y), 1\}$ and $\tau(X) = 1$, we have $\tau(Y) = 1$.

As corollaries to the theorem we have the following.

COROLLARY 1 *KL is consistent.*

PROOF. Since $\tau(\neg t) = 0$, the formula $\neg t$ is not provable in KL. Thus the Kleene logic is consistent.

COROLLARY 2 *The Kleene logic is different from the classical propositional logic (CPL) and the intuitionistic propositional logic (IPL).*

PROOF. If we think of a valuation function τ such that $\tau(p) = 1/2$ for any propositional variable p, then we have that $\tau(p \vee \neg p) = 1/2$ and hence that the formula $p \vee \neg p$ is not provable in KL. Thus the Kleene logic is different from CPL. Next, the formula $\neg\neg A \rightarrow A$ is not provable in IPL but it is provable in the Kleene logic. Thus IPL is not equal to the Kleene logic.

REMARK: Concerning CPL, we have a stronger result: If $\vdash_{KL} A$ then $\vdash_{CPL} A$, where $\vdash_{CPL} A$ means that A is provable in CPL.

4 Completeness Theorem

In this section we shall establish the completeness theorem of the Kleene logic KL, and it is the main theorem of this paper. The completeness theorem of KL means that a formula A is probable in KL provided $\tau(A) = 1$ for any valuation function τ. As a method to show the theorem, we define the Lindenbaum algebra of KL and investigate the property of that algebra.

Let Φ be the set of all formulas of KL. We introduce the relation $\equiv$ on Φ

as follows. For A, $B \in \Phi$,

$$A \equiv B \text{ iff } \vdash_{KL} A \to B \text{ and } \vdash_{KL} B \to A.$$

PROPOSITION 1 *The relation $\equiv$ is a congruent relation on Φ.*

PROOF. We only show that the relation $\equiv$ satisfies the conditions: If $A \equiv X$ and $B \equiv Y$, then

(a) $A \wedge B \equiv X \wedge Y$
(b) $NA \equiv NB$
(c) $A \to B \equiv X \to Y$

It is evident that the conditions (a) and (b) hold from axioms (8), (9), and (10). We prove the condition (c) holds. Suppose that $A \equiv X$ and $B \equiv Y$. For the condition (c), since $\vdash (B \to Y) \to ((A \to B) \to (A \to Y))$, we have $\vdash (A \to B) \to (A \to Y)$ by assumption $\vdash B \to Y$. Similarly it follows that $\vdash (A \to Y) \to ((X \to A) \to (X \to Y))$. By lemma 7, we obtain that $\vdash (A \to B) \to (X \to Y)$. A similar argument yields the converse $\vdash (X \to Y) \to (A \to B)$.

Hence the relation $\equiv$ is the congruent relation.

We put the quotient set L^* of Φ by the congruent relation $\equiv$. That is, we set $L^* = \{[A] \mid A \in \Phi\}$, where $[A] = \{X \in \Phi \mid A \equiv X\}$. We introduce an order relation $\sqsubseteq$ on L^* as follows: For any $[A]$, $[B] \in L^*$,

$$[A] \sqsubseteq [B] \text{ iff } \vdash_{KL} A \to B.$$

Since the relation $\equiv$ is congruent, it is clear that the definition of $\sqsubseteq$ is well-defined and that the relation $\sqsubseteq$ is a partial order. Concerning to this ordr we have

LEMMA 9 $\mathit{inf}\{[A], [B]\} = [A \wedge B]$, $\mathit{sup}\{[A], [B]\} = [A \vee B]$

PROOF. We shall show the first case for the sake of simplicity. The second case can be proved analogously.

Since $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, we obtain $[A \wedge B] \sqsubseteq [A], [B]$. For any $[C]$ such that $[C] \sqsubseteq [A], [B]$, since $\vdash C \to A$ and $\vdash C \to B$, it follows that $\vdash C \wedge B \to A \wedge B$ by $\vdash (C \to A) \to ((C \wedge B) \to (A \wedge B))$. Thus it means $\vdash B \wedge C \to A \wedge B$. On the other hand, $\vdash (C \to B) \to ((C \wedge C) \to (B \wedge C))$ and $\vdash C \to B$ yield $\vdash C \wedge C \to B \wedge C$. So we have $\vdash C \to B \wedge C$. These mean that $\vdash C \to A \wedge B$ and hence that $[C] \sqsubseteq [A \wedge B]$. Thus we have $\inf\{[A], [B]\} = [A \wedge B]$.

By the lemma we can define the operations $\sqcap$ and $\sqcup$ respectively by

$$[A] \sqcap [B] = \inf\{[A], [B]\} = [A \wedge B]$$
$$[A] \sqcup [B] = \sup\{[A], [B]\} = [A \vee B].$$

It is easy to show that the structure $(L^*, \sqcap, \sqcup)$ is a lattice. Moreover, if we put $[t]=1$, $[f]=0$, $N[A]=[\neg A]$, and $[A]\Rightarrow[B]=[A\rightarrow B]$, then the axioms of KL assures that the structure $(L^*, \sqcap, \sqcup, N, \Rightarrow, 0, 1)$ is an i-Kleene algebra. The structure is called a Lindenbaum algebra of KL. Hence we have the theorem.

THEOREM 5 *The Lindenbaum algebra L^* of the Kleene logic KL is the i-Kleene algebra.*

As to that algebra L^*, we have an important lemma.

LEMMA 10 *For every formula A, $\vdash_{KL} A$ iff $[A]=1$ in L^*.*

PROOF. Suppose that $\vdash A$. Since $A\rightarrow(t\rightarrow A)$ is provable in KL, we get that $\vdash t\rightarrow A$, that is, $[A]=1$. Conversely if we assume that $[A]=1$ then it follows $\vdash t\rightarrow A$ by definition. Thus we have $\vdash A$ by $\vdash t$.

Now we shall prove the completeness theorem of KL. In order to show that, it sufficies to indicate the existence of a valuation function τ such that $\tau(A)\neq 1$ if A is not provable in KL. Suppose that a formula A is not provable in KL. In the Lindenbaum algebra L^* of KL, we have $[A]\neq 1$ by the lemma above. It means that $N[A]\neq 0$. By lemma 1, there is a maximal filter M^* in L^* such that $N[A]\in M^*$. Using the filter M^* we define a valuation function τ. For any propositional variable p, we put

$$\tau(p)=\begin{cases} 1 & \text{if } [p]\in M^* \text{ and } N[p]\notin M^* \\ 1/2 & \text{if } [p]\in M^* \text{ and } N[p]\in M^* \\ 0 & \text{if } [p]\notin M^* \text{ and } N[p]\in M^*. \end{cases}$$

As to that function τ, we can show the next lemma without difficulty.

LEMMA 11 *For any formula $B\in\Phi$,*

$$\tau(B)=\begin{cases} 1 & \textit{if } [B]\in M^* \textit{ and } N[B]\notin M^* \\ 1/2 & \textit{if } [B]\in M^* \textit{ and } N[B]\in M^* \\ 0 & \textit{if } [B]\notin M^* \textit{ and } N[B]\in M^*. \end{cases}$$

PROOF. The same proof as that of lemma 6 gives the result.

Well, since $N[A]\in M^*$, it follows that $\tau(A)\neq 1$ by that lemma. Hence we have the completeness theorem of KL.

THEOREM 6 *For any formula A, $\vdash A$ iff $\tau(A)=1$ for every valuation function τ.*

It turns out from the theorem that it is sufficient to calculate the value of $\tau(A)$ whether the formula A is provable or not in KL. Since any formula has at most finite numbers of propositional variables, say n, the possible values of the n-tuple of the propositional variables in that formula are finite (at most 3^n). Thus we can establish that

THEOREM 7 *The Kleene logic is decidable.*

References

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