Mem. Fac. Sci. Shimane Univ.,
26, pp. 39–42 Dec. 25, 1992

External Commutativity and Commutativity in Semigroups

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Communicated by Teruo Imaoka
(Received September 10, 1992)

A semigroup S is called externally commutative if S satisfies the permutation identity $xyz = zyx$. In this paper the structure of externally commutative semigroups will be studied. It will be shown that an externally commutative semigroup is a certain special ideal extension of a commutative semigroup by a null semigroup.

§1. Introduction

Recently, S. Lajos [2] introduced the concept of external commutativity to semigroups, and investigated the class of these semigroups.

A semigroup S is called an externally commutative semigroup (EC-semigroup) if it satisfies the following permutation identity:

$$xyz = zyx.$$

It has been noted in [2] that simple semigroups, regular semigroups and cancellative semigroups are all EC-semigroups.

However, an externally commutative semigroup is not necessarily commutative. We can see this fact from the following simple example:

EXAMPLE. Let T be a null semigroup with $|T| \geqq 3$. Let 0 be the zero element of T, and let $a, b \in T \backslash 0$ such that $a \neq b$. Let $N = \{c, d\}$ be a set, and put $S = N \cup T$. Define multiplication in S as follows: $cd = a$, $dc = b$; $uv = 0$ if $u, v \in T$; $ct = tc = td = dt = 0$ for $t \in T$; and $c^2 = d^2 = 0$.

Then, S is an EC-semigroup but not commutative.

In this paper, we shall consider several conditions for an EC-semigroup S in order that S is commutative. Further, we shall describe every EC-semigroup as a certain special ideal extension of a commutative semigroup by a null semigroup (see [1]).

§2. Some special cases

First of all, it should be noted that an EC-semigroup satisfies all permutation identities $x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$, where $n \geqq 4$ and π is a permutation on $\{1, 2, \cdots n\}$. This was pointed out recently by S. Iyanaga in his letter. Hereafter, we shall say that a semigroup S is universal if it satisfies $S^2 = S$.

By using this result, we have easily the following:

PROPOSITION 1. *A universal EC-semigroup S is commutative.*

PROOF. Let $a, b \in S$. Then, $a = a_1a_2$ and $b = b_1b_2$ for some $a_1, a_2, b_1, b_2 \in S$. Then, $ab = a_1a_2b_1b_2 = b_1b_2a_1a_2 = ba$ (since S satisfies any permutation identity on $\{x_1, x_2, x_3, x_4\}$).

THEOREM 2. *An EC-semigroup is commutative if and only if $ab = ba$ is satisfied for any elements $a, b \in S \backslash S^2$.*

PROOF. The "only if" part is obvious. The "if" part: If $S^2 = S$, then it follows from Proposition 1 that S is commutative. Assume that $S^2 \neq S$. Let $x \in S^2$ and $y \in S \backslash S^2$. **Case 1.** $x \in S^3$. In this case, $x = x_1x_2x_3$ for some x_1, x_2, x_3. Now, $x_1x_2x_3y = yx_1x_2x_3$, and accordingly $xy = yx$. **Case 2.** $x \in S^2 \backslash S^3$. In this case, there exist $x_1, x_2 \in S \backslash S^2$ such that $x = x_1x_2$. Then, $xy = x_1x_2y = yx_2x_1$ (by EC-property) $= yx_1x_2$ (by the assumption) $= yx$. Therefore, $xy = yx$ is satisfied for any $x, y \in S$.

PROPOSITION 3. *An EC-semigroup S is commutative if S^2 is weakly reductive (for definition, see* [1]).

PROOF. For $a, b \in S \backslash S^2$ and for $x \in S^2$, there exist $x_1, x_2 \in S$ such that $x = x_1x_2$. Now, since S is an EC-semigroup, $abx_1x_2 = bax_1x_2$ is satisfied. That is, $abx = bax$. Similarly we have $xab = xba$. Now, $ab, ba \in S^2$ and S^2 is weakly reductive. Accordingly, $ab = ba$. Therefore, it follows from Theorem 2 that S is commutative.

THEOREM 4. *A weakly reductive EC-semigroup is commutative.*

PROOF. We shall show that S^2 is weakly reductive. Let $a, b \in S^2$, and assume that $ax = bx$ and $xa = xb$ for all $x \in S^2$.

There exist $a_1a_2, b_1, b_2, x_1, x_2 \in S$ such that $a = a_1a_2$, $b = b_1b_2$ and $x = x_1x_2$. Now, $a_1a_2x_1x_2 = b_1b_2x_1x_2$ and $x_1x_2a_1a_2 = x_1x_2b_1b_2$. Since S is an EC-semigroup, $x_1x_2a_1a_2 = x_2a_1a_2x_1$ and $x_1x_2b_1b_2 = x_2b_1b_2x_1$. By weakly reductivity, $a_1a_2x_1 = b_1b_2x_1$. Similarly, we have $x_1a_1a_2 = x_1b_1b_2$. Then, again by weakly reductivity, $a_1a_2 = b_1b_2$, that is, $a = b$. Therefore, S^2 is weakly reductive. Then, it follows from Proposition 3 that S is commutative.

THEOREM 5. *Let S be an EC-semigroup. If S satisfies the following condition:*

(C) *for any $a, b \in S^2$, $axy = bxy$ and $xya = xyb$ for all $x, y \in S$ implies $a = b$,* *then S is commutative.*

PROOF. For $a, b \in S$, $abxy = baxy$ and $xyab = xyba$ for all $x, y \in S$ since S is externally commutative. Since $ab, ba \in S^2$, $ab = ba$. Thus, S is commutative.

THEOREM 6. *Let S be an EC-semigroup such that $S \supsetneqq S^2 \supsetneqq \cdots \supsetneqq S^n = S^{n+1}$, where $n \geqq 2$. Then, if the Rees factor semigroup S/S^n is commutative and S^n is weakly reductive, then S is commutative.*

PROOF. Since the proof can be given in an anologous way to that of Propsition 3, we omit the proof.

§3. General case.

Finally, we shall consider the construction of all possible EC-semigroups by using concept of ideal extensions of a commutative semigroup by a null semigroup.

Let T be a commutative semigroup, and N a null semigroup. Put $N^* = N \backslash 0$, where 0 is the zero element of N. Let λ, ρ be mappings of N^* into the translation semigroup $TL(T)$ of T. (Since T is commutative, every left translation is also a right translation, and vice-versa. We shall call it a translation.) Let $A\lambda = \lambda_A$ and $A\rho = \rho_A$ for $A \in N^*$. Let $\phi: N^* \times N^* \to T$ be a ramification (see [1]), and put $(A, B)\phi = [A, B]$. Then,

THEOREM 7. *If the system $\{\rho_A; \lambda_A; \phi: A \in N^*\}$* saisfies the following confitions:

(*C*.1) $\rho_B \rho_A = \rho_{[B,A]}$ *(the inner translation (see* [1]) *induced by* $[B, A]$),

(*C*.2) $\lambda_B \lambda_A = \lambda_{[A,B]}$ *(the inner translation induced by* $[A, B]$),

(*C*.3) $\lambda_A = \rho_A$ *on* T^2,

(*C*.4) $\lambda_{[A,B]} = \rho_{[B,A]}$,

(*C*.5) $\lambda_A \rho_C = \rho_C \lambda_A = \lambda_C \rho_A = \rho_A \lambda_C$, *and*

(*C*.6) $[A, B]\rho_C = [B, C]\lambda_A = [B, A]\lambda_C$,

then $S = T \cup N^*$ is an EC-semigroup under the multiplication $\circ$ defined by

(*N*.1) $s \circ t = st$ *for* $s, t \in T$, (*N*.2) $s \circ A = s\rho_A$, $A \circ s = s\lambda_A$ *for* $s \in T$, $A \in N^*$, (*N*.3) $A \circ B = [A, B]$ *for* $A, B \in N^*$.

Further, every EC-semigroup can be constructed in this way.

PROOF. It is easy to see from the conditions (C.1)-(C.2) that S becomes a semigroup under the multiplication given by (N.1)-(N.3). Next, we shall show that $\alpha \circ \beta \circ \gamma = \gamma \circ \beta \circ \alpha$ for $\alpha, \beta, \gamma \in S$. It is obvious that this is satisfied if two of α, β, γ

are elements of T. Suppose that at least two of α, β, γ are elements of N^*. For $A, B \in N^*$ and $s \in T$, $A \circ B \circ s = s\lambda_{[A,B]}$ and $s \circ B \circ A = s\rho_{[B,A]}$. Since $\lambda_{[A,B]} = \rho_{[B,A]}$ on T, $A \circ B \circ s = s \circ B \circ A$. Finally, $A \circ B \circ C = [A, B]\rho_C = [B, A]\lambda_C = C \circ B \circ A$.

To prove the latter half, let S be an EC-semigroup, and put $T = S^2$ and $N = S/S^2$ (the Rees factor semigroup of modulo S^2). Then, it is obvious that T is commutative and N is a null semigroup. Let $N \backslash 0 = N^*$, and define $\lambda; N^* \to TL(T)$, $\rho: N^* \to TL(T)$ and $\phi: N^* \times N^* \to T$ as follows: $A\lambda = \lambda_A$, where $s\lambda_A = As$; $A\rho = \rho_A$, where $s\rho_A = sA$; and $(A, B)\phi = [A, B] = AB$. Then, it is easy to see that the system $\{\lambda_A; \rho_A; \phi; A \in N^*\}$ satisfies (C.1)-(C.6) and the multiplication in $S = T \cup N^*$ is given by (N.1)-(N.3).

References

[1] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Vol. 1, Amer. Math. Soc., Providence, R.I., 1961.

[2] S. Lajos, Notes on externally commutative semigroups, PU.M.A., Ser. A, 2 (1991), 67–72.