Mem. Fac. Sci. Shimane Univ.
28, pp.27-31 Dec. 28, 1994

Remarks on Projectivity of Subsystems of Homogeneous Systems

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(Received September 5, 1994)

For two regular analytic homogeneous systems in projective relation, we investigate the projectivity for their normal subsystems.

Introduction

In the previous paper [14] we have shown that any homogeneous left loop is characterized by the homogeneous system associated with it, that is, a homogeneous left loop with the unit e is regarded as a homogeneous system with the origin e. Therefore, the theory of homogeneous left Lie loops on analytic manifolds can be replaced by the theory of analytic homogeneous systems with base points as their origins.

We have introduced in [7] the concept of projective relation between homogeneous left Lie loops, and investigated homogeneous left Lie loops in projective relation with certain special homogeneous left loops such as the additive group $\boldsymbol{R}^n$ ([7]), Lie groups ([8], [9], [10], [11]) and symmetric homogeneous Lie loops ([12], [13]). In these cases, the projectivity conditions are given in terms of the properties of their tangent Lie triple algebras.

In this paper, we consider normal subsystems of analytic homogeneous systems on connected and simply connected manifolds. When two homogeneous systems on the same underlying manifold are in projective relation, we investigate the projectivity for their normal subsystems. The results which will be given in terms of certain relations on the tangent Lie triple algebras are valid for normal left Lie subloops of homogeneous left Lie loops, by replacing homogeneous systems with homogeneous left loops as mentioned above.

§ 1. Main theorem

Let G be a connected analytic manifold, (G,η) and $(G,\tilde{\eta})$ analytic homogeneous systems on G with the same origin e. Their canonical connections ∇ and $\tilde{\nabla}$ induce on the tangent space $T_e(G)$ at e the tangent Lie triple algebras $\mathfrak{g}:=\{T_e(G)\,;\,S_e,\,R_e\}$ and $\tilde{\mathfrak{g}}:=\{T_e(G)\,;\,\tilde{S}_e,\,\tilde{R}_e\}$, respectivity, where S (resp.$\tilde{S}$) and R (resp. $\tilde{R}$) are the torsion tensor field and the curvature tensor field of the canonical connection ∇ (resp. $\tilde{\nabla}$). In[7], we have shown the following fact :

Assume that both of the homogeneous systems η and $\tilde{\eta}$ are geodesic, and that they are in projective relation. Then, the tensor fields $T=\nabla-\tilde{\nabla}$ and $-T$ are affine homogeneous structures (see [6]) of ∇ and $\tilde{\nabla}$, respectively. More precisely, the tensor field T is parallel with respect to both of the connections ∇ and $\tilde{\nabla}$ and the tensor field T, S and R satisfy the following relations ;

$$T(X,X)=0 \tag{1.1}$$

$$T(X,T(Y,Z))+T(Y,T(Z,X))+T(Z,T(X,Y))=0 \tag{1.2}$$

$$T(X,S(Y,Z))=S(T(X,Y),Z)+S(Y,T(X,Z) \tag{1.3}$$

$$T(X,R(Y,Z)W)=R(T(X,Y),Z)W+R(Y,T(X,Z))W+R(Y,Z)T(X,W) \tag{1.4}$$

$$R(X,Y)T(Z,W)=T(R(X,Y)Z,W)+T(Z,R(X,Y)W) \tag{1.5}$$

for any vector fields X, Y, Z, W on G.

In this case, the torsion tensor $\tilde{S}$ and $\tilde{R}$ of $\tilde{\nabla}$ are given by the following formulas (cf. [6] Prop. 1.1) ;

$$\tilde{S}(X,Y)=S(X,Y)+2T(X,Y) \tag{1.6}$$

$$\tilde{R}(X,Y)Z=R(X,Y)Z-T(S(X,Y),Z)-T(T(X,Y),Z). \tag{1.7}$$

The relations (1.1) and (1.2) induce on the tangent Lie triple algebra $\mathfrak{g}=\{T_e(G)\ ;S_e,R_e\}$ of η at e a Lie algebra structure $\mathfrak{t}$ with the bracket operation given by the affine homogeneous structure T at e, i. e.,

$$[X,Y]:=T_e(X,Y) \quad \text{for any } X, Y \text{ in } \mathfrak{g}. \tag{1.8}$$

The relations (1.3) and (1.4) show that any inner derivation of $\mathfrak{t}$ is a derivation of the Lie triple algebra $\mathfrak{g}$, while the relation (1.5) shows that any inner derivation of $\mathfrak{g}$ is a dervation of $\mathfrak{t}$.

Let $(H, \eta|_H)$ be a connected and closed normal subsystem of η through the origin e. Then the tangent Lie triple algebra $\mathfrak{h}$ of H is an ideal of the Lie triple algebra $\mathfrak{g}$ (cf. [4], [5]), that is,

$$S_e(\mathfrak{g},\mathfrak{h})\subset\mathfrak{h} \quad \text{and} \quad R_e(\mathfrak{g},\mathfrak{h})\mathfrak{g}\subset\mathfrak{h}. \tag{1.9}$$

Since any ideal of a Lie triple algebra is invariant, it satisfies the following :

$$R_e(\mathfrak{g},\mathfrak{g})\mathfrak{h}\subset\mathfrak{h}. \tag{1.10}$$

The main result of this paper is the following ;

THEOREM. *Let G be a connected and simply connected analytic manifold, η and $\tilde{\eta}$ be two regular geodesic homogeneous systems on G with the same origin e, which are in projective relation. Assume that a connected closed normal subsystem $(H, \eta|_H)$ is also a normal subsytem of $\tilde{\eta}$. Then, the tangent space $\mathfrak{h}$ of H at e forms an ideal of the Lie algebra $\mathfrak{t}$ associated with the affine homogeneous structure $T=\nabla-\tilde{\nabla}$, and two normal subsystems $\eta|_H$ and $\tilde{\eta}|_H$ are in projective relation under the induced affine homogeneous structure $T|_H$ on the submanifold H.*

§ 2. Projective double Lie algebras of Lie triple algebras

Let $\mathfrak{g}=\{V; XY, <X, Y, Z>\}$ be a Lie triple algebra with the underlying vector space V over a field F of characteristic zero. If a Lie algbra $\mathfrak{t}=\{V;[X, Y]\}$ on V satisfies the following relations, we call $\mathfrak{t}$ a *projective double Lie algebra* of the Lie triple algebra $\mathfrak{g}$:

(2.1) $$[X, UV]=[X, U]V+U[X, V]$$

(2.2) $$[X,<U, V, W>]=<[X, U], V, W>+<U, [X, V], W>+<U, V,[X, W]>$$

(2.3) $$<X, Y,[U, V]>=[<X, Y, U>,V]+[U, <X, Y, V>]$$

for any $X, Y, U, V, W\in V$

REMARK 1. If we denote the inner derivations of $\mathfrak{g}$ by

$$D(X, Y)Z=<X, Y, Z>,$$

the relations above can be given by

$$\mathrm{ad}_{\mathfrak{t}}\ \mathfrak{t} \subset \mathrm{Der}\ \mathfrak{g} \text{ and } D(\mathfrak{g}, \mathfrak{g})\subset \mathrm{Der}\ \mathfrak{t}.$$

In the case when the Lie triple algebra $\mathfrak{g}$ is reduced to a Lie algebra, i. e., $<X, Y, Z>=0$ for all X, Y, Z, any projective double Lie algebra on $\mathfrak{g}$ is reduced to one introduced in [11] for Lie algebras (cf. [1], [15]]).

Let $\mathfrak{t}=\{V;[X, Y]\}$ be a projective double Lie algebra of the Lie triple algebra $\mathfrak{g}=\{V; XY,<X, Y, Z>\}$.

PROPOSITION 1. *For any element p of the base field F, the bilinear operation*

(2.4) $$X* Y:=XY+2p[X, Y]$$

and the trilinear operation

(2.5) $$\ll X, Y, Z\gg\ :=<X, Y, Z>-p[XY, Z]-p^2[[X, Y],Z]$$

defined on the vector space V form a Lie triple algebra $\mathfrak{g}_p:=\{V ;\ X*Y, \ll X, Y, Z\gg\}$.

PROOF. Let $\mathfrak{S}_{X,Y,Z}$ denote the cyclic sum with respect to three elements X, Y and Z. By using the relation (2.1) we can show the following relation ;

(2.6) $$\mathfrak{S}_{X,Y,Z}[XY, Z]=\mathfrak{S}_{X,Y,Z}(X[Y,Z]).$$

From this we obtain one of the relations in the axiom of Lie triple algebra for $\mathfrak{g}_p$;

(2.7) $$\mathfrak{S}_{X,Y,Z}\{\ll X, Y, Z\gg+(X* Y)*Z\}=0.$$

On the other hand, we can get the following relation by using (2.2) ;

(2.8) $$\mathfrak{S}_{X,Y,Z}<[X, Y], Z, W>=<[Y, Z], X, W>-[<Y, Z, X>, W].$$

By summing up the left hand side and the right hand side cyclically with respect to X, Y, Z, we have

$$3\mathfrak{S}_{X,Y,Z}<[X, Y], Z, W>=\mathfrak{S}_{X,Y,Z}<[X, Y], Z, W>-[\mathfrak{S}_{X,Y,Z}<X, Y, Z>, W],$$

which shows the other relation of the axiom ;

$$\mathfrak{S}_{X,Y,Z}\ll X*Y, Z, W\gg=0 \tag{2.9}$$

The relations (2.1), (2.2) and (2.3) assure that the endomorphism $D_p(X, Y)$ of V given by

$$D_p(X, Y)Z:=\ll X, Y, Z\gg$$

is a derivation of both of the billinear operation $*$ and the trilinear operation $\ll\ ,\ ,\ \gg$. Thus, the remaining relations of the axiom are obtained.

q. e. d.

Now consider an ideal $\mathfrak{h}$ of the Lie triple algebra $\mathfrak{g}$, i. e., $\mathfrak{h}$ is a subspace of V satisfying

$$\mathfrak{g}\mathfrak{h}\subset\mathfrak{h} \text{ and } <\mathfrak{g}, \mathfrak{h}, \mathfrak{g}>\subset\mathfrak{h}.$$

By (2.4) and (2.5) we can easily show the following

PROPOSITION 2. *Let $\mathfrak{g}_p$ be as above. The ideal $\mathfrak{h}$ of $\mathfrak{g}$ is an ideal of $\mathfrak{g}_p$ if and only if it is an ideal of the projective double Lie algebra $\mathfrak{t}$ of the Lie triple algebra $\mathfrak{g}$.*

§ 3. Proof of the main theorem

Now we prove the main theorem mentioned in Section 1.

Let $\mathfrak{g}$ be the tangent Lie triple algebra of the regular homogeneous system (G, η) at the origin e, with the operations given by ;

$$XY=S_e(X, Y) \text{ and } <X, Y, Z>=R_e(X, Y)Z$$

for any tangent vectors X, Y, Z at e, where S_e and R_e denote the values of the torsion tensor S and the curvature tensor R of the canonical connection ∇ at e. Let $(G, \tilde{\eta})$ be the regular homogeneous system on G in projective relation with (G, η). Then, the (1, 2)-type tensor field $T=\nabla-\tilde{\nabla}$ is an affine homogeneous structure of ∇ satisfying the relations (1.1)-(1.5). Hence, on the tangent space $T_e(G)$ at e, the value of T at e induces a projective double Lie algebra $\mathfrak{t}=\{T_e(G): [X, Y]=T_e(X, Y)\}$ of the tangent Lie triple algebra $\mathfrak{g}$. Let $\tilde{\mathfrak{g}}$ be the tangent Lie triple algebra of $(G, \tilde{\eta})$ at e. By the relations (1.6) and (1.7), we see that $\tilde{\mathfrak{g}}$ is the Lie triple algebra obtained from $\mathfrak{g}$ by the projective double Lie algebra $\mathfrak{t}$ for $p=1$ in Proposition 1.

Let $(H, \eta|_H)$ be a connected closed normal subsystem of (G, η) passing through the origin e. Since η is geodesic (indeed regular), the submanifold H of G is an autoparallel with respect to the canonical connection ∇ of η, and the torsion and curvature of ∇ are induced on H in a natural manner. The Lie algebra of the holonomy group at e is generated by the inner derivation algebra $R_e(\mathfrak{g}, \mathfrak{g})$ because η is regular. In this case, we know that the tangent Lie triple algebra $\mathfrak{h}$ of $\eta|_H$ at e is an ideal of $\mathfrak{g}$. Assume that the ternary operation $\tilde{\eta}$ is closed in H and $(H, \tilde{\eta}|_H)$ is a normal subsystem of $\tilde{\eta}$. Then, H is autoparallel with respect to the canonical connection $\tilde{\nabla}$ too, and the tensor field $T = \nabla - \tilde{\nabla}$ can be restricted on the submanifold H. Therefore, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{t}$. Since $\mathfrak{h}$ is an ideal of the tangent Lie triple algebra $\tilde{\mathfrak{g}}$ of $\tilde{\eta}$, Proposition 2 implies that $\mathfrak{h}$ is an ideal of the projective double Lie algebra $\mathfrak{t}$. The same relations as (1.1)-(1.7) restricted on H must be valid, so the homogeneous systems $\eta|_H$ and $\tilde{\eta}|_H$ on H are in projective relation under the affine homogeneous structure $T|_H$. In fact, the projectivity relations (cf. [7]);

$$\eta(x, y, \tilde{\eta}(u, v, w)) = \tilde{\eta}(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)) \tag{3.1}$$

and

$$\tilde{\eta}(x, y, \eta(u, v, w)) = \eta(\tilde{\eta}(x, y, u), \tilde{\eta}(x, y, v), \tilde{\eta}(x, y, w)) \tag{3.2}$$

are valid on H since they are assured on G.

q. e. d.

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