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Nonlinear Potentials on an Infinite Network

Dedicated to Professor Mitsuru Nakai on his 60th birthday

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The aim of this paper is to study some properties of Dirichlet potentials and pure potentials of order p on an infinite network. It will be shown that several assumptions in the theory of general nonlinear potential theory are fulfilled in our case. The space of Dirichlet potentials of order p will play a crucial role in our study if the network is hyperbolic of order p. A nonlinear version of Cartan's domination principle will be shown with elementary properties of p-superharmonic functions.

§1. Introduction

It has been well-known that a discrete analogue to potential theory on Riemann surfaces (see for instance [5] and [7]) plays important roles in the study of discrete harmonic functions on an infinite network or random walks. Our aim is to investigate nonlinear version of discrete potential theory along the same lines as in [6] or [8]. For notation and terminology, we mainly follow [8] and [10].

More precisely, let $N = \{X, Y, K, r\}$ be an infinite network which is locally finite and has no self-loop. Denote by $L(X)$(resp. $L(Y)$) the set of all real valued functions on the set X (resp. Y) of nodes (resp. arcs) and by $L_0(X)$ the set of all $u \in L(X)$ with finite support. Let p and q be positive numbers such that $1 < p < \infty$ and $1/p + 1/q = 1$. The energy $H_p(w)$ of $w \in L(Y)$ of order p is defined by

$$H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p.$$

The mutual energy $\langle w, w' \rangle$ of $w, w' \in L(Y)$ is defined by

$$\langle w, w' \rangle = \sum_{y \in Y} r(y) w(y) w'(y)$$

if the sum is well-defined. Denote by $L_p(Y; r)$ the set of all $w \in L(Y)$ with finite energy of order p. The mutual energy is well-defined for the pair of elements in $L_p(Y; r)$ and $L_q(Y; r)$.

The Dirichlet sum $D_p(u)$ of $u \in L(X)$ of order p is defined by

$$D_p(u) = H_p(du) = \sum_{y \in Y} r(y)|du(y)|^p,$$

where $du \in L(Y)$ is the discrete derivative of u, i.e.,

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y)u(x).$$

Denote by $\mathbf{D}^{(p)}(N)$ the set of all $u \in L(X)$ with finite Dirichlet sum of order p and by $\mathbf{D}_0^{(p)}(N)$ the closure of $L_0(X)$ in $D^{(p)}(N)$ with respect to the norm:

$$\|u\|_p = [D_p(u) + |u(x_0)|^p]^{1/p},$$

where x_0 is a fixed node. An element of $\mathbf{D}_0^{(p)}(N)$ is called a Dirichlet potential of order p.

The (discrete) p-Laplacian $\Delta_p u \in L(X)$ of $u \in L(X)$ is defined by

$$\Delta_p u(x) := \sum_{y \in Y} K(x, y)|du(y)|^{p-1} \operatorname{sign}(du(y)),$$

where $\operatorname{sign}(t) = 1$ if $t \geq 0$ and $\operatorname{sign}(t) = -1$ if $t < 0$.

A Dirichlet potential u of order p is called a pure potential if it is p-superharmonic on X, i.e., $\Delta_p u(x) \leq 0$ on X. According to the framework of Kenmochi and Mizuta [3], we shall study some properties of pure potentials. Several assumptions in [3] will be verified in Sections 2, 3 and 4. By means of the domination principle for pure potentials and a result in §7 that the lower envelope of two p-superharmonic functions is also p-superharmonic, we shall obtain Cartan's domination principle.

§2. Preliminaries

Let us introduce the following real valued function which plays a central role in our study:

$$\phi_p(t) = |t|^{p-1} \operatorname{sign}(t) = |t|^{p-2} t.$$

Define $\phi_p(w) \in L(Y)$ for $w \in L(Y)$ by

$$\phi_p(w)(y) = \phi_p(w(y)).$$

Then $w \in L_q(Y; r)$ implies $\phi_p(w) \in L_p(Y; r)$. Notice that

$$\Delta_p u(x) = \sum_{y \in Y} K(x, y)\phi_p(du(y)),$$

$$H_p(w) = \sum_{y \in Y} r(y)w(y)\phi_p(w(y)) = \langle w, \phi_p(w) \rangle,$$

$$D_p(u) = \langle \phi_p(du), du \rangle = H_q(\phi_p(du)).$$

For later use, we list the following fundamental properties related to ϕ_p(cf. [8]):

LEMMA 2.1. *$\langle \phi_p(w) - \phi_p(w'), w - w' \rangle \geq 0$ for all $w, w' \in L_q(Y; r)$. The equality holds only if $w = w'$.*

LEMMA 2.2. *Let $u \in \mathbf{D}^{(p)}(N)$. Then*

$$D_p(v) - D_p(u) \geq \langle p\phi_p(du), dv - du \rangle$$

for every $v \in \mathbf{D}^{(p)}(N)$.

LEMMA 2.3. *Let $u, f \in L(X)$. If any one of u and f belongs to $L_0(X)$, then*

$$\langle \phi_p(du), df \rangle = -\sum_{x \in X} [\Delta_p u(x)] f(x).$$

Recall that N is parabolic of order p if the value of the following extremum problem related to a nonempty finite set A of nodes and the point of infinity vanishes:

(P.1) Find $d_p(A, \infty) = \inf\{D_p(u); u \in L_0(X) \text{ and } u = 1 \text{ on } A\}$.

We say that N is hyperbolic of order p if it is not parabolic of order p. The following result is well-known (cf. [6]):

LEMMA 2.4. *The following are equivalent:*

(*a*) *N is parabolic of order p;*

(*b*) $\mathbf{D}_0^{(p)}(N) = \mathbf{D}^{(p)}(N)$;

(*c*) $1 \in \mathbf{D}_0^{(p)}(N)$.

For other practical criteria for parabolicity, we refer to [5] and [6].

By Lemmas 2.1 and 2.4, we have

LEMMA 2.5. *Assume that N is hyperbolic of order p. If two Dirichlet potentials u and v of order p satisfy*

$$\langle \phi_p(du) - \phi_p(dv), du - dv \rangle = 0,$$

then $u = v$.

The following result was shown in the proof of [8; Theorem 2.1]:

LEMMA 2.6. *Assume that N is hyperbolic of order p and let $\{u_n\}$ be a sequence in $\mathbf{D}_0^{(p)}(N)$. If $\{D_p(u_n)\}$ is bounded, then $\{u_n(x)\}$ is bounded for every $x \in X$.*

We proved in [9; Theorem 4.1]

LEMMA 2.7. *Let $\{v_n\}$ be a sequence in $\mathbf{D}_0^{(p)}(N)$ which converges pointwise to $v \in L(X)$. If $\{D_p(v_n)\}$ is bounded, then $v \in \mathbf{D}_0^{(p)}(N)$.*

By the same argument as in [4; Theorem 4.1], we can prove the following nonlinear version of discrete Green's formula:

LEMMA 2.8. *Let* $v \in \mathbf{D}_0^{(p)}(N)$ *and* $u \in \mathbf{D}^{(p)}(N)$. *Then Green's identity*

$$\langle \phi_p(du), dv \rangle = -\sum_{x \in X}[\Delta_p u(x)]v(x)$$

holds if any one of the following conditions is fulfilled:

(i) $\sum_{x \in X} |[\Delta_p u(x)]v(x)| < \infty$;

(ii) $v \in L^+(X)$ *and* $-\Delta_p u \in L^+(X)$.

§3. Functional spaces

In order to apply the theory due to Kenmochi and Mizuta [3] to our study, we shall verify their axiom for functional spaces.

We say that a subspace $\mathscr{X}$ of $\mathbf{D}^{(p)}(N)$ satisfies Axiom (a) if the following condition is fulfilled:

(H.1) For every nonempty finite subset F of X, there exists a constant $M(F)$ such that

$$\sum_{x \in F} |u(x)| \leq M(F)[D_p(u)]^{1/p} \tag{3.1}$$

for all $u \in \mathscr{X}$.

Clearly $\mathbf{D}^{(p)}(N)$ does not satisfy (H.1), since $1 \in \mathbf{D}^{(p)}(N)$.

THEOREM 3.1. $\mathbf{D}_0^{(p)}(N)$ *satisfies Axiom* (a) *if and only if* N *is hyperbolic of order* p.

PROOF. Assume that N is hyperbolic of order p and let F be a nonempty finite subset of X. It suffices to show that there exists a constant $M(F)$ which satisfies (3.1) for all $u \in \mathbf{D}_0^{(p)}(N)$ with $D_p(u) = 1$. Supposing the contrary, we can find a sequence $\{u_n\}$ in $\mathbf{D}_0^{(p)}(N)$ such that $D_p(u_n) = 1$ and $\sum_{x \in F} |u_n(x)| \to \infty$ as $n \to \infty$. Since F is a finite set, we may assume $|u_n(b)| \to \infty$ as $n \to \infty$ for some $b \in F$. This contradicts our assumption by Lemma 2.6. Therefore (H.1) is fulfilled. Next assume that N is parabolic of order p. Then there exists a nonempty finite subset F of X such that $d_p(F, \infty) = 0$. Thus we can find a sequence $\{u_n\}$ in $L_0(X)$ such that $u_n = 1$ on F and $D_p(u_n) \to 0$ as $n \to \infty$, which contradicts (H.1).

THEOREM 3.2. *If* N *is hyperbolic of order* p, *then* $\mathbf{D}_0^{(p)}(N)$ *is a Banach space with respect to the norm* $|u|_p := [D_p(u)]^{1/p}$.

PROOF. Clearly $|u|_p$ is a pseudo-norm on $\mathbf{D}^{(p)}(N)$. In case N is hyperbolic of order p, $|u|_p = 0$ and $u \in \mathbf{D}_0^{(p)}(N)$ imply $u = 0$ by Lemma 2.4. Hence $|u|_p$

is a norm on $\mathbf{D}_0^{(p)}(N)$. To prove the completeness of $\mathbf{D}_0^{(p)}(N)$ with respect to this norm, let $\{u_n\}$ be a sequence in $\mathbf{D}_0^{(p)}(N)$ such that $|u_n - u_m|_p \to 0$ as $n, m \to \infty$. Then $\{|u_n|_p\}$ is bounded, and hence $\{u_n(x_0)\}(x_0 \in X)$ is bounded by Lemma 2.6. Let $\{u'_n\}$ and $\{u''_n\}$ be subsequences of $\{u_n\}$ such that both $\{u'_n(x_0)\}$ and $\{u''_n(x_0)\}$ converge. Recall that $\mathbf{D}^{(p)}(N)$ is a Banach space with respect to the norm $\|\cdot\|_p$ and that $\mathbf{D}_0^{(p)}(N)$ is a closed subspace of $\mathbf{D}^{(p)}(N)$. There exist $u', u'' \in \mathbf{D}_0^{(p)}(N)$ such that

$$\|u'_n - u'\|_p \to 0 \quad \text{and} \quad \|u''_n - u''\|_p \to 0$$

as $n \to \infty$. It follows that

$$|u' - u''|_p \leq |u' - u'_n|_p + |u'_n - u''_n|_p + |u''_n - u''|_p \to 0$$

as $n \to \infty$, so that $u' = u''$. Namely there exists $u \in \mathbf{D}_0^{(p)}(N)$ such that $|u_n - u|_p \to 0$ as $n \to \infty$.

§4. Contractions

We say that a function T on the real line R into itself is a normal contraction of R if $T0 = 0$ and $|Ts - Ts'| \leq |s - s'|$ for every $s, s' \in R$. For $u \in L(X)$, define $Tu \in L(X)$ by $(Tu)(x) = T(u(x))$. If T is a normal contraction, then $D_p(Tu) \leq D_p(u)$, so that $u \in \mathbf{D}^{(p)}(N)$ implies $Tu \in \mathbf{D}^{(p)}(N)$.

We proved in [9; Theorem 4.2]

THEOREM 4.1. *Let T be a normal contraction of R. Then $u \in \mathbf{D}_0^{(p)}(N)$ implies $Tu \in \mathbf{D}_0^{(p)}(N)$.*

Let us consider the following condition related to a contraction T of R:

$$(C_p) \qquad \langle \phi_p(d(u + Tv)) - \phi_p(du), \quad d(v - Tv) \rangle \geq 0$$

for every $u, v \in \mathbf{D}^{(p)}(N)$.

We shall prove

THEOREM 4.2. *Assume that a normal contraction T of R is monotone, i.e.,*

$$(4.1) \qquad (Ts_1 - Ts_2)(s_1 - s_2) \geq 0 \quad \textit{for every } s_1, s_2 \in R.$$

Then Condition (C_p) holds.

PROOF. Take $u, v \in \mathbf{D}^{(p)}(N)$ and $y \in Y$ and put

$$s(y) = du(y), \; s_1(y) = d(Tv)(y), \; s_2(y) = d(v - Tv)(y).$$

To verify (C_p), it suffices to show that

$$(4.2) \qquad \phi_p(s(y) + s_1(y))s_2(y) \geq \phi_p(s(y))s_2(y)$$

holds for every $y \in Y$. Let $e(y) = \{a, b\}$ and set

$$\alpha = Tv(a) - Tv(b) \quad \text{and} \quad \beta = v(a) - v(b).$$

Since $\alpha\beta \geq 0$ by (4.1) and $|\alpha| \leq |\beta|$, we have

$$s_1(y)s_2(y) = r(y)^{-2}(\alpha\beta - \alpha^2) \geq 0.$$

Since $\phi_p(t)$ is an increasing function, (4.2) holds.

Let k be a nonnegative number or ∞. Then the mapping T_k from R to R^+ defined by

$$T_k s = \min\{\max(s, 0), k\}$$

is a monotone contraction of R. In particular, $T_\infty s = \max\{s, 0\} = s^+$. For $v \in L(X)$, put $v^+ = T_\infty v$ and $v^- = T_\infty(-v)$.

COROLLARY 4.3. *For every $u, v \in \mathbf{D}^{(p)}(N)$,*

$$\langle \phi_p(d(u - v^+)), dv^- \rangle \geq \langle \phi_p(du), dv^- \rangle.$$

PROOF. Replacing u by $-u$ in (C_p), we have

$$\langle \phi_p(d(u - T_\infty v)) - \phi_p(du), d(v - T_\infty v) \rangle \leq 0$$

for every $u, v \in \mathbf{D}^{(p)}(N)$.

REMARK 4.4. We see by [3; Proposition 2.1] that the following two conditions are equivalent:

(C_p^k) $$\langle \phi_p(du + dT_k v) - \phi_p(du), dv - dT_k v \rangle \geq 0$$

for every $u, v \in \mathbf{D}_0^{(p)}(N)$;

(D_p^k) $$D_p(u) + D_p(v) \geq D_p(u + T_k(v - u)) + D_p(v - T_k(v - u))$$

for every $u, v \in \mathbf{D}_0^{(p)}(N)$.

§5. Potentials of order p

For $\mu \in L(X)$, denote by $\mathbf{PSD}^{(p)}(\mu)$ the set of solutions of the nonlinear Poisson equation: $\Delta_p u = -\mu$ with finite Dirichlet sum of order p.

We proved in [10; Theorem 3.2]

LEMMA 5.1. *$\mathbf{PSD}^{(p)}(\mu)$ is nonempty if and only if there exists $w \in L_q(Y; r)$ which satisfies the relation:*

$$\partial w(x) := \sum_{y \in Y} K(x, y) w(y) = \mu(x) \quad \textit{on } X.$$

If N is hyperbolic of order p and if $\mathbf{PSD}^{(p)}(\mu)$ is nonempty, then $\mathbf{PSD}^{(p)}(\mu) \cap \mathbf{D}_0^{(p)}(N)$ is a singleton by [10; Theorem 3.3]. In this case, we denote this element by u_μ and call μ the associated measure of u_μ.

By Lemma 2.3, we see easily

LEMMA 5.2. *Let $u \in \mathbf{D}_0^{(p)}(N)$. Then the following are equivalent:*

(a) $\quad u \in \mathbf{PSD}^{(p)}(\mu)$;

(b) $\quad \langle \phi_p(du), df \rangle = -\sum_{x \in X} f(x)\mu(x)$

for all $f \in L_0(X)$.

Denote by $\mathbf{AM}_p(N)$ the set of all associated measures of potentials of order p. In case N is hyperbolic of order p, $L_0(X) \subset \mathbf{AM}_p(N)$ by [10; Theorem 4.1]. Notice that by [10; Theorems 3.1 and 3.2],

$$\{\Delta_p u;\ u \in \mathbf{D}^{(p)}(N)\} \subset \mathbf{AM}_p(N).$$

Clearly, $\mathbf{AM}_p(N)$ is a linear space by Lemma 5.2. Furthermore we proved the following result in [10; Theorem 4.4] as a counterpart of [3; Lemma 3.1].

LEMMA 5.3. *Let $\mu, \nu \in \mathbf{AM}_p(N)$ and $\sigma \in L(X)$. If $\mu(x) \le \sigma(x) \le \nu(x)$ on X, then $\sigma \in \mathbf{AM}_p(N)$.*

We have

LEMMA 5.4. *Assume that $\tilde{w} \in L_q(Y; r)$ and $\langle \tilde{w}, df \rangle \ge 0$ for all $f \in L_0(X) \cap L^+(X)$. Then $\mu := -\partial\tilde{w} \in L^+(X)$ and $\mathbf{PSD}^{(p)}(\mu)$ is nonempty.*

PROOF. By Lemma 5.1, $\mathbf{PSD}^{(p)}(\mu)$ is nonempty. By our assumption,

$$\mu(x) = -\partial\tilde{w}(x) = \langle \tilde{w}, d\varepsilon_x \rangle \ge 0$$

for every $x \in X$, where $\varepsilon_x(z) = 0$ for $z \ne x$ and $\varepsilon_x(x) = 1$.

§6. Pure potentials of order p

We always assume in this section that N is hyperbolic of order p.

We recall

DEFINITION 6.1. A Dirichlet potential u of order p is called a pure potential of order p if it is p-superharmonic on X, i.e., $\Delta_p u(x) \le 0$ on X.

We shall prove

THEOREM 6.2. *Let $\tilde{v} \in \mathbf{D}_0^{(p)}(N)$. Then the following are equivalent:*

(*a*) $\tilde{v}$ *is a pure potential of order* p;

(*b*) $D_p(u+\tilde{v}) \geq D_p(\tilde{v})$ *for all* $u \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$;

(*c*) $\tilde{v}$ *is a unique optimal solution of the following extremum problem*:

(P.2) *Minimize* $D_p(u)$
subject to $u \in \mathbf{D}_0^{(p)}(N)$ *and* $u \geq \tilde{v}$ *on* X;

(*d*) $\langle \phi_p(d\tilde{v}), du \rangle \geq 0$ *for all* $u \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$.

PROOF. Clearly (b) and (c) are equivalent. Assume that $\tilde{v}$ is an optimal solution of (P.2). Then we have by the variational technique

$$\langle \phi_p(d\tilde{v}), du \rangle = \sum_{y \in Y} r(y) \phi_p(d\tilde{v}(y)) du(y) \geq 0 \tag{6.1}$$

for every $u \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$. Taking $u = \varepsilon_x$, we see by (6.1) that $\Delta_p \tilde{v}(x) \leq 0$ on X. Thus (c) implies (a) and (d).

Next we assume that $\tilde{v}$ is a pure potential of order p. Then we have by Lemma 2.2

$$D_p(u+\tilde{v}) - D_p(\tilde{v}) \geq \langle p\phi_p(d\tilde{v}(y)), du(y) \rangle$$

for every $u \in \mathbf{D}^{(p)}(N)$. Let $u \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$. Then we have by Lemma 2.8

$$\langle \phi_p(d\tilde{v}), du \rangle = -\sum_{x \in X} [\Delta_p \tilde{v}(x)] u(x) \geq 0,$$

and hence $D_p(u+\tilde{v}) \geq D_p(\tilde{v})$. Namely (a) implies (b) and (d). Clearly, (d) implies (a) by Lemma 2.3.

THEOREM 6.3. *A pure potential of order* p *is nonnegative, namely,* $u \in \mathbf{D}_0^{(p)}(N)$ *and* $\Delta_p u(x) \leq 0$ *on* X *imply* $u(x) \geq 0$ *on* X.

PROOF. Let v be a pure potential of order p and consider the normal contraction T of R defined by $Ts = |s|$. Then $|v| \in \mathbf{D}_0^{(p)}(N)$ by Theorem 4.1 and $D_p(|v|) \leq D_p(v)$. Since $u = |v| - v \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$, we see by Theorem 6.2

$$D_p(v) \leq D_p(u+v) = D_p(|v|),$$

and hence $D_p(|v|) = D_p(v)$. By the uniqueness of the optimal solution of (P.2), $v = |v| \in L^+(X)$.

We prepare

LEMMA 6.4. *Let* $\mu, \nu \in \mathbf{AM}_p(N)$. *If* $\mu(x) \geq \nu(x)$ *on* X, *then*

$$\langle \phi_p(du_\mu) - \phi_p(du_\nu), dv \rangle \geq 0$$

for every $v \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$.

PROOF. By Lemma 2.8,

$$\begin{aligned}\langle \phi_p(du_\mu), dv \rangle &= -\sum_{x\in X}[\Delta_p u_\mu(x)]v(x) \\ &= \sum_{x\in X}\mu(x)v(x) \\ &\geq \sum_{x\in X}\nu(x)v(x) = \langle \phi_p(du_\nu), dv \rangle.\end{aligned}$$

For $u, v \in L(X)$, define $u \wedge v \in L(X)$ by

$$(u \wedge v)(x) := \min\{u(x), v(x)\}.$$

The following result was proved in [3, Theorem 3.2].

THEOREM 6.5. *Let* $\mu, \nu \in \mathbf{AM}_p(N)$. *If there exists* $\sigma \in \mathbf{AM}_p(N)$ *such that* $(\mu \wedge \nu)(x) \geq \sigma(x)$ *on* X, *then there exists* $\eta \in \mathbf{AM}_p(N)$ *such that* $u_\eta = u_\mu \wedge u_\nu$ *and* $\eta(x) \geq \sigma(x)$ *on* X.

PROOF. By Lemma 5.3, $\tau = \mu \wedge \nu \in \mathbf{AM}_p(N)$. Consider the following extremum problem:

(P.3) Minimize $J(v) := D_p(v) - \langle p\phi_p(du_\tau), dv \rangle$
subject to $v \in C = \{v \in \mathbf{D}_0^{(p)}(N);\, v(x) \geq (u_\mu \wedge u_\nu)(x) \text{ on } X\}$.

For simplicity, put $w_0 = \phi_p(du_\tau)$ and $\alpha = \inf\{J(v);\, v \in C\}$. From the relation

$$|\langle w_0, dv \rangle| \leq M_1[D_p(v)]^{1/p} \quad \text{with} \quad M_1 = [H_q(w_0)]^{1/q} < \infty,$$

$$J(v) \geq D_p(v) - pM_1[D_p(v)]^{1/p},$$

it follows that $\alpha \geq \inf\{t^p - pM_1 t;\, t \in R^+\} > -\infty$. Let $\{v_n\}$ be a minimizing sequence for (P.3). Then $\{J(v_n)\}$ is bounded, i.e., there exists a constant M_2 such that

$$|D_p(v_n)| \leq p|\langle w_0, dv_n \rangle| + M_2$$

for all n. It follows that $\{D_p(v_n)\}$ is bounded. Thus $\{v_n(x)\}$ is bounded for every $x \in X$ by Lemma 2.6. By choosing subsequences if necessary, we may assume that $\{v_n(x)\}$ converges to $\tilde{v} \in L(X)$ for every $x \in X$. Then $\tilde{v} \in \mathbf{D}_0^{(p)}(N)$ by Lemma 2.7, so that $\tilde{v} \in C$. We have

$$\liminf_{n\to\infty} D_p(v_n) \geq D_p(\tilde{v}) \quad \text{and} \quad \lim_{n\to\infty} \langle w_0, dv_n \rangle = \langle w_0, d\tilde{v} \rangle,$$

and hence

$$\alpha = \lim_{n\to\infty} J(v_n) \geq J(\tilde{v}) \geq \alpha.$$

Namely $\tilde{v}$ is an optimal solution of (P.3). Therefore

(6.2) $$0 \leq \langle \phi_p(d\tilde{v}) - \phi_p(du_\tau), dv - d\tilde{v} \rangle$$

for every $v \in C$. In fact, since $tv + (1-t)\tilde{v} \in C$ for any $v \in C$ and $t \in R$ such that $0 < t < 1$, we have

$$J(\tilde{v}) \leq J(tv + (1-t)\tilde{v}).$$

Therefore the derivative of $J(tv + (1+t)\tilde{v})$ (with respect to t) at $t = 0$ is nonnegative and (6.2) follows. Define $Bv \in L(Y)$ for $v \in L(X)$ by

$$Bv(y) := \phi_p(dv(y)) - \phi_p(du_\tau(y))$$

and put $\tilde{W} = B\tilde{v}$. Then $\tilde{W} \in L_q(Y; r)$ and $0 \leq \langle \tilde{W}, df \rangle$ for all $f \in L_0(X) \cap L^+(X)$ by (6.2), since $v = \tilde{v} + f \in C$. By Lemma 5.4, there exists $u_\lambda \in \mathbf{D}_0^{(p)}(N)$ such that $\lambda = -\partial\tilde{W} \in L^+(X)$. Thus

$$\Delta_p \tilde{v} = \partial\phi_p(d\tilde{v}) = \partial\tilde{W} + \partial\phi_p(du_\tau) = -\lambda - \tau,$$

namely $\tilde{v} = u_\eta$ with $\eta = \lambda + \tau$. Since $\eta \geq \tau \geq \sigma$, our proof is complete if we show that $\tilde{v} = u_\mu \wedge u_\nu$.

For simplicity, put $g = u_\mu - \tilde{v}$. Then $g \in \mathbf{D}_0^{(p)}(N)$,

$$\tilde{v} \wedge u_\mu = u_\mu - g^+ \quad \text{and} \quad \tilde{v} - \tilde{v} \wedge u_\mu = g^-.$$

By Corollary 4.3,

$$\begin{aligned} \langle \phi_p(d(\tilde{v} \wedge u_\mu)), dg^- \rangle &= \langle \phi_p(d(u_\mu - g^+)), dg^- \rangle \\ &\geq \langle \phi_p(du_\mu), dg^- \rangle, \end{aligned}$$

so that by Lemma 6.4

$$\begin{aligned} \langle B(\tilde{v} \wedge u_\mu), dg^- \rangle &\geq \langle \phi_p(du_\mu), dg^- \rangle - \langle \phi_p(du_\tau), dg^- \rangle \\ &= \textstyle\sum_{x \in X} g^-(x)\mu(x) - \sum_{x \in X} g^-(x)\tau(x) \geq 0, \end{aligned}$$

since $\mu(x) \geq \tau(x) \geq 0$ on X. By (6.2),

$$\begin{aligned} &\langle \phi_p(d\tilde{v}) - \phi_p(d(\tilde{v} \wedge u_\mu)), d\tilde{v} - d(\tilde{v} \wedge u_\mu) \rangle \\ &\quad = \langle B\tilde{v}, d\tilde{v} - d(\tilde{v} \wedge u_\mu) \rangle - \langle B(\tilde{v} \wedge u_\mu), dg^- \rangle \leq 0, \end{aligned}$$

since $\tilde{v} \wedge u_\mu \in C$. It follows from Lemmas 2.1 and 2.5 that $\tilde{v} = \tilde{v} \wedge u_\mu$. Similarly we obtain $\tilde{v} = \tilde{v} \wedge u_\nu$. Therefore $\tilde{v} = u_\mu \wedge u_\nu$.

COROLLARY 6.6. *If u and v are pure potentials of order p, then $u \wedge v$ is also a pure potential of order p.*

We have by [3; Theorem 3.3]

THEOREM 6.7. *Let $\mu, \nu \in \mathbf{AM}_p(N)$. Assume that there exists $\sigma \in \mathbf{AM}_p(N)$ such that $(\mu \wedge \nu)(x) \geq \sigma(x)$ on X and*

$$\langle \phi_p(du_\mu) - \phi_p(du_\sigma), d(u_\mu - u_\nu)^+ \rangle = 0.$$

Then $u_\mu(x) \leq u_\nu(x)$ on X.

PROOF. Now the modification of the proofs in [3] to our case may be clear. But we give the proof for completeness. By Theorem 6.5, there exists $\eta \in \mathbf{AM}_p(N)$ such that

$$\eta \geq \sigma \quad \text{and} \quad u_\mu \wedge u_\nu = u_\eta.$$

Notice that

$$u_\mu - u_\eta = u_\mu - u_\mu \wedge u_\nu = (u_\mu - u_\nu)^+.$$

By our assumption and Lemma 6.4,

$$\begin{aligned} &\langle \phi_p(du_\mu) - \phi_p(du_\eta), d(u_\mu - u_\eta) \rangle \\ &\quad = \langle \phi_p(du_\mu) - \phi_p(du_\sigma), d(u_\mu - u_\nu)^+ \rangle \\ &\quad\quad + \langle \phi_p(du_\sigma) - \phi_p(du_\eta), d(u_\mu - u_\nu)^+ \rangle \\ &\quad = - \langle \phi_p(du_\eta) - \phi_p(du_\sigma), d(u_\mu - u_\nu)^+ \rangle \leq 0, \end{aligned}$$

since $(u_\mu - u_\nu)^+ \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$. Thus $u_\mu = u_\eta$ by Lemmas 2.1 and 2.5. Hence $u_\mu(x) \leq u_\nu(x)$ on X.

COROLLARY 6.8. *Let $\mu, \nu \in \mathbf{AM}_p(N)$, If $\mu(x) \leq \nu(x)$ on X, then $u_\mu(x) \leq u_\nu(x)$ on X.*

For $f \in L(X)$, denote by Sf the support of f, i.e.,

$$Sf = \{x \in X; f(x) \neq 0\}.$$

COROLLARY 6.9. *Let $f \in L_0^+(X)$ and v be a pure potential of order p. If $u_f(x) \leq v(x)$ on Sf, then the same inequality holds on X.*

PROOF. Take $\sigma = 0$ in Theorem 6.7. Note that $(u_f - v)^+ \in \mathbf{D}_0^{(p)}(N) \cap L^+(X)$ by Theorem 4.1. If $u_f(x) \leq v(x)$ on Sf, then $(u_f - v)^+(x) = 0$ on Sf, so that by Lemma 2.8

$$\langle \phi_p(du_f), d(u_f - v)^+ \rangle = - \textstyle\sum_{x \in S_f} f(x)[u_f(x) - v(x)]^+ = 0.$$

Hence $u_f(x) \leq v(x)$ on X by Theorem 6.7.

We shall prove the following discrete analogue to Cartan's domination principle:

THEOREM 6.10. *Let $f \in L_0^+(X)$ and $h \in \mathbf{D}^{(p)}(N) \cap L^+(X)$ be p-superharmonic on X. If $u_f(x) \leq h(x)$ on Sf, then the same inequality holds on X.*

PROOF. Let $v = u_f \wedge h$. Then

$$v = [u_f + h - |u_f - h|]/2 \in \mathbf{D}^{(p)}(N).$$

Since u_f and h are p-superharmonic, v is also p-superharmonic by Theorem 7.6. Since $0 \leq v \leq u_f$ on X and $u_f \in \mathbf{D}_0^{(p)}(N)$, we see by [8; Theorem 3.2] that $v \in \mathbf{D}_0^{(p)}(N)$. Thus v is a pure potential. By assumption, $v(x) = u_f(x)$ on Sf, and hence $u_f(x) \leq v(x)$ on X by Corollary 6.9. Therefore $u_f(x) \leq h(x)$ on X.

§7. Appendix: p-superharmonic functions

We shall review some properties of p-superharmonic functions on an infinite network. The results in this section are special cases of the general theory due to Maeda [2]. For the study of nonlinear networks, it is worth reproducing some parts of his preliminary manuscript [1].

Denote by $U(a)$ the set of neighboring nodes of a, by $S_a(z)$ the set of arcs between a and z. Then

$$\Delta_p f(a) = -\sum_{z \in U(a)} \sum_{y \in S_a(z)} r(y)^{1-p} \phi_p(f(a) - f(z)).$$

Given $a \in X$ and a real valued function f on $U(a)$, define the function $F_{a,f}(t)$ for $t \in R$ by

$$F_{a,f}(t) = \sum_{z \in U(a)} \sum_{y \in S_a(z)} r(y)^{1-p} \phi_p(t - f(z)).$$

Then $F_{a,f}$ is a continuous and strictly increasing function on R and $\Delta_p f(a) = -F_{a,f}(f(a))$. Furthermore

$$F_{a,f}(t) < 0 \quad \text{if } t < \min\{f(z);\, z \in U(a)\};$$

$$F_{a,f}(t) > 0 \quad \text{if } t > \max\{f(z);\, z \in U(a)\}.$$

Therefore there exists a unique $t_0 \in R$ such that $F_{a,f}(t_0) = 0$. Denote by $m_p(f; a)$ this t_0, i.e.,

$$F_{a,f}(m_p(f; a)) = 0.$$

This $m_p(f; a)$ is regarded as a kind of mean value of f on $U(a)$.

We can easily prove the following:

THEOREM 7.1. *Let $u \in L(X)$ and A be a subset of X. Then the following are equivalent:*

(a) *u is p-harmonic (resp. p-superharmonic) on A, i.e.,*

$$\Delta_p u(x) = 0 \quad (resp.\ \Delta_p u(x) \leq 0) \ on\ A.$$

(*b*) $u(x) = m_p(u; x)$(*resp.* $u(x) \geq m_p(u; x)$) *on* A;

(*c*) $\langle \phi_p(du), df \rangle = 0$ (*resp.* $\langle \phi_p(du), df \rangle \geq 0$) *for every* $f \in L_0^+(X)$ *such that* $Sf \subset A$.

LEMMA 7.2. *Let* $u, v \in L(X)$ *and* $a \in X$.

(i) *If* $u \leq v$ *on* $U(a)$, *then* $m_p(u; a) \leq m_p(v; a)$

(ii) *If* $u \leq v$ *on* $U(a)$ *and* $u(z) < v(z)$ *for some* $z \in U(a)$, *then* $m_p(u; a) < m_p(v; a)$.

PROOF. (i) Since $t - u(z) \geq t - v(z)$ on $U(a)$, $F_{a,u}(t) \geq F_{a,v}(t)$ for every $t \in R$. Thus $F_{a,v}(m_p(u; a)) \leq F_{a,u}(m_p(u; a)) = 0$ and $m_p(u; a) \leq m_p(v; a)$. To prove (ii), let $u \leq v$ on $U(a)$ and $u(z) < v(z)$ for some $z \in U(a)$. Then we see by the above observation that $F_{a,u}(t) > F_{a,v}(t)$ for every $t \in R$. Thus $F_{a,v}(m_p(u; a)) < 0$ and $m_p(u; a) < m_p(v; a)$.

As a fundamental relation of $m_p(f; a)$, we observe

(iii) $m_p(-f; a) = -m_p(f; a)$.

This follows from the relation $F_{a,-f}(t) = -F_{a,f}(-t)$.

LEMMA 7.3. (*Local minimum principle*) *Let* $u, v \in L(X)$ *and* $a \in X$. *If* u *and* v *are* p-*superharmonic at* a *and* $u(z) + v(z) \geq 0$ *for all* $z \in U(a)$, *then* $u(a) + v(a) \geq 0$ *and* $u(a) + v(a) = 0$ *occurs only when* $u(z) + v(z) = 0$ *for all* $z \in U(a)$.

PROOF. Since $u \geq -v$ on $U(a)$, we see by Lemma 7.2 and (iii) that

$$m_p(u; a) \geq m_p(-v; a) = -m_p(v; a).$$

Since u and v are p-superharmonic at a, $u(a) \geq m_p(u; a)$ and $v(a) \geq m_p(v; a)$. Hence $u(a) \geq -v(a)$. If, in addition, $u(z) + v(z) > 0$ for some $z \in U(a)$, then $m_p(u; a) > -m_p(v; a)$ by Lemma 7.2 (ii), so that $u(a) > v(a)$.

COROLLARY 7.4. *Let* $u, v \in L(X)$ *be* p-*superharmonic on* X. *If* $u + v \geq 0$ *on* X *and* $u(a) + v(a) = 0$ *for some* $a \in X$, *then* $u + v = 0$ *on* X.

THEOREM 7.5. (*Minimum principle*). *Let* A *be a finite subset of* X *and let* $u, v \in L(X)$ *be* p-*superharmonic on* A. *If* $u + v \geq 0$ *on* $X - A$, *then* $u + v \geq 0$ *on* X.

PROOF. Let $c = \min\{u(x) + v(x); x \in A\}$. Suppose that $c < 0$ and put $B = \{x \in X; u(x) + v(x) = c\}$. By our assumption, $B \subset A$ and $B \neq \emptyset$. Since $c < 0$ and $u + v \geq 0$ on $X - A$, $(u - c) + v \geq 0$ on X. Noting that $u - c$ is p-superharmonic and that $(u - c) + v = 0$ on B, we see by Lemma 7.3 that $U(x) \subset B$ for all $x \in B$. Since N is connected, it follows that $B = X$, a contradiction. Hence $c \geq 0$, i.e., $u + v \geq 0$ on A.

As a special case of [1; Corollary of Proposition 1.3], we have

THEOREM 7.6. *If* u *and* v *are p-superharmonic on* A, *then so is* $u \wedge v$.

PROOF. Put $f = u \wedge v$. For any $a \in A, f \leq u$ and $f \leq v$ on $U(a)$, so by Lemma 7.2

$$F_{a,f}(u(a)) \geq F_{a,u}(u(a)) \geq 0,$$

$$F_{a,f}(v(a)) \geq F_{a,v}(v(a)) \geq 0.$$

Hence,

$$-\Delta_p f(a) = F_{a,f}(f(a)) \geq \min\{F_{a,f}(u(a)), F_{a,f}(v(a))\} \geq 0.$$

Therefore f is p-superharmonic at a.

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