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On Tangent Algebras of Symmetrizable Homogeneous Left Lie Loops

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After introducing the concept of feigned Lie triple algebras, we investigate local symmetrizability of geodesic homogeneous left Lie loops in terms of their tangent Lie triple algebras.

Introduction

It is well-known that geodesic homogeneous left Lie loops on analytic manifolds are locally characterized by their tangent Lie triple algebras. In [6], the concept of projectivity of geodesic homogeneous left Lie loops is introduced: Two geodesic homogeneous left Lie loops (G, μ) and (G, μ') on the same analytic manifold G is *in projective relation* if they have the same identity element, the same system of geodesics in G with respect to the respective canonical connections ∇ and ∇', and the following relations are valid among them:

$$\eta(u, v, \eta'(x, y, z)) = \eta'(\eta(u, v, x), \eta(u, v, y), \eta(u, v, z)),$$

$$\eta'(u, v, \eta(x, y, z)) = \eta(\eta'(u, v, x), \eta'(u, v, y), \eta'(u, v, z)),$$

where (G, η) (resp. (G, η')) is the homogeneous system of (G, μ) (resp. (G, μ')) (cf. [3], [4], [8]). Especially, if (G, μ) is in projective relation with a symmetric left Lie loop (cf. [1], [2]), it is said to be *symmetrizable* (cf. [9]).

In this paper, symmetrizability and local symmetrizability of geodesic homogeneous left Lie loops are studied in terms of their tangent Lie triple algebras. To do this, the concept of feigned Lie triple algebra is introduced in §1 and some elementary properties of feigned Lie triple algebras are presented. The main theorem is given in §2 (Theorem 2.3): A geodesic homogeneous left Lie loop is locally symmetrizable if and only if its tangent Lie triple algebra is feigned. In §3, a condition is found for a geodesic homogeneous left Lie loop in projective relation with a symmetrizable one to be symmetrizable too. Finally, by applying the result obtained in [12], symmetrizability of odd-dimensional geodesic homogeneous left Lie loops is determined in §4.

§1. Feigned Lie triple algebras

In this section, we consider a particular class of Lie triple algebras composed of Lie triple systems and Lie algebras.

DEFINITION 1.1. A vector space $\mathfrak{g}$ over a field of characteristic 0, equipped with a bilinear system $\mathfrak{g}_B=\{\mathfrak{g}; [X, Y]\}$ given by

$$B: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g};\ B(X, Y)=[X, Y]$$

and a trilinear system $\mathfrak{g}_D=\{\mathfrak{g}; \langle X, Y, Z\rangle\}$ given by

$$D: \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g};\ D(X, Y, Z)=\langle X, Y, Z\rangle,$$

is called a *Lie triple algebra* (*general Lie triple system* of K. Yamaguti [13]) if the following relations hold among $\mathfrak{g}_B$ and $\mathfrak{g}_D$:

$$[X, X]=0, \tag{1.1.1}$$

$$\langle X, X, Y\rangle=0, \tag{1.1.2}$$

$$\mathfrak{S}_{X,Y,Z}\{\langle X, Y, Z\rangle+[[X, Y], Z]\}=0, \tag{1.1.3}$$

$$\mathfrak{S}_{X,Y,Z}\{\langle [X, Y], Z, W\rangle\}=0, \tag{1.1.4}$$

$$\langle U, V, [X, Y]\rangle=[\langle U, V, X\rangle, Y]+[X, \langle U, V, Y\rangle], \tag{1.1.5}$$

$$\langle U, V, \langle X, Y, Z\rangle\rangle=\langle\langle U, V, X\rangle, Y, Z\rangle+\langle X, \langle U, V, Y\rangle, Z\rangle + \langle X, Y, \langle U, V, Z\rangle\rangle \tag{1.1.6}$$

for any $X, Y, Z, W, U, V\in\mathfrak{g}$, where $\mathfrak{S}$ denotes the cyclic sum.

In a Lie triple algebra $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$, we often consider the following endomorphisms;

$$B_X: \mathfrak{g}\to\mathfrak{g};\ B_X Y:=[X, Y],$$

and

$$D(X, Y): \mathfrak{g}\to\mathfrak{g};\ D(X, Y)Z:=\langle X, Y, Z\rangle.$$

The endomorphisms $D(X, Y)$ for $X, Y\in\mathfrak{g}$ are called *inner derivations* of the Lie triple algebra $\mathfrak{g}$. In fact, the relations (1.1.5) and (1.1.6) mean that every inner derivation is a derivation of both of $\mathfrak{g}_B$ and $\mathfrak{g}_D$, that is,

$$D(\mathfrak{g}, \mathfrak{g})\subset \operatorname{Der}\mathfrak{g}_B\cap\operatorname{Der}\mathfrak{g}_D,$$

where $\operatorname{Der}\mathfrak{g}_B$ (resp. $\operatorname{Der}\mathfrak{g}_D$) denotes the Lie algebra of all derivations of $\mathfrak{g}_B$ (resp. $\mathfrak{g}_D$) and $D(\mathfrak{g}, \mathfrak{g})$ the Lie algebra of all inner derivations, called the *inner derivation algebra*.

A Lie algebra $\mathfrak{g}_B$ with the underlying vector space $\mathfrak{g}$ can be considered as a Lie triple algebra $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$ with a flat trilinear system $\mathfrak{g}_D$; $\langle \mathfrak{g}, \mathfrak{g}, \mathfrak{g}\rangle=\{0\}$. On the other hand, if $\mathfrak{g}_B$ is an abelian Lie algebra, a Lie triple algebra $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$ can be considered to be a Lie triple system $\mathfrak{g}_D$, that is, $\mathfrak{g}_D$ satisfies the relations

(1.1.2)′ $$\langle X, X, Z\rangle = 0,$$

(1.1.3)′ $$\mathfrak{S}_{X,Y,Z}\{\langle X, Y, Z\rangle\} = 0,$$

(1.1.6)′ $$D(\mathfrak{g}, \mathfrak{g}) \subset \mathrm{Der}\,\mathfrak{g}_D.$$

It is easy to show the following;

LEMMA 1.2. *Let $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$ be a Lie triple algebra. The bilinear system $\mathfrak{g}_B$ forms a Lie algebra if and only if the trilinear system $\mathfrak{g}_D$ forms a Lie triple system. A pair of a Lie algebra $\mathfrak{g}_B$ and a Lie triple system $\mathfrak{g}_D$ on the same underlying vector space $\mathfrak{g}$ forms a Lie triple algebra if and only if the following relations hold*:

(1.2.1) $$\mathfrak{S}_{X,Y,Z}\{\langle [X, Y], Z, W\rangle\} = 0.$$

(1.2.2) $$D(\mathfrak{g}, \mathfrak{g}) \subset \mathrm{Der}\,\mathfrak{g}_B.$$

DEFINITION 1.3. A Lie triple algebra $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$ will be called to be *feigned* if $\mathfrak{g}_B$ forms a Lie algebra, $\mathfrak{g}_D$ forms a Lie triple system and the relation

(1.3) $$B_{\mathfrak{g}} \subset \mathrm{Der}\,\mathfrak{g}_D$$

holds, where $B_{\mathfrak{g}}$ is the inner derivation algebra $\mathrm{ad}_B\mathfrak{g}$ of the Lie algebra $\mathfrak{g}_B$.

We obtain the following;

LEMMA 1.4. *A pair $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$ of a Lie algebra $\mathfrak{g}_B$ and a Lie triple system $\mathfrak{g}_D$ is a feigned Lie triple algebra if and only if the relations* (1.2.1), (1.2.2) *and* (1.3) *hold.*

PROOF. This is obtained immediately from Definition 1.3 and Lemma 1.2. q.e.d.

THEOREM 1.5. *A pair $\mathfrak{g}=(\mathfrak{g}_B, \mathfrak{g}_D)$ of a Lie algebra $\mathfrak{g}_B$ and a Lie triple system $\mathfrak{g}_D$ forms a feigned Lie triple algebra if and only if the relations*

$$B_{\mathfrak{g}} \subset \mathrm{Der}\,\mathfrak{g}_D, \quad D(\mathfrak{g}, \mathfrak{g}) \subset \mathrm{Der}\,\mathfrak{g}_B$$

and

(1.4) $$D([X, Y], Z)W = [D(X, Y)Z, W]$$

are satisfied.

PROOF. In fact, the relation (1.2.1) is equivalent to (1.4) under the condition (1.3). q.e.d.

COROLLARY 1.6. *Let* $\mathfrak{g} = (\mathfrak{g}_B, \mathfrak{g}_D)$ *be a Lie triple algebra. Assume that* $\mathfrak{g}_B$ *forms a Lie algebra* (*or*, $\mathfrak{g}_D$ *forms a Lie triple system*). *Then the relations* (1.3) *and* (1.4) *are multually equivalent.*

PROOF. Theorem 1.5 shows that (1.3) implies (1.4) under (1.2.1). Conversely, from (1.2.1), we get

$$\langle B_X Y, Z, W\rangle = \langle B_Z Y, X, W\rangle + \langle Y, B_Z X, W\rangle.$$

Then from (1.2.1) and (1.2.2) the following is obtained:

$$\begin{aligned}\langle B_X Y, Z, W\rangle &= B_Z\langle Y, X, W\rangle - \langle Y, X, B_Z W\rangle \\ &= [\langle X, Y, Z\rangle, W].\end{aligned}$$

q.e.d.

Also, we can easily show the following;

PROPOSITION 1.7. *Let* $\mathfrak{g} = (\mathfrak{g}_B, \mathfrak{g}_D)$ *be a Lie triple algebra. Assume that* $\mathfrak{g}_B$ *forms a Lie algebra* (*or*, $\mathfrak{g}_D$ *forms a Lie triple system*). *Then, for any fixed element* α *of the base field, the trilinear system* $\mathfrak{g}_D^\alpha$ *given by*

$$\langle X, Y, Z\rangle^\alpha := \langle X, Y, Z\rangle + \alpha[[X, Y], Z] \tag{1.5}$$

forms a Lie triple system if and only if the following relation holds:

$$B_{[\mathfrak{g},\mathfrak{g}]} \subset \operatorname{Der} \mathfrak{g}_D. \tag{1.6}$$

COROLLARY 1.8. *For a feigned Lie triple algebra, the trilinear system given by* (1.5) *is a Lie triple system.*

§2. Tangent Lie triple algebras of symmetrizable left Lie loops.

In [9] the concept of symmetrizability of geodesic hmogeneous left Lie loops has been introduced. In this section, we consider the tangent Lie triple algebras of symmetrizable left Lie loops.

DEFINITION 2.1. A geodesic homogeneous left Lie loop (G, μ) on a connected analytic manifold G is *symmetrizable* if it is in projective relation with a symmetric left Lie loop on G.

The real vector space $\mathfrak{X}(G)$ of all analytic vector fields on a geodesic homogeneous left Lie loop (G, μ) (or, on any locally reductive space in general) forms a Lie triple algebra $\mathfrak{X}(G) = (\mathfrak{X}_S, \mathfrak{X}_R)$ with the bilinear system

$$S;\ [X, Y] := S(X, Y) \tag{2.1}$$

and the trilinear system

$$R; \langle X, Y, Z\rangle := R(X, Y)Z, \tag{2.2}$$

where S and R denote respectively the torsion and the curvature tensor of the canonical connection ∇ of (G, μ) (cf., e.g. [8]). The following theorem has been proved in [9] (Theorem 4 [9]):

THEOREM 2.2. *The Lie triple algebra $\mathfrak{X}(G) = (\mathfrak{X}_S, \mathfrak{X}_R)$ of a symmetrizable geodesic homogeneous left Lie loop (G, μ) is a feigned Lie triple algebra.*

PROOF. By Theorem 4 in [9], we get

$$\mathfrak{S}_{X,Y,Z}\{S(S(X, Y), Z)\} = 0, \tag{2.3}$$

$$S(X, R(Y, Z)W) = R(S(X, Y), Z)W + R(Y, S(X, Z))W + R(Y, Z)S(X, W). \tag{2.4}$$

Then, we see that $\mathfrak{X}_S$ forms a Lie algebra, and Lemma 1.2 implies that the Lie triple algebra $\mathfrak{X}(G)$ is feigned. q.e.d.

Let $\mathfrak{g} = T_e(G)$ denote the tangent space of a geodesic homogenous left Lie loop (G, μ), at the identity element e. The *tangent Lie triple algebra* of (G, μ) is, by definition, the finite dimensional real Lie triple algebra $\mathfrak{g} = (\mathfrak{g}_S, \mathfrak{g}_R)$ given by evaluating the torsion and the curvature at e in the Lie triple algebra $\mathfrak{X}(G) = (\mathfrak{X}_S, \mathfrak{X}_R)$ above, i.e.,

$$S; [X,Y] = S_e(X, Y), \tag{2.5}$$

$$R; \langle X, Y, Z\rangle = R_e(X, Y)Z. \tag{2.6}$$

THEOREM 2.3. *A geodesic homogeneous left Lie loop (G, μ) on a connected analytic manifold G is locally symmetrizable, that is, there exists a symmetric local Lie loop $\tilde{\mu}$ around the identity e of (G, μ) such that (G, μ) is in projective relation with $\tilde{\mu}$, if and only if the tangent Lie triple algebra $\mathfrak{g} = (\mathfrak{g}_S, \mathfrak{g}_R)$ of (G, μ) is feigned.*

PROOF. Assume that the tangent Lie triple algebra $\mathfrak{g}$ is feigned. Then, by Theorem 1.5, the relation (1.4) holds for bilinear-trilinear system (2.5) and (2.6), where $D(X, Y) = R_e(X, Y)$. Let U be a normal neighborhood of e with respect to the canonical connection ∇. Since the torsion S and the curvature R satisfy $\nabla S = 0$ and $\nabla R = 0$, and since any left translation L_x by $x \in U$ induces the parallel displacement of the tangent vectors along the geodesic arc joning e to x, the relation (1.4) for the tangent Lie triple algebra $\mathfrak{g}$ implies the following formulas for S and R, in the normal neighborhood U:

(2.7) $$R(S(X, Y), Z)W = S(R(X, Y)Z, W).$$

This relation is assured on the whole manifold G by virtue of analyticity of S and R. In the same way, we see that the other relations in Theorem 1.5 hold for S and R in the Lie triple algebra $\mathfrak{X}(G)$. Now, set $T = -S/2$ and $\tilde{\nabla} = \nabla - T$. Then, by using Proposition 1.1 in [5], it can be shown that T (resp. $-T$) satisfies the conditions for affine homogeneous structure of ∇ (resp. $\tilde{\nabla}$), and that the torsion $\tilde{S}$ of $\tilde{\nabla}$ vanishes identically on G. Hence, $(G, \tilde{\nabla})$ is a locally symmetric space with the curvature tensor $\tilde{R}$ given by;

(2.8) $$\tilde{R}(X, Y)Z = R(X, Y)Z + (1/4)S(S(X, Y), Z),$$

which forms a Lie triple system $\mathfrak{X}_{\tilde{R}}$ on $\mathfrak{X}(G)$ (cf. Proposition 1.1 [5]). The neighborhood U can be regarded as a common normal neighborhood of ∇ and $\tilde{\nabla}$, and, for any $x \in U$, there exists a unique affine transformation $\tilde{\eta}(e, x)$ of $\tilde{\nabla}$ which induces the parallel displacement along the geodesic arc from e to x in U (p. 252 [10]). By setting $\tilde{\mu}(x, y) = \tilde{\eta}(e, x)y$, we have a symmetric local loop $\tilde{\mu}$ around e (cf. [1], [11]). Denote by $\eta(e, x) = L_x$ the left translation of (G, μ) by x. Since $\eta(e, x)$ is an affine transformation of the canonical connection ∇, it is an affine transformation of $\tilde{\nabla} = \nabla - S/2$, too. Conversely, since $\nabla S = 0$ holds, any displacement $\tilde{\eta}(e, x)$ is an affine transformation of ∇. Therefore, we see that μ and $\tilde{\mu}$ are in projective relation, which completes the proof of the sufficiency part of the theorem.

The necessity part is proved by taking account of the same results as Theorem 2.2, for geodesic homogeneous local Lie loops. q.e.d.

§3. Projectivity and symmetrizability

In this section, we consider geodesic homogeneous left Lie loops in projective relation with a given locally symmetrizable one.

The following result has been shown essentially in [6] (cf. [5], [6], [8]). We review it in terms of bilinear-trilinear system on the real vector space $\mathfrak{X} = \mathfrak{X}(G)$ of all analytic vector fields on an analytic manifold G:

THEOREM 3.1. *Let* (G, μ) *and* (G, μ') *be geodesic homogeneous left Lie loops on the same connected analytic manifold* G, *and* S, R (resp. S', R') *denote respectively the torsion and the curvature tensor fields of the canonical connection* ∇ (resp. ∇') *of* μ (resp. μ'). *If* (G, μ) *and* (G, μ') *are in projective relation, then the tensor field* $T = (S' - S)/2$ *satisfies the following* :

(3.1.1) $$\mathfrak{S}_{X,Y,Z}\{T(T(X, Y), Z)\} = 0,$$

(3.1.2) $$T_{\mathfrak{X}} \subset \mathrm{Der}\,\mathfrak{X}_S \cap \mathrm{Der}\,\mathfrak{X}_R \cap \mathrm{Der}\,\mathfrak{X}_{S'} \cap \mathrm{Der}\,\mathfrak{X}_{R'},$$

(3.1.3) $$R(\mathfrak{X}, \mathfrak{X}) \cup R'(\mathfrak{X}, \mathfrak{X}) \subset \mathrm{Der}\, \mathfrak{X}_T,$$

where $\mathfrak{X}_T$ *denotes the Lie algebra defined by* T (cf. (3.1.1)), $T_{\mathfrak{X}} = \mathrm{ad}_T \mathfrak{X}$ *the inner derivation algebra of* $\mathfrak{X}_T$, *and* $\mathfrak{X} = (\mathfrak{X}_S, \mathfrak{X}_R)$ (resp. $\mathfrak{X}' = (\mathfrak{X}_{S'}, \mathfrak{X}_{R'})$) *the Lie triple algebra with respect to* ∇ (resp. ∇').

Moreover, the following formula is valid in $\mathfrak{X}(G)$:

(3.2) $$R'(X, Y)Z - R(X, Y)Z = (1/4)(S - S')((S + S')(X, Y), Z).$$

Assume that the geodesic homogeneous left Lie loop (G, μ) is symmetrizable. Then it satisfies;

(3.3) $$S_{\mathfrak{X}} \subset \mathrm{Der}\, \mathfrak{X}_S \cap \mathrm{Der}\, \mathfrak{X}_R.$$

If (G, μ') is in projective relation with (G, μ), the relations (3.1.2) and (3.3) imply

(3.4) $$S'_{\mathfrak{X}} = (2T + S)_{\mathfrak{X}} \subset \mathrm{Der}\, \mathfrak{X}_S \cap \mathrm{Der}\, \mathfrak{X}_R.$$

By considering these relations on the tangent Lie triple algebras and by applying Theorem 2.3, we get;

THEOREM 3.2. *Let* (G, μ) *and* (G, μ') *be geodesic homogeneous left Lie loops on* G, *which are in projective relation, and assume that* (G, μ) *is locally symmetrizable. Then,* (G, μ') *is locally symmetrizable if and only if their tangent Lie triple algebras* $\mathfrak{g} = (\mathfrak{g}_S, \mathfrak{g}_R)$ *and* $\mathfrak{g}' = (\mathfrak{g}_{S'}, \mathfrak{g}_{R'})$ *satisfy one of, hence both of, the following relations*:

(3.5) $$[X, [Y, Z]'] = [[X, Y], Z]' + [Y, [X, Z]]',$$

(3.5)′ $$[X, [Y, Z]]' = [[X, Y]', Z] + [Y, [X, Z]'],$$

where $[X, Y] = S_e(X, Y)$ *and* $[X, Y]' = S'_e(X, Y)$.

PROOF. By assumption, the tangent Lie triple algebra $\mathfrak{g}$ satisfies that $\mathfrak{g}_S$ forms a Lie algebra, $\mathfrak{g}_R$ forms a Lie triple system and

(3.6) $$\mathrm{ad}_S \mathfrak{g} \subset \mathrm{Der}\, \mathfrak{g}_R.$$

Here, ad_S denotes the inner derivation of the Lie algebra $\mathfrak{g}_S$. Besides these, the relations (3.1.1-3) are satisfied by $\mathfrak{g}_S$ and $\mathfrak{g}_R$. By Theorem 2.3, (G, μ') is locally symmetrizable if and only if $\mathfrak{g}_{S'}$ forms a Lie algebra and the following relation holds:

(3.7) $$\mathrm{ad}_{S'} \mathfrak{g} \subset \mathrm{Der}\, \mathfrak{g}_{R'}.$$

By (3.1.1-2), $T = (S' - S)/2$ forms a Lie algebra in $\mathfrak{X}(G)$ whose inner derivations are derivations of both of $\mathfrak{X}_S$ and $\mathfrak{X}_{S'}$. Hence by restricting these on the tangent space $\mathfrak{g} = T_e(G)$, the following relation holds if and only if $\mathfrak{g}_{S'}$ forms a Lie algebra:

(3.8) $$S_e(X, T_e(Y, Z)) = T_e(S_e(X, Y), Z) + T_e(Y, S_e(X, Z))$$

for any $X \in T_e(G)$, which is equivalent to (3.5) since $T=(S'-S)/2$. We see that the relation (3.5)' is also equivalent to (3.8) under the relation (3.1.1). On the other hand, by (3.2), the curvature R' is expressed in terms of R, S and S'. Hence (3.1.2), (3.5)' and (3.6) imply

$$\mathrm{ad}_S\mathfrak{g} \cup \mathrm{ad}_T\mathfrak{g} \subset \mathrm{Der}\,\mathfrak{g}_{R'}.$$

This shows that (3.7) is always assured by (3.5)'. q.e.d.

§4. Projective double Lie algebras on feigned Lie triple algebras

In [12], we have introduced the concept of projective double Lie algebras on a given Lie algebra. For feigned Lie triple algebras, Theorem 3.2 suggests us that the projective double Lie algebras should play an important role on the tangent Lie triple algebras of symmetrizable geodesic left Lie loops.

DEFINITION 4.1. Let $\mathfrak{g}$ be a Lie algebra. A Lie algebra $\mathfrak{h}$ with the same underlying vector space as $\mathfrak{g}$ is called a *projective double Lie algebra* on $\mathfrak{g}$ if the inner derivation algebra $\mathrm{ad}_{\mathfrak{h}}\mathfrak{h}$ of $\mathfrak{h}$ is a subalgebra of $\mathrm{Der}\,\mathfrak{g}$ (cf. [12]).

Under the assumption of Theorem 3.2, it is asserted that the geodesic homogeneues left Lie loop (G, μ') is locally symmetrizable if and only if $\mathfrak{g}_{S'}$ is a projective double Lie algebra on $\mathfrak{g}_S$. Therefore, by Theorem 1 in [12] and Theorem 3.1, we get;

THEOREM 4.2. *Under the same assumption as* Theorem 3.2, *if the manifold G is of odd-dimension and if the Lie algebra $\mathfrak{g}_S$ is simple, then the torsion tensor S' of locally symmetrizable (G, μ') is given by*

$$S' = pS, \quad \textit{for same real constant } p. \tag{4.1}$$

In this case, the curvature tensor R' is given by

$$R'(X, Y)Z = R(X, Y)Z + (1-p^2)/4\, S(S(X, Y), Z). \tag{4.2}$$

PROOF. In fact, Theorem 1 in [12] asserts that any projective double Lie algebra $\mathfrak{h}=\{\mathfrak{V}; [X, Y]'\}$ on an odd-dimensional real simple Lie algebra $\mathfrak{g}=\{\mathfrak{V}; [X, Y]\}$ with the same underlying vector space $\mathfrak{V}$ should satisfy;

$$[X, Y]' = p[X, Y], \quad X, Y \in \mathfrak{V} \tag{4.3}$$

for some real number p. Since the tensor fields S and S' are parallel tensor fields with respect to both of the canonical connections ∇ and ∇', (4.1) holds on whole manifold G. The relation (4.2) follows immediately from (3.2) in Theorem 3.1.

q.e.d.

REMARK. The tangent Lie triple algebra $\mathfrak{g}' = (\mathfrak{g}_{S'}, \mathfrak{g}_{R'})$ in Theorem 4.2 above is a special case of feigned Lie triple algebra in Proposition 1.7. In fact, (4.2) is reduced to the Lie triple system (1.5) for $\alpha = (1 - p^2)/4$.

References

[1] M. Kikkawa, On some quasigroups of algebraic models of symmetric spaces, Mem. Fac. Lit. & Sci. Shimane Univ., Nat. Sci **6** (1973), 9–13; II, ibid. **7** (1974), 29–35; III, ibid. **9** (1975), 7–12.

[2] ———, Geometry of homogeneous Lie loops, Hiroshima Math. J. **5** (1975), 141–179.

[3] ———, On the left translations of homogeneous loops, Mem. Fac. Lit. & Sci. Shimane Univ., Nat. Sci. **10** (1976) 19–25.

[4] ———, On homogeneous systems I, ibid. **11** (1977), 9–17; II, Mem. Fac Sci. Shimane Univ. **12** (1978), 5–13; III, ibid. **14** (1980), 41–46; IV, ibid. **15** (1981), 1–7; V, ibid. **16** (1982), 23–27.

[5] ———, Affine homogeneous structures on analytic loops, ibid. **21** (1978), 1–15.

[6] ———, Projectivity of left loops on $\mathbf{R}^n$, ibid. **22** (1988), 33–41.

[7] ———, Projectivity of left loops on Lie groups II, ibid. **24** (1990), 1–16.

[8] ———, Projectivity of homogeneous left loops, Non-associative Algebras and Related Topics, World Scientific 1991, 77–99.

[9] ———, Symmetrizability of geodesic homogeneous left Lie loops, Proc. 3rd Congress of Geometry, Thessaloniki Univ. 1991, 230–238.

[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry I, Interscience 1963. (11] O. Loos, Symmetric Spaces I, Benjamin 1969.

[12] M. Sanami and M. Kikkawa, A class of double Lie algebras on simple Lie algebras and projectivity of simple Lie groups, Mem. Fac. Sci, Shimane Univ. **25** (1991), 39–44.

[13] K. Yamaguti, On the Lie triple system and its generalization, J. Sci. Hiroshima Univ. **A-21** (1958), 155–160.