Mem. Fac. Sci. Shimane Univ.
28, pp.19-26 Dec. 28, 1994

Representation Extension Properties of *CN*-Bands

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(Received September 5, 1994)

This is a continuation of the paper "*CN*-band which are semigroup almalgamation bases". In that paper, we found necessary conditions for *CN*-bands to have the represenatation extension propery. In this paper, we prove that a finite *CN*-band S has the representation extension property if S satisfies all of those necessary conditions.

§ 1. Introduction and the main theorem

T. E. Hall [4] originally studied the representation extension property (REP) of semigroups in connection with semigroup amalgamations. Since then, several authors ([1], [2], [5], [6], [7], [8]) have studied the properties (REP) of specific semigroups (inverse semigroups, primitive regular semigroups, commutative semigroups and so on). Among them, S. Bulman-Fleming and K. McDowell [2] determined the structure of normal bands with (REP). In the previous paper, we introduction *CN*-bands whose class is slightly larger than one of normal bands and investigated the property (REP) and its dual $(REP)^{OP}$ of them. Consequently, we found the five necessary conditions $(C.1)$ through $(C.5)$ for *CN*-bands to have (REP) and proved that any finite *CN*-band S has (REP) if S satisfies the five conditions $(C.1)$ through $(C.5)$ and an additional condition $(C.0)$. The purpose of the present paper is to prove that

THE MAIN THEOREM. *Let S be a finite CN-band.*

Then S has (REP) if and only if S satisfies the following conditions $(C.1)$ through $(C.5)$:

$(C.1)$ *If $\alpha>\beta$, and Mul (S_α, S_β) is normal, then there exists $r\in S$ such that*

$$ru=ur=a \text{ for all } u\in S_\alpha \text{ and } a\in S_\beta \text{ with } u>a.$$

$(C.2)$ *If $\alpha>\beta(\alpha,\beta\in\Lambda)$, S_α contains distinct u, v with $u\mathscr{R}v$, S_β contains distinct a, b with $a\mathscr{L}b$ and Mul (S_α, S_β) is commutative, then there exists $\gamma\in\Lambda$ such that $\alpha>\gamma>\beta$ and Mul (S_γ, S_β) is commutative.* *)

$(C.3)$ *For any distinct $u_1, u_2, \cdots, u_n (n\geq 2)$, $a\in S$ such that $u_i\mathscr{R}u_j$ $(1\leq i,j\leq n)$, $u_n>a$, there exists $\gamma\in S$ such that $ru_1=ru_i$ $(1\leq i\leq n)$, but $ru_n\neq ra$.*

$(C.4)$ *For any pair of disjoint subsets A_1, A_2 from an $\mathscr{R}$-class of S such that $|A_1|\geq 2$ or $|A_2|\geq 2$, then there exists $\gamma\in S$ such that $|rA_1|=|rA_2|=1$, but $rA_1\neq rA_2$.*

$(C.5)$ *For any distinct a, b, $c\in S$ with $a\mathscr{R}b\mathscr{L}c$, there exists $r\in S$ such that rb, $rc\in Sa$, but rb*

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*) This is a corrected form of the statement of [8, condition $(C.2)$].

$\neq rc$.

Our proof of the main theorem is obtained by improving the proof of [8, Theorem 2].

§ 2. Definitions and preliminary results.

Throughout this paper, let S denote a semigroup and S^1 the semigroup with adjoined identity 1 if S do not have identity. We will use the notations and conventions from Clifford & Preston's book [3] for semigroup theory. Let S-*Ens* (*Ens*-S, S-*Ens*-S) denote the category of all left S-sets (right S-sets, S-bisets). Let $X\in Ens\text{-}S$, $Y\in S\text{-}Ens$. The *tensor product* over S of X and Y is denoted by $X\otimes_S Y$ (simply, $X\otimes Y$ if there is no confusion). Also, any element of $X\otimes Y$ is written in a form $x\otimes y (x\in X, y\in Y)$. For brevity, $X\supset Y$ (X, $Y\in S\text{-}Ens(Ens\text{-}S, S\text{-}Ens\text{-}S)$) means that Y is a left S-(right S-, S-bi) subset of X.

RESULT 1 ([2, Lemma 1.2]). *Let $A\in Ens\text{-}S$, $B\in S\text{-}Ens$. Then $a\otimes b=a'\otimes b'$ in $A\otimes B$ if and only if there exists $a_1, \cdots, a_n\in A$, $b_2, \cdots, b_n\in B$, $s_1, \cdots, s_n$ and $t_1, \cdots, t_n\in S^1$ such that*

$$(1.1)\qquad \begin{array}{rlrl} a &= a_1s_1, & s_1b &= t_1b_2 \\ a_1t_1 &= a_2s_2, & s_2b_2 &= t_2b_3 \\ &\vdots & &\vdots \\ a_{n-1}t_{n-1} &= a_ns_n, & s_nb_n &= t_nb' \\ a_nt_n &= a' & & \end{array}$$

Then we call the system of equations (1.1) *a scheme of length n over A and B joining (a, b) to (a', b').*

DEFINITION ([1] [4] [5]). We say that a semigroup S has the *representation extension property* (*REP*) if for every emedding $S \longrightarrow T$ of semigroups and every right S-set X, the canonical map : $X \longrightarrow X\otimes T^1$ is injective. The left-right dual of (*REP*) is denoted by $(REP)^{op}$.

RESULT 2 ([7, Theorem 2.1]). *A monoid (semigroup) S has (REP) if and only if for each $M\in S\text{-}Ens$ with $M\supset S(S^1)$ and each $X\in Ens\text{-}S$, the map : $X \longrightarrow X\otimes M$ ($x \longrightarrow x\otimes 1$) is injective.*

DEFINITION ([8]). Let S be a band and $\cup\{S_\alpha : \alpha\in\Lambda\}$ the semilattice decomposition of S. Then S is called a *CN-band* if for each α, $\beta\in\Lambda$ with $\alpha>\beta$, there are only the following two types of multiplication between S_α and S_b :

I. (Commutative type) $ua=au$ for all $u\in S_\alpha$ and $a\in S_\beta$.

II. (Normal type) $|uS_\beta u|=1$ for all $u\in S_\alpha$.

Hereafter we will describe that $Mul(S_\alpha, S_\beta)$ is *commutative* [ressp. *normal*] if there happens multipication of type I [resp. II].

§ 3. The proof of the main theorem.

Throughout this section, we let S be a finite *CN*-band and $S=\cup\{S_\lambda \mid \lambda\in\Lambda\}$ the semilattice decomposition of S.

Before proceed to prove the main theorem, we whall give preliminary lemmas.

LEMMA 1. *Let S be a finite CN-band satisfying $(C.2)$. Then S satisfies the following condition :*

$(C.2.1)$ If $\alpha>\beta$, S_α is not an $\mathscr{L}$-class, S_β is not an $\mathscr{R}$-class, and $Mul(S_\alpha, S_\beta)$ is commutative, then there exists $\gamma\in\Lambda$ such that $\alpha>\gamma>\beta$, $|S_\gamma|=1$, $Mul(S_\alpha, S_\gamma)$ is normal and $Mul(S_\gamma, S_\beta)$ is commutative.

PROOF. Since S is a finite *CN*-band, $(C.2.1)$ follows immediately from $(C.2)$.

REMARK. The condition $(C.2.1)$ is a slightly generalization of [8, Condition $(C.2)$']．

The following lemma was proved in [8].

LEMMA 2. *Let S be as above and $a, u\in S$ with $\mathscr{J}_u>\mathscr{J}_a$. Let $X\in Ens\text{-}S$, $Y\in S\text{-}Ens$, $x, x'\in X$, $y, y'\in Y$.*

Then (i) $xu=x'v$ $(v\in\mathscr{J}_u)$ implies $xuau=x'vav$.

(ii) $uy=vy'$ $(v\in\mathscr{J}_u)$ implies $uauy=vavy'$.

LEMMA 3. *Let S be as above. Given $x\otimes y=x'\otimes y'$ in $X\otimes_S Y$. There exists a scheme over X and Y joining (x, y) to (x', y') as follows :*

$x_1, \cdots, x_n\in X$, $y_2, \cdots, y_n\in Y$, $s_1, \cdots, s_n$ *and* $t_1, \cdots, t_n\in S^1$ *such that*

$$(2.1)\qquad \begin{array}{ll} x=x_1s_1, & s_1y=t_2y_2 \\ x_1t_1=x_2s_2, & s_2y_2=t_2y_3 \\ \vdots & \vdots \\ x_{n-1}t_{n-1}=x_ns_{n'} & s_ny_n=t_ny' \\ x_nt_n=x' & \end{array}$$

and

$$s_1\geq_{\mathscr{J}} t_1\geq_{\mathscr{J}} \cdots \geq_{\mathscr{J}} s_i\geq_{\mathscr{J}} t_i\leq_{\mathscr{J}} s_{i+1}\leq_{\mathscr{J}} t_{i+1}\leq_{\mathscr{J}} \cdots \leq_{\mathscr{J}} s_n \leq_{\mathscr{J}} t_n$$

$$(or\ s_1\geq_{\mathscr{J}} t_1\geq_{\mathscr{J}} \cdots \geq_{\mathscr{J}} s_i\leq_{\mathscr{J}} t_i\leq_{\mathscr{J}} s_{i+1}\leq_{\mathscr{J}} t_{i+1}\leq_{\mathscr{J}} \cdots \leq_{\mathscr{J}} s_n\leq_{\mathscr{J}} t_n)$$

In this case, we say that shceme (2.1) is V-formed.

PROOF. As shown in the proof of [8, Lemma 8], we can assume that any adjacent two elements of sequence $s_1, t_1, \cdots, s_n, t_n$ are comparable w.r. t. $\leq_{\mathscr{J}}$. We deal next with the following cases :

$$\text{Case 1}: s_i <_{\mathcal{J}} t_i \mathcal{J} \cdots \mathcal{J} t_{j-1} \mathcal{J} s_j >_{\mathcal{J}} t_j.$$

Set

$$t'_k = t_k s_i t_k, \qquad s'_{k+1} = s_{k+1} s_i s_{k+1'}$$

$$t''_k = t_k t_j t_{k'}, \qquad s''_{k+1} = s_{k+1} t_j s_{k+1}$$

$$t^*_k = t_k s_i s_j t_j t_k, \qquad s^*_{k+1} = s_{k+1} s_i s_j t_j s_{k+1} \quad (i \leq k \leq j-1).$$

Then by Lemma 2 we have

$$x_i t'_i = x_{i+1} s'_{i+1}, \qquad s'_{i+1} y_{i+1} = t'_{i+1} y_{i+2}$$

$$\vdots \qquad\qquad \vdots$$

$$x_{j-1} t'_{j-1} = x_j s'_j, \qquad s'_j y_j = (s'_j s_j y_j = s'_j s_j t_j s_j y_j) = s^*_j y_j$$

$$x_j s^*_j = x_{j-1} t^*_{j-1}, \qquad t^*_{j-1} y_j = s^*_{j-1} y_{j-1}$$

$$\vdots \qquad\qquad \vdots$$

$$x_{i+2} s^*_{i+2} = x_{i+1} t^*_{i+1}, \qquad t^*_{i+1} y_{i+2} = s^*_{i+1} y_{i+1}$$

$$x_{i+1} s^*_{i+1} = x_i t^*_i.$$

Here we shall show

$$t^*_i y_{i+1} = t''_i y_{i+1}. \tag{2.2}$$

Proof of (2.2) :

Subcase 1.1 : $Mul(\mathcal{J}_{t_i}, \mathcal{J}_{s_i t_j})$ is normal. Then $t_i(s_i s_j t_j) t_i = t_i (t_j s_i) t_i$. In this case

$$t^*_i y_{i+1} = t_i t_j s_i t_i y_{i+1} = t_i t_j t_i y_{i+1} = t''_i y_{i+1}.$$

Subcase 1.2 : $Mul(\mathcal{J}_{t_i}, \mathcal{J}_{s_i t_j})$ is commutative and $\mathcal{J}_{s_i t_j} = \mathcal{R}_{s_i t_j}$. Then

$$t^*_i y_{i+1} = t_i s_i s_j t_j t_i y_{i+1} = (t_i s_i s_j t_j)(t_i s_i t_i) y_{i+1} = t_i (t_j t_i s_i) t_i y_{i+1} = t_i t_j t_i y_{i+1} = t''_i y_{i+1}.$$

Subcase 1.3 : $Mul(\mathcal{J}_{t_i}, \mathcal{J}_{s_i t_j})$ is commutative and $\mathcal{J}_{s_i t_j} \neq \mathcal{R}_{s_i t_j}$. If $\mathcal{J}_{t_i} = \mathcal{L}_i$, then it follows from (2.1) that $t_k s_i y_i = t_k y_{k+1}$, $s_{k+1} s_i y_i = s_{i+1} y_{k+1}$ $(i+1 \leq k \leq j-1)$. Hence $s_i y_i = t_i y_{i+1} = t_i y_i = t_i t_j y_{j+1}$. So we get

$$t^*_i y_{i+1} = (t_i s_i s_j t_j t_i) y_{i+1} = (t_i s_i s_j t_j)(s_i y_i) = (t_i s_i s_j t_j)(t_i t_j s_i) y_i = (t_i s_i)(s_i) y_i = s_i y_i$$

$$= t_i y_{i+1} = (t_i t_j)(t_i y_{i+1}) = t''_i y_{i+1}.$$

so we can assume that $\mathcal{J}_{t_i} \neq \mathcal{L} t_{t_i}$. By (C.2.1) there exists $\lambda \in \Lambda$ such that $S_\lambda \mathcal{J}_{t_i} \subset S_\lambda$, S_λ is an $\mathcal{L}$-class and $Mul(S_\lambda, \mathcal{J}_{s_i t_j})$ is commutative. Let $c \in S_\lambda$ be fixed. Since $S_\lambda = \mathcal{L}_c$, we have $c = cu = cv$ for all u, $v \in \mathcal{J}_{t_i}$. So it follows from (2.1) that

$$cs_iy_i=ct_iy_{i+1}=\cdots=ct_jy_j=ct_jy_{j+1},\ \text{say},\ z.$$

Consequently, $z=ct_iz=cs_jz=ct_jz$. Let $e=cs_ict_j\in\mathscr{J}_{s_it_j}$. Then by the above, $z=ez$. Now we can show that $z=dz$ for all $d\in\{s_i,t_i,s_j,t_j\}$. For, since $de\in\mathscr{J}_{s_it_j}$ and $Mul(S_\lambda,\mathscr{J}_{s_it_j})$ is commutative, by [8, Lemma 1 (i)], we get $cde=de$, so that $z=ez=cdez=dez=dz$. Eventually

$$t_i^*y_{i+1}=(t_is_is_jt_jt_i)y_{i+1}=((t_it_jt_is_i)c)t_iy_{i+1}=(t_is_is_jt_j)z=z$$

and

$$t_i''y_{i+1}=(t_it_jt_i)(t_iy_{i+1})=(t_it_jt_i)(t_is_it_i)y_{i+1}=((t_it_jt_i)c)(t_iy_{i+1})=(t_it_jt_is_i)z=z$$

so that $t_i^*y_{i+1}=t_i''y_{i+1}$. In all the cases $t_i^*y_{i+1}=t_i''y_{i+1}$.

Furthermore by Lemma 2, we have

$$\begin{array}{ll} x_it_i''=x_{i+1}s_{i+1}'', & s_{i+1}''y_{i+1}=t_{i+1}''y_{i+2} \\ \vdots & \vdots \\ x_{j-1}t_{j-1}''=x_js_j'', & s_j''y_j\,(=s_jy_j)=t_jy_{j+1}. \end{array}$$

Thus we obtain the required scheme.

Case 2 : $t_i<_{\mathscr{J}} s_{i+1}\ \mathscr{J}\ \cdots\ \mathscr{J}\ t_{j-1}\ \mathscr{J}\ s_j>_{\mathscr{J}} t_j$.

Set

$$\begin{array}{ll} s_{k+1}'=s_{k+1}t_is_{k+1}, & t_{k+1}'=t_{k+1}t_it_{k+1} \\ s_{k+1}''=s_{k+1}t_js_{k+1}, & t_{k+1}''=t_{k+1}t_jt_{k+1} \\ s_{k+1}^*=s_{k+1}t_is_jt_js_{k+1}, & t_{k+1}^*=t_{k+1}t_is_jt_jt_{k+1}\quad (i\leq k\leq j-1) \end{array}$$

Then by Lemma 2 we have

$$\begin{array}{ll} x_it_i=x_{i+1}s_{i+1}', & s_{i+1}'y_{i+1}=t_{i+1}'y_{i+2} \\ \vdots & \vdots \\ x_{j-1}t_{j-1}'=x_js_j', & s_j'y_j=(s_j's_jy_j=s_j's_jt_js_jy_j)=s_j^*y_j \\ x_js_j^*=x_{j-1}t_{j-1}^*. & t_{j-1}^*y_j=s_{j-1}^*y_{j-1} \\ \vdots & \vdots \\ x_{i+2}s_{i+2}^*=x_{i+1}t_{i+1}^*, & t_{i+1}^*y_{j+2}=s_{i+1}^*y_{i+1}. \end{array}$$

Here we shall show

(2.3) $$x_{i+1}s_{i+1}^*=x_{i+1}s_{i+1}''$$

Proof of (2.3) :

Subcase 1.1 : $Mul(\mathscr{J}_{s_{i+1}}, \mathscr{J}_{s_it_j})$ is normal. Then $s_{i+1}(t_is_jt_j)s_{i+1}=s_{i+1}(t_jt_i)s_{i+1}$ and so

$$x_{i+1}s^*_{i+1}=x_{i+1}s_{i+1}(t_is_jt_j)s_{i+1}=x_{i+1}(s_{i+1}t_it_js_{i+1})=x_{i+1}(s_{i+1}t_js_{i+1})=x_{i+1}s''_{i+1}.$$

Subcase 1.2 : $Mul(\mathscr{J}_{s_{i+1}}, \mathscr{J}_{s_it_j})$ is commutative. Then $s_jt_jt_is_j=t_jt_i$. Hence $t_is_jt_j=(t_is_jt_j)(t_is_jt_j)=t_i(t_jt_i)t_j=t_it_j$. Thus

$$x_{i+1}s^*_{i+1}=x_{i+1}(s_{i+1}t_is_jt_js_{i+1})=x_{i+1}(s_{i+1}t_it_js_{i+1})=x_{i+1}(s_{i+1}t_js_{i+1})=x_{i+1}s''_{i+1}.$$

In all the cases, (2.3) holds.

By Lemma 2, we have

$$\begin{array}{ll} x_it''_i=x_{i+1}s''_{i+1}, & s''_{i+1}y_{i+1}=t''_{i+1}y_{i+2} \\ \vdots & \vdots \\ x_{j-1}t''_{j-1}=x_js''_j, & s''_jy_j(=s_jy_j)=t_jy_{j+1}. \end{array}$$

Thus we obtain the required scheme.

Case 3 : $s_i<_{\mathscr{J}} t_i \mathscr{J} \cdots \mathscr{J} t_{j-1}>_{\mathscr{J}} s_j$. By reversive ordering the equations (2.1), we reduce the case to Case 2.

Case 4 : $t_i<_{\mathscr{J}} s_{i+1} \mathscr{J} \cdots \mathscr{J} t_{j-1}>_{\mathscr{J}} s_j$. Again, this case is essentially the same as case 1. The proof of the lemma is complete.

We are now in a position to prove the main theorem.

The proof of the main thorem : Assume that S satisfies the following conditions $(C.1)$, $(C.2)$, $(C.3)$, $(C.4)$, $(C.5)$. We shall show that S has (REP). Let W be a left S-set containing S as a S-subset. Assume that $x\otimes 1=x'\otimes 1$ in $X\otimes W$. By lemma 3, there exists a V-formed scheme joining $(x,1)$ to $(x',1)$; that is, there exist $x_1, \cdots, x_n\in X$, $y_2, \cdots, y_n\in W$, $s_1, \cdots, s_n$ and $t_1, \cdots, t_n\in S^1$ such that

$$\text{(2.4)}\quad \begin{array}{ll} x=x_1s_1, & s_1=t_1y_2 \\ x_1t_1=x_2s_2, & s_2y_2=t_2y_3 \\ \vdots & \vdots \\ x_{n-1}t_{n-1}=x_ns_n, & s_ny_n=t_n \\ x_nt_n=x' & \end{array}$$

Then we may assume that scheme (2.4) is a V-formed one of the shortest length. Moreover we may assume that

(2.5) the number of the $\mathscr{J}$-classes which contain any one of s_i, t_i appearing in (2.4) is the least.

Now we proceed to prove that $x=x'$. Suppose first that all s_i 's, t_i 's belong to an $\mathscr{J}$-class, say, J. Then we can assume

(2.6) $$s_i \ \mathcal{R}\ t_i\ (1 \leq i \leq n-1),\ s_n\ \mathcal{R}\ t_n.$$

For, it is clear that $s_1 \mathcal{R} t_1$. If $s_i \mathcal{R} t_i\,(1\leq i\leq k-1)$, but $s_k \not\mathcal{R}\, t_k$, then $s_k s_i \mathcal{R} s_k t_i\,(1\leq i\leq k-1)$. $s_k \neq t_k s_k$. By $(C.4)$, there exists $a\in S$ such that $as_k s_1 = as_k t_1 = as_k s_2 = as_k t_2 = \cdots = as_k s_{k-1} = as_k t_{k-1} = as_k s_k \neq at_k s_k$. By (2.4), we have $as_k s_1 = at_k s_k$, which is a contradiction. This proves that $s_i \mathcal{R} t_i\,(1 \leq i\leq n-1)$, $s_n \mathcal{R} t_n$. Then by virtue of (2.6), we get from (2.4)

$$xs = x_1 s = x_2 s = \cdots = x_{n-1} s = x_n s \ \text{ for all } s \in J.$$

Hence

$$xt_i = xs_{i+1} = x' t_i = x' s_{i+1} \quad (1 \leq i \leq n-1).$$

Let $A = \{s_1, t_1, \cdots, s_n, t_n\}$ and ρ an equivalence on A generated by $\{(t_i, s_{i+1}) \mid 1 \leq i \leq n-1\}$. If $s_1 \rho t_n$, then it follows from the above that $xs_1 = xt_n$, so that $x = xs_1 = xt_n = x' t_n = x'$. So we can assume that $(s_1, t_n) \notin \rho$. Let A_1 be the ρ-class containing s_1 and $A_2 = A - A_1$. If $A_1 = \{s_1\}$, $A_2 = \{t_1\}$, then $t_i = s_{i+1}\,(1 \leq i \leq n-1)$, and by (2.4), $s_1 = t_1 y_2 = s_2 y_2 = \cdots = t_{n-1} y_n = s_n y_n = t_n$, a contradiction. Hence we have $|A_1| \geq 2$ or $|A_2| \geq 2$. By $(C.4)$ there exists $t \in S$ such that $|tA_1| = |tA_2| = 1$ but $tA_1 \neq tA_2$. Note that $|tA_1| = |tA_2| = 1$ implies that $tt_i = ts_{i+1}\,(1 \leq i \leq n-1)$. So we obtain $ts_1 = tt_1 y_2 = ts_2 y_2 = \cdots = tt_{n-1} y_n = ts_n y_n = tt_n$. This is a contradiction.

Suppose next that all s_i's, t_i's do not belong to a $\mathcal{J}$-class. Since (2.4) is V-formed, we can assume that $s_1 \not\leq t_n$. So there happens following two cases :

Case 1 : $s_1\ \mathcal{J}\ t_1\ \mathcal{J} \cdots \mathcal{J}\ s_{p-1}\ \mathcal{J}\ t_{p-1} >_{\mathcal{J}} s_p$. By the same way as in the proof of (2.6), we can prove that

$$s_1\ \mathcal{R}\ t_1\ \mathcal{R} \cdots \mathcal{R}\ s_{p-1}\ \mathcal{R}\ t_{p-1}.$$

By multiplying the right side of (2.4) by s_1 from the left, we get $x = x_p s_p s_1$, so that $x = x_1(s_1 s_p s_1)$. Set

$$s_k' = s_k s_p s_k,\ t_k' = t_k s_p t_k \ (1 \leq k \leq p-1).$$

By applying Lemma 3 to (2.4), we obtain

$$\begin{aligned} x &= x_1 s_1', & s_1' &= t_1' y_2 \\ x_1 t_1' &= x_2 s_2', & s_2' y_2 &= t_2' y_3 \\ &\ \vdots & &\ \vdots \\ & & s_{p-1}' y_{p-1} &= t_p' y_p \\ x_{p-1} t_{p-1}' &= x_p s_p. & & \end{aligned}$$

Then we can reduce the number of the $\mathcal{J}$-classes $\mathcal{J}_{s_i}$, $\mathcal{J}_{t_i}$. By assumption (2.5), we are done.

Case 2 : $s_1\ \mathcal{J}\ t_1\ \mathcal{J} \cdots \mathcal{J}\ s_{p-1}\ \mathcal{J}\ t_{p-1}\ \mathcal{J}\ s_p >_{\mathcal{J}} t_p$.

Set

$$\hat{s}_k = s_k t_p s_k,\ \hat{t}_k = t_k t_p t_k\quad (1 \leq k \leq p).$$

By applying Lemma 3 to the right side of (2.4), we obtain

$$s_1 = s_1 t_1 y_2,\ s_1 s_2 y_2 = s_1 t_2 y_3,\ \cdots,\ s_1 s_p y_p = s_1 \hat{s}_p y_p$$

(2.7) and

$$s_1 \hat{t}_{p-1} y_p = s_1 \hat{s}_{p-1} y_{p-1},\ \cdots,\ s_1 \hat{t}_1 y_2 = \hat{s}_1.$$

Now by $(C.3)$, there exists $b \in S$ such that

$$bs_1 s_j = bs_1 t_j = bs_1 s_p\quad (1 \leq j \leq p),\ \text{but}\ bs_1 \neq b\hat{s}_1.$$

Then from (2.7) we get

$$bs_1 (= bs_1 t_2 y_3 = \cdots = bs_1 s_p y_p) = bs_1 \hat{s}_p y_p.$$

In particular, $bs_1 \mathscr{J} b\hat{s}_1$.

Here we divide argument into two parts :

Subcase 1 : $Mul(\mathscr{J}_{s_1}, \mathscr{J}_{bs_1})$ is normal. Then $bs_1 = bs_1 bs_1 = bs_1 (bs_1) s_1 = bs_1 (\hat{s}_1 b_{s_1}) s_1 = b\hat{s}_1 b\hat{s}_1 = b\hat{s}_1$, a contradiction.

Subcase 2 : $Mul(\mathscr{J}_{s_1}, \mathscr{J}_{bs_1})$ is commutative. Then we get $bt_p = bt_b u = b(ut_p u)$ for all $u \in \mathscr{J}_{s_1}$, since $but_p = (but_p)^2 = b(ut_p bu)t_p = b(t_p b)t_p = bt_p$. Especially, we have $b\hat{s}_j = b\hat{t}_j\ (1 \leq j \leq p)$. Whence it follows from (2.7) that $bs_1 = b\hat{s}_p y_p = b\hat{t}_p y_{p+1} = b\hat{s}_{p-1} y_{p-1} = \cdots = b\hat{t}_1 y_2 = b\hat{s}_1$, which is a contradiction, again. The proof of the theorem is complete.

In a sequent paper, we will study the problem “Is a CN-band with (REP) and $(REP)^{OP}$ a semigroup amalgamation base?”.

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