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On the p-Systems in a Straight Locally Inverse Semigroup whose Idempotents form a Band

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In the previous paper [4], the concept of a p-system in a regular semigroup has been introduced ; and it has been shown that every $*$-operation in a regular semigroup S is determined by some p-system in S, and conversely in a regular semigroup S with $*$-operation the projections of S form a p-system in S. On the other hand, the structure of orthodox SLI (straight locally inverse)-semigroups has been clarified in [5]. In this short note, the p-systems in an orthodox SLI-semigroup will be studied.

§ 0. Introduction

Let S be a regular semigroup equipped with a unary operation $*$: $S \rightarrow S$ satisfying the following three axioms :

(1) $xx^*x=x$ for $x \in S$,

(2) $(x^*)^*=x$ for $x \in S$,

(3) $(xy)^*=y^*x^*$ for $x, y \in S$.

In this case, S is called *a regular $*$-semigroup*. An element x of S is called *a projection* if $x=x^*$ and $x^2=x$.

Let S ba a regular semigroup, and $E(S)$ the set of idempotents of S. A subset P of $E(S)$ is called *a p-system* in S if it satisfies the following : Let $V(a)$ be the set of all inverses of a for $a \in S$.

(1) For any $a \in S$, there exists a unique $a^* \in V(a)$ such that aa^*, $a^*a \in P$,

(2) for any $a \in S$, $a^*Pa \subset P$ and $aPa^* \subset P$, where $*$ is the unary operation in S determined.by (1),

(3) $P^2 \subset E(S)$.

It has been shown in [4] that if P is a p-system in S then S becomes a regular $*$-semigroup under the unary operation $*$ determined by (1) above. In this case, the unary oprration $*$ above is called *the $*$-operation* determined by P. It is easy to see that the set of projections of the regular $*$-semigroup S concides with P. Conversely, if S is a regular $*$-semigroup with $*$-operation $\#$, then the set P of all projections is a p-system in S, and the $*$-operation $*$ determined by P coincides with $\#$; that is $\# = *$. If there exists at least one p-

system in a regular semigroup S, then S is called *involutive*. Therefore, we can say that S is involutive if and only if a $*$-operation is well-defined in S.

Now, let S be *a locally inverse semigroup*, that is a regular semigroup such that

(1) eSe is an inverse semigroup for each $e \in E(S)$.

If S further satisfies the following :

(2) $E(S)$ is a disjoint union of maximal subsemilattices, then S is called *a straight locally inverse semigroup* (SLI-semigroup) (see[2]).

It has been shown in [2] that if S is a SLI-semigroup then S is a rectangular band χ of subsemigroups $\{S_\alpha : \alpha \in \chi\}$, where $E(S_\alpha)$ is commutative for each $\alpha \in \chi$. The main purpose of this paper is to find out all the p-systems, accordingly all the $*$-operations, in an orthodox SLI-semigroup.

§ 1. Involutive orthodox SLI-semigroups.

Let S be an orthodox SLI-semigroup. Then, S is a rectangular band χ of $\{S_\alpha : \alpha \in \chi\}$, where S_α is a subsemigroup of S for $\alpha \in \chi$ such that $E(S_\alpha)$ is a subsemilattice. Since $E(S)$ is a band, $E(S)$ is a rectangular band χ of $\{E_\alpha : \alpha \in \chi\}$, where $E_\alpha = E(S_\alpha) = E(S) \cap S_\alpha$. Let ϕ be the homomorphism of S onto χ defined by $x\phi = \alpha$ if $x \in S_\alpha$, and τ the congruence on S induced by ϕ. Let ϕ_E, τ_E be the restrictions of ϕ, τ to $E = E(S)$ respectively. On the other hand, since S is an orthodox semigroup, there exists an inverse semigroup $\Gamma(\Lambda)$ (where Λ is the semilattice of idempotents of Γ) and a surjective homomotphism $\Psi : S \to \Gamma(\Lambda)$ such that $\lambda\Psi^{-1}$ is a rectangular subband of E for each $\lambda \in \Lambda$ (see [1]). Let $\gamma\Psi^{-1} = T_\gamma$ for $\gamma \in \Gamma$. Then, every T_λ is a rectangular band for $\lambda \in \Lambda$. Let ξ be the congruence on S induced by Ψ; that is, $x\xi y$ if and only if $x\Psi = y\Psi$. Let Ψ_E, ξ_E be the restrictions of Ψ, ξ to E respectively. Since χ is a rectangular band, χ is the direct product of a left zero semigroup I and a right zero semigroup J : $\chi \times I \times J$. Now, $\tau_E \cap \xi_E = \iota_E$ (the equality on E ; further, in fact $\tau \cap \xi = \iota_S$ (the equality on S)), and hence E is isomorphic to a subdirect product $E/\tau_E \divideontimes E/\xi_E$ of E/τ_E and E/ξ_E. It is obvious that $E/\tau_E \cong \chi$ (isomorpic) and $E/\xi_E \cong \Lambda$. Hence, $E \cong \chi \divideontimes \Lambda$, and an isomorphism θ is goven by $e\theta = (e\phi, e\Psi) \in \chi \divideontimes \Lambda$ for $e \in E$. Now, identify e with $(e\phi, e\Psi)$.

Then, we can assume that $E = \chi \divideontimes \Lambda$. Let P be a p-system in E. Let $\#$ be the $*$-operation in E determined by P. Let e, f be elements of E such that $e\tau_E f$. Then, since $e, f \in E_\alpha$ for some α, e and f have forms $e = ((i,j),\lambda)$ and $f = ((i,j),\delta)$, where $(i,j) \in I \times J$ and $\lambda, \delta \in \Lambda$. Let $e^\# = ((k,s),\varepsilon)$ and $f^\# = ((u,v),\sigma)$. Since $e^\#$ is an inverse of e, $\varepsilon = \lambda$.

Similarly, $\sigma = \delta$. Now, $((i,j),\lambda)((k,s),\lambda) = ((i,s),\lambda) \in P$, $((k,s),\lambda)((i,j),\lambda) = ((k,j),\lambda) \in P$, $((i,j),\delta)((u,v),\delta) = ((i,v),\delta) \in P$ and $((u,v),\delta)((i,j),\delta) = ((u,(j),\delta) \in P$. Hence, $((i,s),\lambda)((u,v),\delta)((i,s),\lambda) = ((i,s),\lambda\delta) \in P$. Similarly, $((k,j),\lambda\delta)$, $((i,v),\lambda\delta)$, $((u,j),\lambda\delta) \in P$. Since each L-class of E contains only one element of P, it follows that $k = u$.

Dually, we have $s = v$. Therefore, $f^\# = ((k,s),\sigma)$. Hence, $e^\# \tau_E f^\#$. Thus, we have the following :

LEMMA 1.1. *The congruence* τ_E *is a* $*$*-congruence* (*see*[3]), *and accordingly* $P\phi_E$ *is a* p*-system in* χ.

Since χ is involutive, $|I|=|J|$ (where $|\ \ |$ means cardinality), that is χ is a square band, and every L- [R-] class contains just one element of $Q=P\phi_E$. Now, it is easy to see that for $e, f\in E=\chi \divideontimes \Lambda=(I\times J)\divideontimes \Lambda$, where $e=((i,j),\lambda)$ and $f=((k,s),\delta)$, eLf if and only if $\lambda=\delta$ and $s=j$[eRf if and only if $\lambda=\delta$ and $k=i$] (where $L[R]$ is Green's $L[R]$-relation).

LEMMA 1.2. *Let* $e=((i,j),\alpha)$ *and* $f=((i,j),\beta)$ (*hence*, e, $f\in E_{(i,j)}$). *Then*, $e\leqq f$ *if and only if* $\alpha\leqq\beta$.

PROOF. Since $((i,j),\alpha)((i,j),\beta)=((i,j),\beta)((i,j),\alpha)=((i,j),\alpha)$ implies $\alpha\beta=\beta\alpha=\alpha$, that is, $\alpha\leqq\beta$. Conversely, $\alpha\leqq\beta$ implies $((i,j),\alpha)=((i,j),\alpha\beta)=((i,j),\beta\alpha)$, and accordingly $e\leqq f$.

LEMMA 1.3. *If* $e=((i,j),\alpha)\in P$ *and* $\beta\leqq\alpha$, *then* $((i,j),\beta\in P$. *That is*, $e\in P$, $f\in E$ *and* $f\leqq e$ *implies* $f\in P$.

PROOF. Since each R-class contains an element of P and there exists an element having the form $((k,s),\beta)$, it follows that $((k,u),\beta)\in P$ for some u. Then, since P is a p-system, we have $((i,j),\alpha)((k,u),\beta)((i,j),\alpha)=((i,j),\beta)\in P$.

Let $P_\delta=E_\delta\cap P$ for $\delta\in\chi$.

LEMMA 1.4. *For any* $e\in E_\delta$, *there exist* $p\in P_\sigma$ *and* $q\in P_\gamma$, *where* $\delta L\sigma$ *and* $\delta R\gamma$, *such that* $epe=e$ *end* $eqe=e$.

PROOF. There exists $p\in P$ such that pLe. Let $p=((k,s),\beta)$ and $e=((i,j),\alpha)$. Since $pe=p$ and $ep=e$, $((k,s),\beta)((i,j),\alpha)=((k,s),\beta)$ and $((i,j),\alpha)((k,s),\beta)=((i,j),\alpha)$, and accordingly $((k,j),\alpha\beta)=((k,s),\beta)$ and $((i,s),\alpha\beta)=(((i,j),\alpha)$. Hence, $\beta=\alpha\beta=\alpha$ and $s=j$.

Thus, $p=((k,j),\alpha)$. Now, $p\phi_E=(k,j)$, and hence $p\in P_{(k,j)}$. It is easy to see that $(k,j)L(i,j)$ in χ.

Further, it is also obvious that $epe=e$. Dually, there exists $q\in P_\gamma$, where $\gamma R\delta$, such that $eqe=e$.

Since $P\phi_E=Q$ is a p-system in the square band χ, for any ε of χ there exist a unique δ of Q and a unique σ of Q such that $\varepsilon L\delta$ and $\delta R\sigma$ respectively. Denote δ, σ by $\delta=\varepsilon_1$ and $\sigma=\varepsilon_r$. Under these notations the lemma above is rewritten as follows :

(C.1) For any $e\in E_\varepsilon$, there exist $p\in P_{\varepsilon_1}$ and $q\in P_{\varepsilon_r}$ such that $epe=e$ and $eqe=e$.

LEMMA 1.5. *For any* $\varepsilon\in Q$, $P_\varepsilon=E_\varepsilon$.

PROOF. For any $\varepsilon\in Q$, $\varepsilon_1=\varepsilon$ and $\varepsilon_r=\varepsilon$. Hence, for any $e\in E_\varepsilon$, there exists $p\in P_\varepsilon$ such that $epe=e$. Let $\varepsilon=(i,j)$, $e=((i,j),\beta)$ and $p=((i,j),\alpha)$. Then, $epe=e$ implies $\beta\leqq\alpha$.

Since $p\in P$ and $\beta\leqq\alpha$, it follows from Lemma 1.3 that $e=((i,j),\beta)\in P$.

THEOREM 1.6. *Let F be a subset of* $E(S)$, *where S is the above-mentioned orthodox SLI-semigroup. Then, F is a p-system in* $E(S)$ *if and only if it is a p-system in S.*

PROOF. The "if" part is obvious. The "only if" part : We have already seen that S is isomorphic to a subdirect product $\chi\divideontimes\Gamma(\Lambda)$ under the isomorphism θ defined by $x\theta=(x\phi, x\Psi)$ since $\tau\cap\xi=\iota_S$. Hence, we can assume that $S\divideontimes\Gamma(\Lambda)$ by identifying x and $(x\phi, x\Psi)$. Let $F\subset E(S)=\chi\divideontimes\Lambda\subset\chi\divideontimes\Gamma(\Lambda)$ be a p-system in $E(S)$. Let $x\in S$. There exists an inverse x° of x. Since $xx^\circ\in E(S)$, there exists $p\in F$ such that $p\boldsymbol{R}xx^\circ$. Since $x\boldsymbol{R}xx^\circ$, it follows that $x\boldsymbol{R}p$. Similary, there exists $q\in F$ such that $x\boldsymbol{L}q$.

Now, there exists a unique inverse x^* of x such that $x^*\boldsymbol{L}p$, $x^*\boldsymbol{R}q$, $xx^*=p$ and $x^*x=q$. Let $x=((i,j),\alpha)$ and $x^*=((k,s),\beta)$. Then, $\alpha=\beta$. Take any $h=((u,v),\gamma)$ from F. Then, $xhx^*=((i,j),\alpha)((u,v),\gamma)((k,s),\alpha)=((i,s),\alpha\gamma\alpha)$. Since $p=xx^*=((i,s),\alpha)\in F$ and $\alpha\gamma\alpha\leqq\alpha$, it follows from Lemma 1.3 that $((i,s),\alpha\gamma\alpha)\in F$. Hence, $xhx^*\in F$. Similarly, we have $x^*Fx\subset F$. Finally, $F^2\subset E(S)$ is obvious. Thus, F is a p-system in S.

§ 2. The p-systems in S.

In this section, we shall determine all the p-systems in the above-mentioned orthodox SLI-semigroup $S=\chi\divideontimes\Gamma(\Lambda)=(I\times J)\divideontimes\Gamma(\Lambda)$, where $|I|=|J|$.

By Theorem 1.6, we need only to find out the p-systems in $E(S)=E=\chi\divideontimes\Lambda$. Let $P\subset E$ be a p-system in E. Then, it follows from the results above that there exists a p-system Q of $\chi=I\times J$ such that

(C.2) (1) $P=\sum\{E_\delta:\delta\in Q\}$ (disjoint union), and

(2) for any $e\in E_\varepsilon$, where $\varepsilon\in\chi$, there exist $p\in E_{\varepsilon_l}$ and $q\in E_{\varepsilon_r}$ (where ε_l $[\varepsilon_r]$ is the element of Q such that $\varepsilon_l\boldsymbol{L}\varepsilon\,[\varepsilon_r\boldsymbol{R}\varepsilon\,]$) satisfying $epe=e$ and $eqe=e$.

Conversely, let Q be a p-system in $\chi=I\times J$. Let $P=\sum\{E_\delta:\delta\in Q\}$. Then, if P satisfies (2) of (C.2), then P is a p-system in E. In fact : $p^2\subset E(S)$ is obvious. First, we shall show that if $e=((i,j),\alpha)\in E_\mu$, where $\mu=(i,j)$, then there exists $((k,j),\alpha)\in E_{\mu_l}$ and $((i,v),\alpha)\in E_{\mu_r}$, where $(k,j)=\mu_l$ and $(i,v)=\mu_r$. By (2) of (C.2), there exist $p\in E_{\mu_l}$ and $q\in E_{\mu_r}$ such that $e=epe$ and $e=eqe$.

Let $p=((k,j),\beta)$, where $(k,j)=\mu_l$. Then, $((i,j),\alpha)=((i,j),\alpha)((k,j),\beta)((i,j),\alpha)$ implies $\alpha\leqq\beta$. Now, consider $pep=((k,j),\beta)((i,j),\alpha)((k,j),\beta)=((k,j),\alpha)$. Since $pep\in E_{\mu_l}$, it follow that $((k,j),\alpha)\in E_{\mu_l}$. Similarly, there exists $((i,v),\alpha)\in E_{\mu_r}$. Now, let $e=((i,j),\alpha)\in E_\mu$, where $\mu=(i,j)$. There exists $p\in E_{\mu_l}$ and $q\in E_{\mu_r}$ such that $p=((k,j),\alpha)$ and $q=((i,v),\alpha)$. Then, $e\boldsymbol{L}p$ and $e\boldsymbol{R}q$. Therefore, there exists an inverse e^* of e such that $ee^*=q$ and $e^*e=p$. Hence, of course ee^*, $e^*e\in P$. Now, let $h=((w,t),\gamma)$ be an element of $E_{(w,t)}$, where $(w,t)\in Q$. Put $e^*=((a,b),\alpha)$. Since $ee^*=q$ and $e^*e=p$, $((i,j),\alpha)((a,b),\alpha)=((i,v),\alpha)$ and $((a,b),\alpha)((i,j),\alpha)=((k,j),\alpha)$, and accordingly $b=v$ and $a=k$. Thus,

$e^* = ((k, v), \alpha)$. Therefore, $ehe^* = ((i, j), \alpha)((w, t), \gamma)((k, v), \alpha) = ((i, v), \alpha\gamma\alpha)$.

Since $((i, v), \alpha) \in E_{\mu r}$, we have $((i, v), \alpha\gamma\alpha) \in E_{\mu r}$. That is, $ehe^* \in P$. Similarly, $e^*he \in P$. Thus, P is a p-system in $E(S)$.

Accordingly, we have the following :

THEOREM 2.1. *Let S be an orthodox SLI-semigroup. Then, S is a rectangular band χ of subsemigroups $\{S_\delta : \delta \in \chi\}$, where $E(S_\delta)$ is commutative : $S = \Sigma\{S_\delta : \delta \in \chi\}$. Let $\chi = I \times J$, where I is a left zero semigroup and J is a right zero semigroup.*

(1) *If S is involutive, then χ is a square band, that is, $|I| = |J|$.*

(2) *Every possible p-system in $E(S)$ is also a p-system in S, and vice-versa.*

(3) *In case $|I| = |J|$, take a p-system Q in χ, and put $P = \Sigma\{E_\delta : \delta \in Q\}$. For any $\varepsilon \in \chi$, let ε_1, ε_r be the elements of Q such that $\varepsilon L \varepsilon_1$ and $\varepsilon \boldsymbol{R} \varepsilon_r$ respectively. Then, if P satisfies the condition (2) of (C.2) then P is a p-system in $E(S)$ (accordinfly in S). Further, every p-system in S can be obtained in this way.*

REMARK. Let I, J be a left zero semigroup and a right zero semigroup respectively such that $|I| = |J|$. Let $\chi = I \times J$ be the direct product of I and J. Of course, χ is a square band. Let η be a 1-1 mapping of I onto J. Then, $\{[a, a\eta) : a \in I\}$ becomes a p-system in χ. Further, every p-system in χ is constructed in this way.

EXAMPLES. Let I ba a set consisting of two elements a and b ; that is, $I = \{a, b\}$. Then $\chi = I \times I$ becomes a square band under the multiplication defined by $(i, j)(k, s) = (i, s)$ for $(i, j), (k, s) \in I \times I = \chi$. Let $\Lambda = \{0, 1\}$ be a semilattice in which multiplication is given by $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Consider the direct product $\chi \times \Lambda = (I \times I) \times \Lambda$ of χ and Λ.

(I) Let $E = \{((a, a), 0), ((a, b), 0), ((b, a), 0), ((b, b), 0, ((b, b), 1)\}$. Then, E is a subdirect product $\chi \divideontimes \Lambda$ of χ and Λ. Since the p-systems in χ are (1) $\{(a, a), (b, b)\}$ and (2) $\{(a, b), (b, a)\}$, the possible p-systems in E are (1) $\{((a, a), 0), (b, b), 0), (b, b), 1)\}$ and (2) $\{(a, b), 0), (b, a), 0)\}$. The first one is a p-system in E, but the second one is not a p-system in E since the set does not satisfy (2) of (C.2).

(II) Let $E = \{((a, a), 0), (a, b), 0), (b, a), 0), (b, b), 0), (b, a), 1), (b, b), 1)\}$. Then E is a subdirect product $\chi \divideontimes \Lambda$ of χ and Λ. It is easy to see that the possible p-systems in E are (1) $\{((a, a), 0), (b, b), 0), (b, b), 1)\}$ and (2) $\{((a, b), 0), ((b, a), 0), ((b, a), 1)\}$. However, each of them does not satisfy the condition (2) of (C.2). Therefore, in this E there is no p-system.

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