Mem. Fac. Sci. Shimane Univ.
28, pp.1-8 Dec. 28, 1994

Some Properties of Reproducing Kernels on an Infinite Network

Dedicated to Professor Fumi-Yuki Maeda on his 60th birthday

Atsushi MURAKAMI[a], Maretsugu YAMASAKI[b] and Yoshinori YONE-E[b]

a *Department of Mathematics, Hiroshima Institute of Technology, Hiroshima,* 731-51 *JAPAN*

and

b *Department of Mathematics, Faculty of Science, Shimane University, Matsue,* 690 *JAPAN*

(Received September 5, 1994)

It is shown that the extremal length of an infinite network satisfies a triangle inequality. A new formula for the extremal distance between two nodes will be given with the aid of a reproducing kernel. We shall discuss extremum problems related to Green functions.

§ 1. Introduction

Let $N=\{X, Y, K, r\}$ be an infinite network which is connected and locally finite and has no self-loop as in [5]. Here X is the countable set of nodes, Y is the countable set of arcs, K is the node-arc incidence matrix and r is a strictly positive real valued function on Y. Denote by $L(X)$ (resp. $L(Y)$) the set of all real valued functions on X (resp. Y) and by $L_0(X)$ (resp. $L_0(Y)$) the set of $u\in L(X)$ (resp. $w\in L(Y)$) with finite support.

The energy $H(w)$ of $w\in L(Y)$ is defind by

$$H(w):=\sum_{y\in Y}r(y)w(y)^2.$$

Let $L_2(Y;r)$ be the set of all $w\in L(Y)$ such that $H(w)<\infty$. For w_1, $w_2\in L_2(Y;r)$, their inner product $<w_1, w_2>$ is defined by

$$<w_1, w_2>:=\sum_{y\in Y}r(y)w_1(y)w_2(y).$$

For $u\in L(X)$, its discrete derivative $du\in L(X)$, its Dirichlet sum $D(u)$ and its Laplacian $\Delta u\in L(X)$ are defined by

$$du(y):=-r(y)^{-1}\sum_{x\in X}K(x,y)u(x)$$
$$=r(y)^{-1}[u(x^-(y))-u(x^+(y))],$$
$$D(u):=H(du)=\sum_{y\in Y}r(y)[du(y)]^2,$$
$$\Delta u(x):=\sum_{y\in Y}K(x,y)[du(y)],$$

where $x^-(y)$ (the initial node of y) and $x^+(y)$ (the terminal node of y) are determined uniquely by the relation :

$$K(x^-(y), y) = -1, K(x^+(y), y) = 1.$$

Denote by $\boldsymbol{D}(N)$ the set of all $u \in L(X)$ with finite Dirichlet sum, i.e.,

$$\boldsymbol{D}(N) := \{u \in L(X); D(u) < \infty\}.$$

For u, $v \in \boldsymbol{D}(N)$, the mutual Dirichlet sum $D(u, v)$ is defined by

$$D(u, v) := \langle du, dv \rangle = \sum_{y \in Y} r(y)[du(y)][dv(y)].$$

It is well-known that $\boldsymbol{D}(N)$ is a Hilbert space with respect to the inner product :

$$(\!(u, v)\!) := D(u, v) + u(x_0)v(x_0)$$

with a fixed node x_0. Denote by $\boldsymbol{D}_0(N)$ the closure of $L_0(X)$ in $\boldsymbol{D}(N)$ with respect to the norm :

$$\|u\| := [(\!(u, u)\!)]^{1/2} = [D(u) + u(x_0)^2]^{1/2}.$$

Note that $\boldsymbol{D}_0(N)$ does not depend on the choice of x_0.

For a fixed node x_0, put

$$\boldsymbol{D}(N; x_0) := \{u \in \boldsymbol{D}(N); u(x_0) = 0\}.$$

It is well-known that $\boldsymbol{D}(N; x_0)$ is a Hilbert space with respect to the inner product $D(u, v)$ (cf. [5]). For simplicity, put $X_0 = X - \{x_0\}$. For any $a \in X_0$, $u(a)$ is a continuous linear functional on $\boldsymbol{D}(N; x_0)$. In fact, there exists a constant M by [5; Lemma 1] which depends only on $\{x_0, a\}$ such that

$$|u(a)| \le M[D(u)]^{1/2} \text{ for every } u \in \boldsymbol{D}(N; x_0).$$

By Riesz representation theorem, there exists a unique reproducing kernel $k_a \in \boldsymbol{D}(N; x_0)$ such that

(K.1) $$u(a) = D(u, k_a) \text{ for every } u \in \boldsymbol{D}(N; x_0).$$

This k_a is called the Kuramochi kernel in [4].

In case N is of hyperbolic type (of order 2), there exists a unique function $g_a \in \boldsymbol{D}_0(N)$ such that

(G.1) $$v(a) = D(v, g_a) \text{ for every } v \in \boldsymbol{D}_0(N).$$

This g_a is called the Green function of N with pole at a in [6].

We shall show some roles of reproducing kernels k_a and g_a in the study of extremum problems on an infinite network.

§ 2. A triangle inequality for the extremal distance

Let a and b be two distinct nodes and denote by $P_{a,b}$ the set of all paths form a to b (cf. [5]).

The extremal distance $\lambda(a, b)$ of N between a and b is defined by

$$\lambda(a, b)^{-1} := \inf\{H(W);\ W \in EL(P_{a,b})\},$$

where $EL(P_{a,b})$ is the set of all non-negative $W \in L(Y)$ such that

$$\textstyle\sum_{y \in C_Y(P)} r(y) W(y) \geq 1 \text{ for all } P \in P_{a,b},$$

where $C_Y(P)$ is the ordered set of arcs in P (cf. [5]).

By definition, $\lambda(a, b) = \lambda(b, a)$.

We shall be concerned with the following triangle inequality for extremal distance :

$$\lambda(a, b) \leq \lambda(a, c) + \lambda(c, b) \tag{2.1}$$

for every distinct three nodes a, b and c.

In case N is a finite network, Duffin [1] showed a triangle inequality for the joint resistance.

Let us consider the following extremum problem :

(2.2) Minimize $D(u)$

subject to $u \in L(X)$, $u(a) = 0$, $u(b) = 1$.

Denote by $d(a, b)$ the value of Problem (2.2). Sometimes we call the optimal solution of Problem (2.2) the optimal solution for $d(a, b)$.

The following result was proved by Duffin [2] in case N is a finite network and by Yamasaki [5] in case N is an infinite network.

LEMMA 2.1. $\lambda(a, b) = d(a, b)^{-1}$.

We prepare

LEMMA 2.2. *There exists a unique optimal solutlon $\tilde{u}$ for $d(a, b)$. This $\tilde{u}$ has the following properties* :

$$D(\tilde{u}) < \infty,\ \tilde{u}(a) = 0,\ \tilde{u}(b) = 1; \tag{2.3}$$

$$0 \leq \tilde{u}(x) \leq 1 \text{ on } X; \tag{2.4}$$

$$\Delta \tilde{u}(x) = D(\tilde{u})[\varepsilon_a(x) - \varepsilon_b(x)], \tag{2.5}$$

where ε_a denotes the characteristic function of $\{a\}$, i.e., $\varepsilon_a(x) = 0$ *for* $x \neq a$ *and* $\varepsilon_a(a) = 1$.

PROOF. The existence and uniqueness of the optimal solution and Properties (2.3) and (2.5) follow from [5; Theorem 2]. Let $\tilde{u}$ be the optimal solution of Problem (2.2). Then

$|\tilde{u}|$ and $\min(\tilde{u}, 1)$ are also feasible solutions of Problem (2.2). By the relations :

$$D(|\tilde{u}|) \leq D(\tilde{u}) \text{ and } D(\min(\tilde{u}, 1)) \leq D(\tilde{u})$$

([5 ; Lemma 2]) and by the uniqueness of the optimal solution of Problem (2.2), we obtain $\tilde{u} = |\tilde{u}| = \min(\tilde{u}, 1)$, so that $0 \leq \tilde{u}(x) \leq 1$ on X.

The following result was shown in [4; Theorem 3.1].

LEMMA 2.3. *The function k_a has the following properties :*

(2.6) $\quad k_a(a) = D(k_a)$;

(2.7) $\quad k_a(b) = k_b(a)$ *for every* $a, b \in X_0$;

(2.8) $\quad \Delta k_a(x) = -\varepsilon_a(x) + \varepsilon_{x_0}(x)$ *for* $x \in X$.

LEMMA 2.4. *$k_a/k_a(a)$ is the optimal solution for $d(x_0, a)$ and $\lambda(x_0, a) = k_a(a)$. Furthermore,*

$$0 \leq k_a(x) \leq k_a(a) \text{ on } X.$$

PROOF. Let us put $\tilde{v} := k_a/k_a(a)$. Then by Lemma 2.3

$$\tilde{v}(x_0) = 0, \; \tilde{v}(a) = 1, \; D(\tilde{v}) = 1/D(k_a) = 1/k_a(a).$$

For any $u \in \boldsymbol{D}(N)$ such that $u(x_0) = 0$ and $u(a) = 1$, we have

$$D(u, \tilde{v}) = D(u, k_a)/k_a(a)^2 = 1/k_a(a) = D(\tilde{v})$$

by (K.1). Since $|D(u, \tilde{v})| \leq [D(u)]^{1/2}[D(\tilde{v})]^{1/2}$, we obtain $D(\tilde{v}) \leq D(u)$. Hence $d(x_0, a) = D(\tilde{v})$. By (2.4), $0 \leq \tilde{v} \leq 1$ on X, or $0 \leq k_a(x) \leq k_a(a)$ on X.

We shall prove

THEOREM 2.5. *Let $a, b \in X_0$ be two distinct nodes and put $u_{ab} := k_b - k_a$. Then $v_{ab} := [u_{ab} - u_{ab}(a)]/D(u_{ab})$ is the optimal solution for $d(a, b)$ and the following relation holds :*

$$\lambda(a, b) = D(u_{ab}) = k_a(a) - 2k_a(b) + k_b(b).$$

PROOF. Note that $D(v_{ab}) = 1/D(u_{ab}) < \infty$, $v_{ab}(a) = 0$. By the relation

$$u_{ab}(b) - u_{ab}(a) = k_b(b) - 2k_a(b) + k_a(a) = D(u_{ab}),$$

we have $v_{ab}(b) = 1$. Let $u \in \boldsymbol{D}(N)$ satisfy $u(a) = 0$ and $u(b) = 1$. Then $u' = u - u(x_0) \in \boldsymbol{D}(N; x_0)$ and

$$\begin{aligned} D(u, v_{ab}) &= D(u, u_{ab})/D(u_{ab}) \\ &= [D(u', k_b) - D(u', k_a)]/D(u_{ab}) \\ &= [u'(b) - u'(a)]/D(u_{ab}) \end{aligned}$$

$$=1/D(u_{ab})=D(v_{ab})$$

by (K.1). Thus $D(v_{ab})\leq D(u)$ and v_{ab} is the optimal solution for $d(a,b)$.

By Lemma 2.4 and Theorem 2.5, we have

THEOREM 2.6. *For distinct three nodes x_0, a and b.*

$$\lambda(a,b)=\lambda(x_0,a)+\lambda(x_0,b)-2k_a(b).$$

Since $k_a(x)\geq 0$ by Lemma 2.4, we obtain

THEOREM 2.7. *Let a, b and c be three distinct nodes. Then the triangle inequality (2.1) holds.*

§ 3. Extremum problems related to Green functions

In this section, *we always assume that N is of hyperbolic type*. Let a and b be two distinct nodes and consider the following extremum problem similar to Problem (2.2):

(3.1) Minimize $D(u)$

subject to $u\in \boldsymbol{D}_0(N)$ and $u(b)-u(a)=1$.

Denote by $\rho(a,b)$ the value of Problem (3.1).
We have

THEOREM 3.1. *Let $u^*_{ab}:=g_b-g_a$ and $v^*_{ab}:=u^*_{ab}/D(u^*_{ab})$. Then v^*_{ab} is an optimal solution of Problem* (3.1).

PROOF. Clearly, $v^*_{ab}\in \boldsymbol{D}_0(N)$ and $D(v^*_{ab})=1/D(u^*_{ab})$. We have

$$u^*_{ab}(b)-u^*_{ab}(a)=g_b(b)-2g_a(b)+g_a(a)$$
$$=D(g_b-g_a)=D(u^*_{ab})$$

by (G.1) and the symmetry of $g_a(b)$ (cf. [6; Theorem 3.3]). Therefore v^*_{ab} is a feasible solution of Problem (3.1). Let u be any feasible solution of Problem (3.1). By (G.1), we have

$$D(u,v^*_{ab})=[D(u,g_b)-D(u,g_a)]/D(u^*_{ab})$$
$$=[u(b)-u(a)]/D(u^*_{ab})$$
$$=1/D(u^*_{ab})=D(v^*_{ab}).$$

By the relation : $|D(u,v^*_{ab})|\leq[D(u)]^{1/2}[D(v^*_{ab})]^{1/2}$, we obtain $D(v^*_{ab})\leq D(u)$. Thus $\rho(a,b)=D(v^*_{ab})$ and v^*_{ab} is an optimal solution of Problem (3.1).

Related to Problems (2.2) and (3.1), we consider extremum problems on the sets of flows as in [5].

Let a and b be distinct nodes. We say that $w\in L(Y)$ is a flow from a to b if it satisfies

(3.2) $$\sum_{y\in Y}K(x,y)w(y)=0 \text{ for } x\in X-\{a, b\};$$

(3.3) $$I(w):=-\sum_{y\in Y}K(a, y)w(y)=\sum_{y\in Y}K(b, y)w(y).$$

We call $I(w)$ the strength of w.

Denote by $F(a, b)$ the set of all flows from a to b and by $F_2(a, b)$ the closure of $F(a, b)\cap L_0(Y)$ in the Hilbert space $L_2(Y;r)$.

Let us consider the following extremum problems :

(3.4) Minimize $H(w)$

subject to $w\in F_2(a, b)$ and $I(w)=1$.

(3.5) Minimize $H(w)$

subject to $w\in F(a, b)$ and $I(w)=1$.

Denote by $d_0^*(a, b)$ and $d^*(a, b)$ the values of Problems (3.4) and (3.5) respectively. We have by [5; Theorem 11]

LEMMA 3.2. $d(a, b)d_0^*(a, b)=1$.

We shall prove

THEOREM 3.3. *$\rho(a, b)d^*(a, b)=1$ holds and $\tilde{w}:=-d(g_b-g_a)$ is an optimal solution of Problem* (3.5).

PROOF. Let $w\in F(a, b)\cap L_2(Y;r)$ and $u\in \boldsymbol{D}_0(N)$ satisfy the conditions : $I(w)=1$ and $u(b)-u(a)=1$. Then

$$\begin{aligned}1=u(b)-u(a)&=\sum_{x\in X}u(x)\sum_{y\in Y}K(x, y)w(y)\\&=\sum_{y\in Y}w(y)\sum_{x\in X}K(x, y)u(x)\\&\leq[H(w)]^{1/2}[D(u)]^{1/2}\end{aligned}$$

by the same reasoning as in the proof of [5; Theorem 5]. Therefore we have

$$1\leq d^*(a, b)\rho(a, b).$$

Since $\tilde{w}\in F(a, b)$ and $I(\tilde{w})=1$, we have

$$d^*(a, b)\leq H(\tilde{w})=D(g_b-g_a)=\rho(a, b)^{-1}.$$

This completes the proof.

THEOREM 3.4. *$d(g_b-g_a)\in F_2(a, b)$ if and only if $d^*(a, b)=d_0^*(a, b)$ holds.*

PROOF. Note that $d^*(a, b)\leq d_0^*(a, b)$ holds in general. The "only if" part is clear. We prove the "if" part. There exists an optimal solution w_1^* of Problem (3.4). We see by[5 ;

Lemma 11] that there exists $v^*\in \boldsymbol{D}(N)$ such that $w_1^*=-dv^*$. Let $\tilde{w}:=-d(g_b-g_a)$ and $w_2^*:=\tilde{w}-w_1^*$. Then we see easily that there exists $h^*\in \boldsymbol{D}(N)$ such that $w_2^*=-dh^*$. Since $I(\tilde{w})=I(w_1^*)=1$, we have

$$\Delta h^*(x)=-\sum_{y\in Y}K(x,y)w_2^*(y)=0,$$

so that $h^*\in \boldsymbol{HD}(N):=\{u\in \boldsymbol{D}(N); \Delta u=0\}$. Since $g_b-g_a\in \boldsymbol{D}_0(N)$, we have by [6; Lemma 1.3]

$$<\tilde{w}, w_2^*>=D(g_b-g_a, h^*)=0,$$

so that $<w_1^*, w_2^*>=-H(w_2^*)$. Therefore

$$d^*(a,b)=H(\tilde{w})=<\tilde{w}, w_1^*>$$
$$=H(w_1^*)-H(w_2^*)=d_0^*(a,b)-H(w_2^*).$$

If $d^*(a,b)=d_0^*(a,b)$ holds, then $H(w_2^*)=0$, so that $w_2^*=0$, or equivalently, $\tilde{w}=w_1^*\in F_2(a,b)$.

§ 4. Miscellaneous remarks

Let us study some properties of the function defined by

$$\varphi(x):=k_x(x)=\lambda(x_0,x) \text{ for } x\in X_0;$$
$$\varphi(x_0):=0.$$

Let us introduce the following coefficients :

$$t(x,a):=\sum_{y\in Y}r(y)^{-1}|K(x,y)K(a,y)| \quad \text{for } x\neq a,$$
$$t(a,a)=0,$$
$$t(a):=\sum_{y\in Y}r(y)^{-1}|K(a,y)|.$$

Then we have $t(a)=\sum_{x\in X}t(x,a)$, $t(x,a)=t(a,x)$ and

$$\Delta u(a)=-t(a)u(a)+\sum_{x\in X}t(x,a)u(x). \tag{4.1}$$

THEOREM 4.1. *The following relations hold :*

$$\Delta\varphi(a)=\sum_{x\in X}t(x,a)\lambda(x,a)-2 \text{ for } a\neq x_0 \tag{4.2}$$

$$\Delta\varphi(x_0)=\sum_{x\in X}t(x,x_0)\lambda(x_0,x). \tag{4.3}$$

PROOF. (4.3) follows from (4.1) and $\varphi(x_0)=0$. To prove (4.2), let $a\in X_0$. By Lemma 2.4 and Theorem 2.6, for $x\neq a$

$$\varphi(x)=\lambda(a,x)-k_a(a)+2k_a(x). \tag{4.4}$$

We have by (4.1) and (4.4)

$$\begin{aligned}\Delta\varphi(a) &= -2t(a)k_a(a)+\sum_{x\in X}t(x,a)\lambda(a,x)\\ &\quad +\sum_{x\in X}t(x,a)2k_a(x)\\ &=\sum_{x\in X}t(x,a)\lambda(a,x)+2\Delta k_a(a)\\ &=\sum_{x\in X}t(x,a)\lambda(a,x)-2.\end{aligned}$$

This completes the proof.

In the case where N is the 2-dimensional lattice domain with $r=1$, it is well-known as in [3] that $\lambda(x,a)=1/2$ if x is a neighboring node of a. Since $t(x,a)=1$ if x is a neighboring node of a, we see that

$$\sum_{x\in X}t(x,a)\lambda(x,a)=2.$$

Flanders [3] called $\varphi(x)$ the fundamental solution.

References

[1] R. J. Duffin, Distributed and lumped networks, J. Math. Mech. 8 (1959), 793-826.

[2] R. J. Duffin, The extremal length of a network, J. Math. Anal. Appl. 5 (1962), 200-215.

[3] H. Flanders, Infinite networks : II—Resistance in an infinite grid, J. Math. Anal. Appl. 40 (1972), 30-35.

[4] A. Murakami, Kuramochi boundaries of infinite networks and extremal problems, Hiroshima Math. J. 24 (1994), 243-256.

[5] M. Yamasaki, Extremum problems on an infinite network, Hiroshima Math. J. 5 (1975), 223-250.

[6] M. Yamasaki, Discrete potentials on an infinite network, Mem. Fac. Sci. Shimane Univ. 13 (1979), 31-44.