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Linearized Stability for Nonlinear Evolution Equations with Subdifferential Operators

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We will present a principle of linearized stability of stationary solutions to nonlinear evolution eqautions having possibly multi-valued subdifferential operators in Hilbert spaces, with the help of 'nonsmooth analysis.'

Introduction

Let H be a Hilbert space and $\varphi: H \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. In this paper, we give a principle of linearized stability of stationary solutions of nonlinear evolution equations of the form

$$\frac{du}{dt}(t) + \partial\varphi(u(t)) \ni 0,$$

where $\partial\varphi$ is the subdifferential of φ.

Subdifferentials are typical and important examples of maximal monotone operators in Hilbert spaces. Since a maximal monotone operator generates a nonexpansive semigroup, it is evident that all the stationary solutions of the above equation are stable. But the asymptotic stability cannot always be obtained. Our purpose is to investigate the asymptotic stability of stationary solutions by the method of 'linearization'. Of course, the subdifferential $\partial\varphi$ is not Fréchet differentiable in general, and so we need to introduce a different notion of differentiability in order to consider the 'linearization' of the operator. We adopt the idea of tangent cones from 'nonsmooth analysis' and use the proto-differentiation introduced by Rockafellar [7]. In [6], we have considered the similar equations having single-valued quasi-m-accretive operators in Banach spaces. There, the single-valuedness of the operators is essentially used. One of the purposes of this paper is to treat multi-valued operators. We remark that the subdifferential operators bring to the parabolic regularity to the solution and this feature plays an important role in our argument.

We prepare some notations and preliminary lemmas in §1. Our main result is stated in §2. In §3, we analyze the regularized equation and its linearization. The proof of our main theorem is contained in §4.

1. Preliminaries

In order to formulate our results, we have to explain the notion of derivatives which we will adopt from 'nonsmooth analysis'. Let U be a metric space and X be a Banach space. For $F: U \to 2^X$, define

$$\liminf_{u \to u_0 (in\ U)} F(u) := \bigcap_{\varepsilon > 0} \bigcup_{\eta > 0} \bigcap_{u \in B(u_0, \eta)} (F(u) + \varepsilon B),$$

$$\limsup_{u \to u_0 (in\ U)} F(u) := \bigcap_{\varepsilon > 0} \bigcap_{\eta > 0} \bigcup_{u \in B(u_0, \eta)} (F(u) + \varepsilon B),$$

where $B(u_0, \eta)$ denotes the η-ball centered at u_0 in U and εB denotes the ε-ball centered at 0 in X. If $\liminf_{u \to u_0} F(u) = \limsup_{u \to u_0} F(u)$, then we denote it by $\lim_{u \to u_0} F(u)$.

Now, let X and Y be Banach spaces and let $F: X \to 2^Y$ be a multi-valued operator, which domain, range, and graph are defined as follows:

$$D(F) := \{x \in X \mid F(x) \neq \emptyset\}, \ R(F) := \bigcup_{x \in D(F)} F(x),$$

$$G(F) := \{(x, y) \in X \times Y | x \in D(F), y \in F(x)\}.$$

For $(x, y) \in G(F)$, we define multi-valued operators $\partial_i F(x, y)$ and $\partial_s F(x, y): X \to 2^Y$ by

$$G(\partial_i F(x, y)) = \liminf_{t \downarrow 0} t^{-1} [G(F) - (x, y)],$$

$$G(\partial_s F(x, y)) = \limsup_{t \downarrow 0} t^{-1} [G(F) - (x, y)],$$

in other words,

$$(u, v) \in G(\partial_i F(x, y)) \Longleftrightarrow \forall t_n \downarrow 0 \, \exists (u_n, v_n) \to (u, v) \text{ in } X \times Y:$$
$$(x + t_n u_n, y + t_n v_n) \in G(F),$$
$$(u, v) \in G(\partial_s F(x, y)) \Longleftrightarrow \exists t_n \downarrow 0 \, \exists (u_n, v_n) \to (u, v) \text{ in } X \times Y:$$
$$(x + t_n u_n, y + t_n v_n) \in G(F).$$

The operators $\partial_i F(x, y)$ and $\partial_s F(x, y)$ are called the intermediate derivative and the contingent derivative of F at (x, y), respectively in 'nonsmooth analysis'. For further details, see e.g. [1, 4]. If $\partial_i F(x, y) = \partial_s F(x, y)$, then we denote it simply by $\partial F(x, y)$. If this is the case, F is said to be proto-differentiable at x relative to y and $\partial F(x, y)$ is called the proto-derivative. See [7]. When F is single-valued, we write $\partial F(x) := \partial F(x, F(x))$.

Let F be a single-valued mapping from X to Y with $D(F) = X$. We say that F is Gâteaux differentiable at $x \in X$ if there exists a $dF(x) \in L(X, Y)$ ($\equiv$ the space of all continuous linear mappings from X into Y) such that

$$(Y\text{-})\lim_{t \downarrow 0} t^{-1}[F(x + th) - F(x)] = dF(x)h \quad \forall h \in X.$$

$dF(x)$ is called the Gâteaux derivative of F at x.

The following lemma is shown in [6, Lemma 1.1].

Lemma 1.1. *Let X and Y be Banach spaces. Let $F: X \to Y$ be a single-valued Lipschitz continuous mapping with $D(F) = X$. If $\partial_i F(x) \in L(X, Y)$, then F is Gâteaux differentiable at x and $\partial_i F(x) = dF(x)$.*

Let H be a Hilbert space. Denote by $\langle \cdot, \cdot \rangle$ the inner product. A possibly multi-valued operator $Q: H \to 2^H$ is said to be monotone if

$$\langle x - y, x' - y' \rangle \geq 0 \quad \forall (x, x'), (y, y') \in G(Q).$$

A monotone operator Q is called maximal monotone if there is no monotone extention of Q. For a maximal monotone operator Q, we define the resolvent and the Yosida approximation by $J_\lambda^Q := (I + \lambda Q)^{-1}$ and $Q_\lambda := (1/\lambda)(I - J_\lambda^Q)$ for $\lambda > 0$, respectively. It is well known that Q is maximal monotone iff J_λ^Q is a nonexpansive mapping defined on all of H for all $\lambda > 0$. Q_λ is a Lipschitz mapping with constant $2/\lambda$ (in fact, $1/\lambda$) and is also a maximal monotone operator. See [3].

The following lemma is a special case of [6, Lemma 1.2] and so the proof is omitted.

Lemma 1.2 *Let Q be a maximal monotone operator in H. Assume that $\partial_i Q(x, y)$ is also maximal monotone. Then $(\partial_i Q(x, y))_\lambda = \partial_i Q_\lambda(x + \lambda y, y)$ for $(x, y) \in G(Q)$. The same fact is also true for ∂_s.*

Finally we, give the definition of subdifferential operator. For a proper lower semicontinuous (l.s.c.) convex function $\varphi: H \to (-\infty, +\infty]$, define

$$\partial\varphi(x) := \{y \in H \mid \varphi(z) - \varphi(x) \geq \langle y, z - x \rangle, \quad \forall z \in H\}.$$

The possibly multi-valued operator $\partial\varphi$ is called the subdifferential of φ and this is the important example of maximal monotone operators. See [3].

2. Main result

Let H be a Hilbert space and denote by $\|\cdot\|$ its norm. Let A be a possibly multi-valued maximal monotone operator in H. It is well known that A generates a nonlinear nonexpansive semigroup $\{S(t)\}_{t\geq 0}$ on $\overline{D(A)}$. If we put $u(t) \equiv S(t)u_0$, $u_0 \in \overline{D(A)}$, then $u \in C([0, \infty); \overline{D(A)})$ gives a unique 'generalized or weak' solution of the Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}u(t) + Au(t) \ni 0, & 0 \leq t < \infty, \\ u(0) = u_0. \end{cases} \qquad (E; 0, u_0)$$

See [2, 3].

In what follows, we assume the following hypotheses:

(H1) The operator A has the multi-valued linear maximal monotone proto-derivative at $x \in D(A)$ relative to $y \in Ax$, that is, for each $(x, y) \in G(A)$, there exists a multi-valued linear maximal monotone operator $\partial A(x, y): H \to 2^H$ such that

$$G(\partial A(x, y)) = \lim_{t \downarrow 0} t^{-1}[G(A) - (x, y)].$$

(H2) The correspondence $(x, y) \mapsto \partial A(x, y)$ is lower semicontinuous in the following sense (cf. [1]):

$$\liminf_{(x,y)\to(z,w) \text{ in } G(A)} G(\partial A(x, y)) \supset G(\partial A(z, w)).$$

(H3) For any $(u, 0) \in G(A)$, there exist a neighborhood $\mathscr{U}$ of $(u, 0)$ and a nondecreasing function $L: [0, \infty) \to [0, \infty)$ such that

$$\| J_\lambda^{\partial A(x,y)} v - J_\lambda^{\partial A(u,0)} v \| \leq \lambda(\|x - u\| + \|y\|) L(\|v\|)$$

for all $(x, y) \in G(A) \cap \mathscr{U}$, $v \in H$, and $\lambda > 0$.

(H4) A is the subdifferential of some proper l.s.c. convex function $\varphi: H \to (-\infty, +\infty]$.

REMARK 2.1. In separable Hilbert space, we can show that any maximal monotone operator has deriative $\partial A(x, y)$ in the sense of (H1) for dense (x, y) in $G(A)$. See [5].

Now, we state our main theorem:

THEOREM 2.1. *Let* (H1)–(H4) *be satisfied. Let* $\bar{u} \in D(A)$ *be a stationary solution of* (E), *i.e.,* $A\bar{u} \ni 0$, *or equivalently,* $S(t)\bar{u} = \bar{u}$. *If there exists an* $\omega > 0$ *such that* $\partial A(\bar{u}, 0) - \omega I$ *is maximal monotone, then* $\bar{u}$ *is asymptotically stable in the sense that there exist constants* $\eta > 0$, $C > 0$, $\gamma > 0$ *such that*

$$\|S(t)u_0 - \bar{u}\| \le Ce^{-\gamma t}\|u_0 - \bar{u}\|, \quad \forall t \ge 0,$$

whenever $u_0 \in \overline{D(A)}$ *and* $\|u_0 - \bar{u}\| < \eta$.

3. Linearized equation

In this section we assume the hypotheses (H1) and (H2) only. At first, we note that A_λ is Gâteaux differentiable at each $u \in H$ by the hypothesis (H1) and Lemmas 1.1 and 1.2. Thus, $dA_\lambda(u) = \partial A_\lambda(u) = (\partial A(J_\lambda^A u, A_\lambda u))_\lambda$.

Let $\{S_\lambda(t)\}$ be a nonlinear semigroup on H generated by $-A_\lambda$. Then $u_\lambda(t) \equiv S_\lambda(t)x$, $x \in H$, gives a unique classical solution $u_\lambda \in C^1([0, \infty); H)$ of the regularized equation of (E), i.e., u_λ satisfies

$$\begin{cases} \dfrac{d}{dt} u_\lambda(t) + A_\lambda u_\lambda(t) = 0, & 0 \le t \le T, \\ u_\lambda(0) = x. \end{cases} \qquad (E_\lambda;\, 0, x)$$

for all $T > 0$. Furthermore, recall that $\lim_{\lambda \downarrow 0} S_\lambda(\cdot)u_0 = S(\cdot)u_0$ in $C([0, T]; H)$ for $u_0 \in \overline{D(A)}$, where $\{S(t)\}$ is a semigroup generated by $-A$. (See [2, 3].)

Let $0 \le s < T$ and consider the linearized equation of (E_λ):

$$\begin{cases} \dfrac{d}{dt} v_\lambda(t) + dA_\lambda(S_\lambda(t)x)v_\lambda(t) = 0, & s \le t < T, \\ v_\lambda(s) = w \in H. \end{cases} \qquad (L_\lambda;\, s, w)$$

To show that $(L_\lambda; s, w)$ has a solution, we need the following two lemmas.

LEMMA 3.1. $\liminf_{u \to v} G(dA_\lambda(u)) \supset G(dA_\lambda(v))$.

PROOF. Take $(\alpha, \beta) \in G(dA_\lambda(v)) = G((\partial A(J_\lambda^A v, A_\lambda v))_\lambda)$. Then $(\alpha - \lambda\beta, \beta)$ belongs to $G(\partial A(J_\lambda^A v, A_\lambda v))$. Let $u_n \to v$. Then $(J_\lambda^A u_n, A_\lambda u_n) \to (J_\lambda^A v, A_\lambda v)$ in $G(A)$. Hence, by (H2), there is a sequence $(\xi_n, \eta_n) \in G(\partial A(J_\lambda^A u_n, A_\lambda u_n))$ such that $(\xi_n, \eta_n) \to (\alpha - \lambda\beta, \beta)$. Note that $(\xi_n + \lambda\eta_n, \eta_n) \in G((\partial A(J_\lambda^A u_n, A_\lambda u_n))_\lambda) = G(dA_\lambda(u_n))$. Since $(\xi_n + \lambda\eta_n, \eta_n) \to (\alpha, \beta)$, this shows that (α, β) belongs to $\liminf_{u_n \to v} G(dA_\lambda(u_n))$. □

LEMMA 3.2. *For any* $x \in H$ *and* $\lambda > 0$, $u \mapsto dA_\lambda(u)x$ *is a continuous mapping from* H *to* H.

PROOF. Since $dA_\lambda(u) = (\partial A(J_\lambda^A u, A_\lambda u))_\lambda$, we notice that $dA_\lambda(u) \in L(H, H)$ and it is maximal monotone by (H1). Especially, we have $\|dA_\lambda(u)x\| \le (2/\lambda)\|x\|$. Let $u_n \to v$. Then by Lemma 3.1,

$$\liminf_{u_n \to v} G(dA_\lambda(u_n)) \supset G(dA_\lambda(v)) \ni (x, dA_\lambda(v)x).$$

Thus there exists a sequence $(\xi_n, dA_\lambda(u_n)\xi_n) \in G(dA_\lambda(u_n))$ converging to $(x, dA_\lambda(v)x)$. Therefore, we have

$$\|dA_\lambda(u_n)x - dA_\lambda(v)x\| \leq \|dA_\lambda(u_n)x - dA_\lambda(u_n)\xi_n\| + \|dA_\lambda(u_n)\xi_n - dA_\lambda(u_n)x\|$$

$$\leq \frac{2}{\lambda}\|x - \xi_n\| + \|dA_\lambda(u_n)\xi_n - dA_\lambda(u_n)x\| \to 0$$

as $n \to \infty$. □

At this stage we can show the existence of solutions to $(L_\lambda; s, w)$:

PROPOSITION 3.1. *There exists a unique strong solution* $v_\lambda \in W^{1,1}(s, T; H)$ *of* $(L_\lambda; s, w)$, *for which the equation is satisfied* a.e. $t \in (s, T)$, *and* $\|v_\lambda(t)\| \leq \|w\|$.

PROOF. Set $\partial A_\lambda^x(t) := \partial A(J_\lambda^A S_\lambda(t)x, A_\lambda S_\lambda(t)x)$. Note that

$$dA_\lambda(S_\lambda(t)x) = (\partial A_\lambda^x(t))_\lambda = \frac{1}{\lambda}(I - J_\lambda^{\partial A_\lambda^x(t)})$$

and $\partial A_\lambda^x(t)$ is maximal monone by (H1). Thus $v \mapsto J_\lambda^{\partial A_\lambda^x(t)} v$ is nonexpansive. Also, by Lemma 3.2, for any $v \in H$ and $\lambda > 0$, $t \mapsto J_\lambda^{\partial A_\lambda^x(t)} v$ is continuous. Thus we can use [3, Corollary 1.1, p.11] and conclude that $(L_\lambda; s, w)$ has a unique strong solution $v_\lambda \in W^{1,1}(s, T; H)$. Since $dA_\lambda(S_\lambda(t)x)$ is a linear maximal monotone operator,

$$0 \leq \langle dA_\lambda(S_\lambda(t)x)v_\lambda(t), v_\lambda(t)\rangle = -\frac{1}{2}\frac{d}{dt}\|v_\lambda(t)\|^2, \quad a.e.\ t,$$

from which one can easily deduce that $\|v_\lambda(t)\| \leq \|v_\lambda(s)\|$. □

REMARK 3.1. As a matter of fact, $v_\lambda \in C^1([s, T]; H)$ and it is a classical solution since v_λ satisfies the relation

$$v_\lambda(t) = e^{-\frac{t-s}{\lambda}} w + \int_s^t e^{\frac{\tau - t}{\lambda}} J_\lambda^{\partial A_\lambda^x(\tau)} v_\lambda(\tau) d\tau,$$

as shown in [3, Corollary 1.1, p.11] and $t \mapsto J_\lambda^{\partial A_\lambda^x(t)} v_\lambda(t)$ is continuous.

Let us define $\mathscr{S}_\lambda: H \to W^{1,1} := W^{1,1}(0, T; H)$ by

$$(\mathscr{S}_\lambda x)(t) := S_\lambda(t)x \quad \text{for } t \in [0,T],$$

where $T > 0$ is given arbitrarily. The next proposition characterizes the solution of $(L_\lambda; 0, v_0)$ by the proto-derivative of the solution of (E_λ). The basic idea is due to [4].

PROPOSITION 3.2. $\partial\mathscr{S}_\lambda(x)(v_0) = v_\lambda$, *the unique solution of* $(L_\lambda; 0, v_0)$, *where* ∂ *is taken in* $H \times W^{1,1}$.

PROOF. Firstly, we will take $v \in \partial_s \mathscr{S}_\lambda(x)(v_0)$ and show that $v = v_\lambda$. Next, let $v = v_\lambda$. We will show that $v \in \partial_i \mathscr{S}_\lambda(x)(v_0)$.

(i) Take $v \in \partial_s \mathscr{S}_\lambda(x)(v_0)$. Then there exist $t_n \downarrow 0$ and $(w_n, z_n) \to (v_0, v)$ in $H \times W^{1,1}$ such that $(x + t_n w_n, \mathscr{S}_\lambda x + t_n z_n) \in G(\mathscr{S}_\lambda)$. Therefore,

$$t_n^{-1}[\mathscr{S}_\lambda(x + t_n w_n) - \mathscr{S}_\lambda x] = z_n \to v \text{ in } W^{1,1}(\subset C([0, T]; H)).$$

Then we can extract a subsequence $\{k\} \subset \{n\}$ such that

$$t_n^{-1}[S_\lambda(t)(x + t_n w_n) - S_\lambda(t)x] = z_n(t) \to v(t), \quad \forall t \in [0, T],$$

$$t_k^{-1}[S'_\lambda(t)(x + t_k w_k) - S'_\lambda(t)x] = z'_k(t) \to v'(t), \quad a.e\ t \in (0, T).$$

The latter reads

$$t_k^{-1}[A_\lambda S_\lambda(t)(x + t_k w_k) - A_\lambda S_\lambda(t)x] = -z'_k(t) \to -v'(t), \quad a.e.\ t \in (0, T).$$

Noting that $A_\lambda S_\lambda(t)(x + t_k w_k) = A_\lambda S_\lambda(t)x - t_k z'_k(t)$ and $S_\lambda(t)(x + t_k w_k) = S_\lambda(t)x + t_k z_k(t)$, we have

$$(S_\lambda(t)x + t_k z_k(t), A_\lambda S_\lambda(t)x - t_k z'_k(t)) \in G(A_\lambda).$$

Hence $(v(t), -v'(t)) \in G(\partial_s A_\lambda(S_\lambda(t)x)) = G(dA_\lambda(S_\lambda(t)x))$ and so $v(t)$ satisfies

$$v'(t) + dA_\lambda(S_\lambda(t)x)v(t) = 0, \quad a.e.\ t \in (0, T).$$

Moreover, since $z_n(0) = w_n \to v_0$ in H, the initial condition $v(0) = v_0$ is fulfilled. By the uniqueness of the solution, we conclude that $v = v_\lambda$ as claimed.

(ii) Next, let $v = v_\lambda \in W^{1,1}(0, T; H)$ be the strong solution of $(L_\lambda; 0, v_0)$. Let $t_n \downarrow 0$ and put $\beta_n(t) := t_n^{-1}[A_\lambda(S_\lambda(t)x + t_n v(t)) - A_\lambda S_\lambda(t)x]$. Then $\beta_n(t) \to dA_\lambda(S_\lambda(t)x)v(t) = -v'(t)$ a.e. $t \in (0, T)$. Since $\|\beta_n(t)\| \leq (2/\lambda)\|v(t)\|$, we have $\beta_n \to -v'$ in $L^1(0, T; H)$ by the Lebesgue dominated convergence theorem. Then, putting $\pi_n(t) := -\int_0^t \beta_n(\tau)d\tau + v_0$, we have

$$\sup_{0 \leq t \leq T} \|\pi_n(t) - v(t)\| \leq \int_0^T \|\beta_n(\tau) + v'(\tau)\| d\tau \to 0 \text{ as } n \to \infty.$$

Therefore, $\pi_n \to v$ in $W^{1,1}(0, T; H)$. Now, put $x_n(t) := S_\lambda(t)(x + t_n v_0)$. Then $x_n \in C^1([0, T]; H)$ and x_n satisfies

$$\begin{cases} x'_n(t) + A_\lambda x_n(t) = 0, & t \in [0, T], \\ x_n(0) = x + t_n v_0. \end{cases}$$

On the other hand, if we set $y_n(t) := S_\lambda(t)x + t_n \pi_n(t)$, then $y_n \in W^{1,1}(0, T; H)$ and y_n satisfies

$$\begin{cases} y'_n(t) + A_\lambda y_n(t) = A_\lambda y_n(t) - A_\lambda(S_\lambda(t)x + t_n v(t)), & a.e.\ t \in (0, T), \\ y_n(0) = x + t_n v_0, \end{cases}$$

because $y_n'(t) = S_\lambda'(t)x - t_n\beta_n(t) = -(A_\lambda S_\lambda(t)x + t_n\beta_n(t)) = -A_\lambda(S_\lambda(t)x + t_n v(t))$. By the well-known estimate for solutions of inhomogeneous evolution equations (see [2, 3]),

$$\begin{aligned}\|x_n(t) - y_n(t)\| &\le \int_0^t \|A_\lambda y_n(\tau) - A_\lambda(S_\lambda(\tau)x + t_n v(\tau))\|\,d\tau \\ &\le \frac{2t_n}{\lambda}\int_0^t \|\pi_n(\tau) - v(\tau)\|\,d\tau. \end{aligned}\tag{3.1}$$

Noting that $\|y_n'(t) + A_\lambda y_n(t)\| = \|-A_\lambda(S_\lambda(t)x + t_n v(t)) + A_\lambda(S_\lambda(t)x + t_n\pi_n(t))\| \le (2/\lambda)t_n\|v(t) - \pi_n(t)\|$, we have

$$\begin{aligned}\|x_n'(t) - y_n'(t)\| &\le \|-A_\lambda x_n(t) + A_\lambda y_n(t)\| + \|-A_\lambda y_n(t) - y_n'(t)\| \\ &\le \frac{2}{\lambda}\|x_n(t) - y_n(t)\| + \frac{2t_n}{\lambda}\|\pi_n(t) - v(t)\| \\ &\le \frac{2t_n}{\lambda^2}\int_0^t \|\pi_n(\tau) - v(\tau)\|\,d\tau + \frac{2t_n}{\lambda}\|\pi_n(t) - v(t)\|. \end{aligned}\tag{3.2}$$

Next, we set $v_n(t) := (x_n(t) - S_\lambda(t)x)/t_n$. Then by (3.1),

$$\begin{aligned}\|v_n(t) - v(t)\| &= \frac{1}{t_n}\|x_n(t) - S_\lambda(t)x - t_n v(t)\| \\ &\le \frac{1}{t_n}\|x_n(t) - S_\lambda(t)x - t_n\pi_n(t)\| + \|\pi_n(t) - v(t)\| \\ &= \frac{1}{t_n}\|x_n(t) - y_n(t)\| + \|\pi_n(t) - v(t)\| \\ &\le \frac{2}{\lambda}\int_0^t \|\pi_n(\tau) - v(\tau)\|\,d\tau + \|\pi_n(t) - v(t)\|. \end{aligned}$$

From this estimate, we conculde that $v_n \to v$ in $C([0, T]; H)$ as $n \to \infty$.

On the other hand, by (3.2)

$$\begin{aligned}\|v_n'(t) - v'(t)\| &= \frac{1}{t_n}\|x_n'(t) - S_\lambda'(t)x - t_n v'(t)\| \\ &\le \frac{1}{t_n}\|x_n'(t) - S_\lambda'(t)x - t_n\pi_n'(t)\| + \|\pi_n'(t) - v'(t)\| \\ &= \frac{1}{t_n}\|x_n'(t) - y_n'(t)\| + \|\pi_n'(t) - v'(t)\| \end{aligned}$$

$$\leq \frac{2}{\lambda^2}\int_0^t \|\pi_n(\tau) - v(\tau)\|d\tau + \frac{2}{\lambda}\|\pi_n(t) - v(t)\| + \|\pi_n'(t) - v'(t)\|.$$

Hence, by intergrating the above over $[0, T]$,

$$\int_0^T \|v_n'(\tau) - v'(\tau)\|d\tau \leq \frac{2T}{\lambda^2}\int_0^T \|\pi_n(\tau) - v(\tau)\|d\tau$$
$$+ \frac{2}{\lambda}\int_0^T \|\pi_n(\tau) - v(\tau)\|d\tau + \int_0^T \|\pi_n'(\tau) - v'(\tau)\|d\tau \to 0$$

as $n \to \infty$. Thus we obtain $v_n \to v$ in $W^{1,1}(0, T; H)$ as $n \to \infty$. Finally, noting that $\mathscr{S}_\lambda(x + t_n v_0)(\cdot) = x_n(\cdot) = S_\lambda(\cdot)x + t_n v_n(\cdot) = (\mathscr{S}_\lambda x)(\cdot) + t_n v_n(\cdot)$, we have $(x, \mathscr{S}_\lambda x) + t_n(v_0, v_n) \in G(\mathscr{S}_\lambda)$, and so $(v_0, v) \in G(\partial_i \mathscr{S}_\lambda(x))$ holds. □

LEMMA 3.3. *The operater $\mathscr{S}_\lambda$: $H \to W^{1,1}(0, T; H)$ is Gâteaux differentiable at each $x \in H$, and $d\mathscr{S}_\lambda(x) = \partial\mathscr{S}_\lambda(x)$.*

PROOF. $\mathscr{S}_\lambda$ is Lipschitz continuous since

$$\|\mathscr{S}_\lambda z - \mathscr{S}_\lambda x\|_{W^{1,1}} = \int_0^T \|S_\lambda(t)z - S_\lambda(t)x\|dt + \int_0^T \|S_\lambda'(t)z - S_\lambda'(t)x\|dt$$
$$= \int_0^T \|S_\lambda(t)z - S_\lambda(t)x\|dt + \int_0^T \|-A_\lambda S_\lambda(t)z + A_\lambda S_\lambda(t)x\|dt$$
$$\leq T\|z - x\| + \frac{2T}{\lambda}\|z - x\| = \left(1 + \frac{2}{\lambda}\right)T\|z - x\|.$$

Next we will show that $D(\partial\mathscr{S}_\lambda(x)) = H$ and $\partial\mathscr{S}_\lambda(x)$: $H \to W^{1,1}(0, T; H)$ is a bounded linear mapping. Then we can use Lemma 1.1 to reach the assertion. By Proposition 3.2, $\partial\mathscr{S}_\lambda(x)v_0 \equiv v_\lambda(\cdot; v_0)$ is a unique strong solution of $(L_\lambda; 0, v_0)$. Since $dA_\lambda(S_\lambda(t)x)$ is a bounded linear maximal monotone operator defined on H, $\alpha\partial\mathscr{S}_\lambda(x)v_0 + \beta\partial\mathscr{S}_\lambda(x)w_0$ becomes the strong solution of $(L_\lambda; 0, v_0)$ with initial value $\alpha v_0 + \beta w_0$. Then the uniqueness of the solution implies that $\partial\mathscr{S}_\lambda(x)(\alpha v_0 + \beta w_0) = \alpha\partial\mathscr{S}_\lambda(x)v_0 + \beta\partial\mathscr{S}_\lambda(x)w_0$ (linearity). Furthermore,

$$\|\partial\mathscr{S}_\lambda(x)v_0\|_{W^{1,1}} = \int_0^T \|v_\lambda(t; v_0)\|dt + \int_0^T \|v_\lambda'(t; v_0)\|dt$$
$$\leq \int_0^T \|v_0\|dt + \int_0^T \|dA_\lambda(S_\lambda(t)x)v_\lambda(t; v_0)\|dt$$
$$\leq T\|v_0\| + \frac{2T}{\lambda}\|v_0\| = \left(1 + \frac{2}{\lambda}\right)T\|v_0\|.$$

This completes the proof. □

REMARK 3.2. The conclusion of Lemmas 3.2 and 3.3 is stronger than [6, Proposition 3.2] since $W^{1,1} \subset C([0, T]; H)$. It is not necessary to be stronger to proceed the argument.

LEMMA 3.4. *The mapping* $z \mapsto d\mathscr{S}_\lambda(z)v$ *is continuous from* H *to* $C([0, T]; H)$ *for every* $v \in H$, $T > 0$.

This lemma is shown by the same way as [6, Lemma 3.3], using Kisyński's technique. Hence the proof is omitted. The next lemma is also proved in the same way as in [6, Lemma 3.4].

LEMMA 3.5. *The following equality holds*:

$$\mathscr{S}_\lambda y - \mathscr{S}_\lambda x = \int_0^1 d\mathscr{S}_\lambda(\theta y + (1-\theta)x)(y-x)d\theta \quad \text{in } C([0, T]; H), \tag{3.3}$$

where the intergral is taken in the sense of Bochner.

4. Proof of Theorem

Let $T > 0$ and fix $t_0 \in (0, T)$ arbitrarily. Let $u \in A^{-1}0$. When (H4) holds, the following esitimate concerning the parabolic regularity is valid by virtue of [3, p.59 (22)]:

$$\|J_\lambda^A S_\lambda(t)x - u\| + \|A_\lambda S_\lambda(t)x\| \leq \|x - u\| + \|A^0 u\| + \frac{1}{t}\|x - u\|$$

$$= \left(1 + \frac{1}{t}\right)\|x - u\| \leq \left(1 + \frac{1}{t_0}\right)\|x - u\|, \quad t \in [t_0, T]. \tag{4.1}$$

Hence, there exists a $\delta > 0$ such that $\|x - u\| < \delta$ implies

$$(J_\lambda^A S_\lambda(t)x, A_\lambda S_\lambda(t)x) \in G(A) \cap \mathscr{U} \quad \text{for } t \in [t_0, T], \tag{4.2}$$

where $\mathscr{U}$ is the one appeared in the hypothesis (H3). We need the following lemma, which is derived from (H3). In fact, it is where (H3) is used.

LEMMA 4.2. *Let* t_0, u, δ *be as above. Take* x *satisfying* $\|x - u\| < \delta$ *and* $w \in H$. *Let* $v_\lambda^x(t) = v_\lambda^x(t; t_0, w)$ *and* $v_\lambda^u(t) = v_\lambda^u(t; t_0, w)$ *be solutions of* $(L_\lambda; t_0, w)$ *corresponding to the operators* $dA_\lambda(S_\lambda(t)x)$ *and* $dA_\lambda(S_\lambda(t)u)(\equiv dA_\lambda(u))$, *respectively. Then we have*

$$\|v_\lambda^x(t) - v_\lambda^u(t)\| \leq (t - t_0)\left(1 + \frac{1}{t_0}\right)\|x - u\| L(\|w\|). \tag{4.3}$$

PROOF. For simplicity, we set $J_\lambda^x(t) := J_\lambda^{\partial A_\lambda^x(t)}$, where $\partial A_\lambda^x(t)$ is the one in the proof of Proposition 3.1. Then v_λ^x satisfies

$$v_\lambda^x(t) = e^{-\frac{t-t_0}{\lambda}} w + \frac{1}{\lambda}\int_{t_0}^t e^{\frac{\tau-t}{\lambda}} J_\lambda^x(\tau) v_\lambda^x(\tau)\,d\tau \tag{4.4}$$

as mentioned in Remark 3.1. It follows from (4.1), (4.2), (4.4) and (H3) that

$$\|v_\lambda^x(t) - v_\lambda^u(t)\| \le \frac{1}{\lambda}\int_{t_0}^t e^{\frac{\tau-t}{\lambda}} \|J_\lambda^x(\tau)v_\lambda^x(\tau) - J_\lambda^u(\tau)v_\lambda^u(\tau)\|\,d\tau$$

$$\le \frac{1}{\lambda}\int_{t_0}^t e^{\frac{\tau-t}{\lambda}} \|J_\lambda^x(\tau)v_\lambda^x(\tau) - J_\lambda^u(\tau)v_\lambda^x(\tau)\|\,d\tau + \frac{1}{\lambda}\int_{t_0}^t e^{\frac{\tau-t}{\lambda}} \|J_\lambda^u(\tau)v_\lambda^x(\tau) - J_\lambda^u(\tau)v_\lambda^u(\tau)\|\,d\tau$$

$$\le \int_{t_0}^t e^{\frac{\tau-t}{\lambda}} (\|J_\lambda^A S_\lambda(\tau)x - u\| + \|A_\lambda S_\lambda(\tau)x\|) L(\|v_\lambda^x(\tau)\|)\,d\tau + \frac{1}{\lambda}\int_{t_0}^t e^{\frac{\tau-t}{\lambda}} \|v_\lambda^x(\tau) - v_\lambda^u(\tau)\|\,d\tau$$

$$\le \left(1 + \frac{1}{t_0}\right)\|x - u\| L(\|w\|) \int_{t_0}^t e^{\frac{\tau-t}{\lambda}}\,d\tau + \frac{1}{\lambda}\int_{t_0}^t e^{\frac{\tau-t}{\lambda}} \|v_\lambda^x(\tau) - v_\lambda^u(\tau)\|\,d\tau$$

$$= \left(1 + \frac{1}{t_0}\right)\|x - u\| L(\|w\|)\lambda(1 - e^{-\frac{t-t_0}{\lambda}}) + e^{-\frac{t}{\lambda}} \int_{t_0}^t \frac{1}{\lambda} e^{\frac{\tau}{\lambda}} \|v_\lambda^x(\tau) - v_\lambda^u(\tau)\|\,d\tau.$$

By Gronwall's lemma, we achieve the desired inequality (4.3). □

PROOF OF THEOREM 2.1. Let $\bar{u}$ be a stationary solution of (E), i.e., $A\bar{u} \ni 0$. By the assumption of Theorem 2.1, there exists an $\omega > 0$ such that $\partial A(\bar{u}, 0) - \omega I$ is maximal monotone. Then it follows that $A_\lambda \bar{u} = 0$ and $dA_\lambda(\bar{u}) - \omega_\lambda I$ is maximal motone with $\omega_\lambda := \omega/(1 + \lambda\omega)$. Then we have the estimate

$$\|v_\lambda^{\bar{u}}(t; t_0, w)\| \le e^{-\omega_\lambda(t-t_0)}\|w\|, \quad w \in H,\ t \ge t_0, \tag{4.5}$$

where $v_\lambda^{\bar{u}}(t; t_0, w)$ is a solution of $(L_\lambda; t_0, w)$ with the operator $dA_\lambda(S_\lambda(t)\bar{u}) \equiv dA_\lambda(\bar{u})$. For, by the fact that $dA_\lambda(\bar{u}) - \omega_\lambda I$ is a linear maximal monotone operator,

$$\omega_\lambda \|v_\lambda^{\bar{u}}(t)\|^2 \le \langle dA_\lambda(\bar{u}) v_\lambda^{\bar{u}}(t), v_\lambda^{\bar{u}}(t)\rangle \le -\frac{1}{2}\frac{d}{dt}\|v_\lambda^{\bar{u}}(t)\|^2, \quad \text{a.e. } t \ge t_0.$$

From this one easily sees that $\|v_\lambda^{\bar{u}}(t)\| \le e^{-\omega_\lambda(t-t_0)}\|v_\lambda^{\bar{u}}(t_0)\|$.

Now take $\varepsilon_0 \in (0, \omega)$. Since $\omega_\lambda \uparrow \omega$, there is a $\lambda_0 > 0$ such that $0 < \lambda < \lambda_0$ implies that $0 < \omega - \omega_\lambda < \varepsilon_0$. Let $0 < \lambda < \lambda_0$. Then there exists an $\alpha > 0$ such that $0 < \alpha < \omega - \varepsilon_0 < \omega_\lambda < \omega$. Furthermore, we can take $\varepsilon > 0$ such that $0 < \varepsilon < e^{-\alpha t_0} - e^{-(\omega - \varepsilon_0)t_0}$. Now let $\delta > 0$ be a constant depending on t_0 and $\bar{u}$, for which (4.2) with $\bar{u}$ in place of u is satisfied. Besides, take $\eta > 0$ such that $0 < \eta < \min\{\delta, \varepsilon/\tilde{L}(1)t_0\}$, where $\tilde{L}(r) := \left(1 + \frac{1}{t_0}\right)L(r)$.

Put $\bar{t} := 2t_0 (> t_0)$ and observe that by Lemma 4.2 and (4.5), the following estimate holds:

$$\begin{aligned} \|[d\mathscr{S}_\lambda(x)v_0](\bar{t})\| &= \|v_\lambda^x(\bar{t}; t_0, [d\mathscr{S}_\lambda(x)v_0](t_0))\| \\ &\le \|v_\lambda^x(\bar{t}; t_0, [d\mathscr{S}_\lambda(x)v_0](t_0)) - v_\lambda^{\bar{u}}(\bar{t}; t_0, [d\mathscr{S}_\lambda(x)v_0](t_0))\| \\ &\quad + \|v_\lambda^{\bar{u}}(\bar{t}; t_0, [d\mathscr{S}_\lambda(x)v_0](t_0))\| \\ &\le t_0\|x - \bar{u}\|\tilde{L}(\|[d\mathscr{S}_\lambda(x)v_0](t_0)\|) + e^{-\omega_\lambda t_0}\|[d\mathscr{S}_\lambda(x)v_0](t_0)\| \\ &\le t_0\eta\tilde{L}(\|v_0\|) + e^{-(\omega-\varepsilon_0)t_0}\|v_0\|, \end{aligned} \tag{4.6}$$

provided $\|x - \bar{u}\| < \eta$. From (4.6), we know that if $\|x - \bar{u}\| < \eta$, then

$$\sup_{\|v_0\| \le 1} \|[d\mathscr{S}_\lambda(x)v_0](\bar{t})\| \le t_0\eta\tilde{L}(1) + e^{-(\omega-\varepsilon_0)t_0} < \varepsilon + e^{-(\omega-\varepsilon_0)t_0} < e^{-\alpha t_0}.$$

Since $v \mapsto d\mathscr{S}_\lambda(x)v$ is linear, we obtain

$$\|[d\mathscr{S}_\lambda(x)v](\bar{t})\| \le e^{-\alpha t_0}\|v\| \quad \text{for } v \in H, \tag{4.7}$$

whenever $\|x - \bar{u}\| < \eta$, $0 < \lambda < \lambda_0$.

Let $\|x - \bar{u}\| < \eta$ and $\theta \in [0, 1]$. Since $\|\theta x + (1-\theta)\bar{u} - \bar{u}\| = \theta\|x - \bar{u}\| < \eta$, by (4.7),

$$\|[d\mathscr{S}_\lambda(\theta x + (1-\theta)\bar{u})(x - \bar{u})](\bar{t})\| \le e^{-\alpha t_0}\|x - \bar{u}\|.$$

Noting that $S_\lambda(t)\bar{u} = \bar{u}$, it follows from Lemma 3.5 that if $\|x - \bar{u}\| < \eta$, then

$$\|S_\lambda(\bar{t})x - \bar{u}\| \le e^{-\alpha t_0}\|x - \bar{u}\|. \tag{4.8}$$

Let $u_0 \in \overline{D(A)}$. $\|u_0 - \bar{u}\| < \eta$. For any integer k,

$$\|S_\lambda^k(\bar{t})u_0 - \bar{u}\| = \|S_\lambda^k(\bar{t})u_0 - S_\lambda^k(\bar{t})\bar{u}\| \le \|u_0 - \bar{u}\| < \eta.$$

Accordingly, we can use (4.8) repeatedly, and have

$$\begin{aligned} \|S_\lambda(k\bar{t})u_0 - \bar{u}\| &= \|S_\lambda^k(\bar{t})u_0 - \bar{u}\| = \|S_\lambda(\bar{t})(S_\lambda^{k-1}(\bar{t})u_0) - \bar{u}\| \\ &\le e^{-\alpha t_0}\|S_\lambda^{k-1}(\bar{t})u_0 - \bar{u}\| \le \cdots \le e^{-\alpha k t_0}\|u_0 - \bar{u}\|. \end{aligned} \tag{4.9}$$

Now, take any $t > \bar{t} (= 2t_0)$ and put

$k := [t/\bar{t}]$ ([ ] denotes the Gaussian bracket), $t^* := t - k\bar{t}$.

Then $0 < t - \bar{t} < k\bar{t} \leq t$, especially $0 \leq t^* < \bar{t}$. Hence we have

$$\|S_\lambda(t)u_0 - \bar{u}\| = \|S_\lambda(t^*)S_\lambda(k\bar{t})u_0 - \bar{u}\| \leq \|S_\lambda(k\bar{t})u_0 - \bar{u}\|$$
$$\leq e^{-\alpha k t_0}\|u_0 - \bar{u}\| = e^{-\frac{\alpha}{2}k\bar{t}}\|u_0 - \bar{u}\| \quad \text{(by (4.9))}$$
$$\leq e^{-\frac{\alpha}{2}(t-\bar{t})}\|u_0 - \bar{u}\| = e^{\frac{\alpha}{2}\bar{t}}e^{-\frac{\alpha}{2}t}\|u_0 - \bar{u}\|.$$

Consequently, setting $\gamma = \dfrac{\alpha}{2} > 0$ and $C = e^{\frac{\alpha}{2}\bar{t}} = e^{\alpha t_0}$, we have

$$\|S_\lambda(t)u_0 - \bar{u}\| \leq Ce^{-\gamma t}\|u_0 - \bar{u}\|.$$

Since $u_0 \in \overline{D(A)}$ letting $\lambda \downarrow 0$, we acheive

$$\|S(t)u_0 - \bar{u}\| \leq Ce^{-\gamma t}\|u_0 - \bar{u}\|, \quad t > \bar{t}.$$

For $0 \leq t \leq \bar{t}$, it is easily verified that $\|S(t)u_0 - \bar{u}\| \leq e^{\bar{t}}e^{-t}\|u_0 - \bar{u}\|$. Thus the proof is complete. □

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