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Identities for Idempotents of Generalized Inverse [∗-] Semigroups

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It is well-known that an orthodox semigroup S is a generalized inverse semigroup if the set $E(S)$ of idempotents of S satisfies one of the following identities : (I.1) $x_1x_2x_3x_4=x_1x_3x_2x_4$, (I.2) $x_1x_2x_3x_1=x_1x_3x_2x_1$ and (I.3) $x_1x_2x_1x_3x_1=x_1x_3x_1x_2x_1$ (see [4]). In this paper, we shall show that a regular [∗-] semigroup S is a generalized inverse [∗-] semigroup if $E(S)$ [the set $P(S)$ of projections of S] satisfies the identity (I.2) [(I.1)], but it is not necessarily a generalized inverse [∗-] semigroup even if $E(S)$ [$P(S)$] satisfies (I.3) [(I.2)].

A regular semigroup S is called an *orthodox semigroup* if $E(S)$ is a band. An orthodox semigroup is called a *generalized inverse semigroup* if $E(S)$ satisfies the identity $x_1x_2x_3x_4=x_1x_3x_2x_4$, that is, $E(S)$ forms a normal band. The followings are well-known results, due to Yamada[4].

RESULT 1 ([4]). *For an orthodox semigroup S, the following conditions are equivalent.*

(1) *S is a generalized inverse semigroup,*

(2) *$E(S)$ satisfies the identity $x_1x_2x_3x_1=x_1x_3x_2x_1$,*

(3) *$E(S)$ satisfies the identity $x_1x_2x_1x_3x_1=x_1x_3x_1x_2x_1$.*

RESULT 2 ([4]). *A regular semigroup S is a generalized inverse semigroup if and only if $E(S)$ satisfies the identity $x_1x_2x_3x_4=x_1x_3x_2x_4$.*

THEOREM 3. *A regular semigroup S is a generalized inverse semigroup if and only if $E(S)$ satisfies the identity $x_1x_2x_3x_1=x_1x_3x_2x_1$.*

PROOF. The "only of" part is obvious. Assume that $efge=egfe$ for all $e, f, g\in E(S)$. Let $e\in E(S)$ and x any inverse of e. Since e, ex, $xe\in E(S)$, we have

$$x^2=(xex)^2=x\{e(ex)(xe)e\}x=x\{e(xe)(ex)e\}x=x.$$

Thus $x\in E(S)$. It follows from Lemma 1.3 of [3] that S is an orthodox semigroup. By Result 1, S is a generalized inverse semigroup.

REMARK. A regular semigroup S in which $E(S)$ satisfies the identity $x_1x_2x_1x_3x_1=$

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$x_1x_3x_1x_2x_1$ is not always a generalized inverse semigroup. Let $S=\left\{0=\begin{bmatrix}1&2&3\\2&2&2\end{bmatrix}, e=\begin{bmatrix}1&2&3\\1&2&2\end{bmatrix},\right.$

$\left. f=\begin{bmatrix}1&2&3\\3&2&3\end{bmatrix}, g=\begin{bmatrix}1&2&3\\1&2&1\end{bmatrix}, h=\begin{bmatrix}1&2&3\\2&2&3\end{bmatrix}, a=\begin{bmatrix}1&2&3\\3&2&2\end{bmatrix}, b=\begin{bmatrix}1&2&3\\2&2&1\end{bmatrix}\right\}$ and its multiplication table is as follows :

	0	e	f	g	h	a	b
0	0	0	0	0	0	0	0
e	0	e	a	e	0	a	0
f	0	0	f	g	f	0	g
g	0	g	f	g	0	f	0
h	0	0	h	b	h	0	b
a	0	0	a	e	a	0	e
b	0	b	h	b	0	h	0

Clearly $E(S)=\{0, e, f, g, h\}$ and a is an inverse of b, so S is a regular semigroup. For any x, y, $z\in E(S)$, xyx is either x or 0, and hence $xyxzx=(xyx)(xzx)=(xzx)(xyx)=xzxyx$. Thus $E(S)$ satisfies the identity $x_1x_2x_1x_3x_1=x_1x_3x_1x_2x_1$. However, $E(S)$ is not a band, since $ef=a\notin E(S)$, and hence S is not a generalized inverse semigroup.

A semigroup S with a unary operation $*: S\to S$ is called a *regular* $*$-*semigroup* if it satisfies (1) $(x^*)^*=x$. (2) $(xy)^*=y^*x^*$, (3) $xx^*x=x$. Let S be a regular $*$-semigroup. An idempotent e of S is called a *projection* if $e^*=e$. Denote the set of projections of S by $P(S)$. A regular $*$-semigroup S is called a *generalized inverse* $*$-*semigroup* if $E(S)$ is a normal band. Since $E(S)=P(S)^2$ for a regular $*$-semigroup S (see [1]), we can easily verify the following theorem.

THEOREM 4. *A regular $*$-semigroup S is a generalized inverse $*$-semigroup if and only if $efgh=egfh$ for all e, f, g, $h\in P(S)$.*

REMARK. A regular $*$-semigroup S in which $P(S)$ satisfies the identity $x_1x_2x_3x_1=x_1x_3x_2x_1$ is not necessarily a generalized inverse $*$-semigroup. Let $S=M(G; I, I; Q)$ where $G=\{e, a\}$ ia a two element group, $I=\{1,2,3\}$ and $Q=(q_{ij})=\begin{bmatrix}e&a&e\\a&e&e\\e&e&e\end{bmatrix}$, and a unary operation is defined by $*: (x; i, j)\to(x; j, i)$. By Theorem 2 of [2], S is a regular $*$-semigroup and $P(S)=\{(e; i, i): i\in I\}$. Since $q_{ij}=q_{ji}$ and G is an abelian group, $(e; i, i)(e; j, j)(e; k, k)(e; i, i)=(q_{ij}q_{jk}q_{ki}; i, i)=(q_{ik}q_{kj}q_{ji}; i, i)=(e; i, i)(e; k, k)(e; j, j)(e; i, i)$. Therefore, $P(S)$ satisfies the identity $x_1x_2x_3x_1=x_1x_3x_2x_1$. But, for idempotents $g=(a; 1,$

2) and $h=(e;2,3)$, $gh=(a;1,3)\not\in E(S)$. Hence S is not a generalized inverse $*$-semigroup.

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